EXACT COMPUTATION OF THE ASYMPTOTIC EFFICIENCY OF MAXIMUM LIKELIHOOD ESTIMATORS OF A DISCONTINUOUS SIGNAL IN A GAUSSIAN WHITE NOISE

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Abstract

In this paper, the problem of computing the exact value of the asymptotic efficiency of maximum likelihood estimators of a discontinuous signal in a Gaussian white noise is considered. A method based on constructing difference equations for the appropriate moments is presented and used to show that the exact variance of the Pitman estimator is $16\zeta(3)$, where ζ is the Riemann zeta function.

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1. Introduction.

Consider the problem of estimating a one-dimensional parameter θ based on observations of the process X(t) satisfying the stochastic differential equation

$$dX(t) = \frac{1}{\epsilon}S(t-\theta)dt + dW(t),\ t \in [0,1] \tag{1}$$

where S is a function possessing at least one discontinuity of the first kind in the interval of observations, ϵ is a small parameter, and $W(t)$ is a standard Wiener process. This estimation problem may also be referred to as estimation of a change-point as it is the continuous analog of the classical change-point problems in the regression context.

The problem considered by many comes from considering the asymptotic situation, in which S can be taken to be constant except for one discontinuity, and instead of using $[0,1]$ we use $(-\infty,\infty)$. In this case the problem is location invariant, and Pitman (1938) showed that the best invariant procedure for such a problem is the formal Bayes procedure with a uniform "prior" on the entire real line.

As this is not a regular estimation problem, the maximum likelihood estimator is, as is usually the case here, not asymptotically efficient. It is natural, therefore, to ask to compare the variances of the maximum likelihood estimator and that of the best invariant estimator.

The formal setup is that specified in Ibragimov and Has'minskii (1981). We wish to obtain the ratio $\kappa = E\xi_2^2/E\xi_1^2$, where ξ_1 and ξ_2 are defined as follows:

$$\xi_1 = \arg\max_{t\in R_1} Z_0(t), \tag{2}$$

$$\xi_2 = \int_{-\infty}^{+\infty} tZ_0(t)\,dt\,(\int_{-\infty}^{+\infty} Z_0(t)dt)^{-1}, \tag{3}$$

$$Z_0(t) = \exp(W(t) - \frac{1}{2}|t|). \tag{4}$$

where $W(t)$ is a two-sided Brownian motion defined as

$$W(t) = \begin{cases} W_1(t) & \text{if } t \geq 0 \\ W_2(-t) & \text{if } t < 0 \end{cases}$$

and $W_i(t)$ are independent standard Wiener processes defined for $t \geq 0$ and such that $W_i(0) = 0$.

The exact evaluation of $E\xi_1^2$ is not too difficult. Ibragimov and Has'minskii (1981) showed that $E\xi_1^2 = 26$. An attempt to evaluate $E\xi_2^2$ was also made in their monograph. Unfortunately, as they stated in the book, it seems that it is difficult to evaluate $E\xi_2^2$ explicitly. Instead they obtained a method for an approximate calculation of $E\xi_2^2$ and obtained through statistical simulation a value of 19.5 ± 0.5. Golubev (1979) proved that $E\xi_2^2$ is the second derivative of an improper integral of a composite function of modified Hankel and Bessel functions with respect to a parameter μ evaluated at 0. Again, the exact evaluation of the result has only been obtained by computer assistance. In this paper, we shall present a method based on constructing difference equations for the appropriate moments to compute the exact value of $E\xi_2^2$.

For an early discussion of the above problem, see Rubin (1961). A related problem of the Pitman estimator for the absolute error-loss function was considered by Paranjape and Rubin (1975). The exact distribution of the estimator was obtained in that paper. It may be worth mentioning that the problem of determining the distribution of the Pitman estimator for the quadratic loss function remains unsolved.

2. Main result.

In this section, we first state the main result and prove it through a series of Lemmas.

Theorem 1.

$$E\xi_2^2 = 16\zeta(3) \tag{5}$$

where ζ is Riemann's zeta function defined as $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$

Let $X_\lambda, Y_\lambda, Z_\lambda$ be defined as

$$X_\lambda = \int_0^\infty \exp(W_1(t) - \lambda t)dt \tag{6}$$

$$Y_\lambda = \int_0^\infty t \exp(W_1(t) - \lambda t)dt \tag{7}$$

$$Z_\lambda = \int_0^\infty t^2 \exp(W_1(t) - \lambda t)dt \tag{8}$$

where $\lambda > 0$.

Lemma 1. *The reciprocal of the random variable X_λ has a gamma distribution with the density defined by the formula*

$$f(x) = \frac{2^{2\lambda} x^{2\lambda-1} e^{-2x}}{\Gamma(2\lambda)}$$

where $\lambda \geq 1/2$

Proof. Consider the random process

$$X(t) = \exp(-W(t) - \lambda t) \int_{-\infty}^{t} \exp(W(s) + \lambda s)ds \tag{9}$$

Observe that for any t the distribution of $X(t)$ is the same and coincides with the distribution of X_λ (See Ibragimov and Has'minskii (1981)). Using Ito's formula, we obtain the following stochastic differential for the process $X(t)$

$$dX(t) = -X(t)dW(t) + (1 - (\lambda - 1/2)X(t))dt \tag{10}$$

It follows that the stationary density $g(x)$ of the process $X(t)$ satisfies the differential equation

$$\frac{1}{2}\frac{d^2}{dx^2}(x^2 g) - \frac{d}{dx}(1 - (\lambda - 1/2)x)g = 0 \tag{11}$$

subject to the constraint

$$\int_0^\infty g(x)dx = 1$$

Solving the above equation, we find

$$g(x) = \frac{2^{2\lambda} x^{-(2\lambda+1)} e^{(-2/x)}}{\Gamma(2\lambda)} \tag{12}$$

which completes the proof. $\square$

Lemma 2. *Let p be any nonnegative integer, then $EY_\lambda/X_\lambda^p < \infty$, where $\lambda > 1/2$.*

Proof. For any given $\lambda > 1/2$, choose a τ such that $1/2 < \tau < \lambda$, then

$$\frac{X_\tau - X_\lambda}{\lambda - \tau} > Y_\lambda.$$

By Lemma 1, X_λ has finite moments of all orders $k < 2\lambda$ and X_τ has finite moments of all orders $k < 2\tau$. Let $k = 1+\delta$, where $0 < \delta < 2\tau - 1$, then Minkowski's inequality implies

$$E(X_\tau - X_\lambda)^{1+\delta} \leq 2^{1+\delta} E X_\tau^{1+\delta}.$$

Therefore Y_λ has finite moments of all orders $1+\delta$. An application of Holder's inequality finishes the proof. □

Lemma 3. *Let $A_{\alpha,n}(\lambda) = EX_\lambda^\alpha(\frac{Y_\lambda}{X_\lambda})^n$, α is a nonpositive integer, $\lambda > 1/2$, then $A_{\alpha,1}(\lambda)$ satisfies the following difference equation:*

$$\frac{1}{2}\alpha(\alpha - 2\lambda)A_{\alpha,1}(\lambda) + (\alpha - 1)A_{\alpha-1,1}(\lambda) + A_{\alpha,0}(\lambda) = 0 \tag{13}$$

and the unique solution of the equation for $\alpha = -q$ is given by

$$A_{-q,1}(\lambda) = \frac{2\Gamma(q + 2\lambda)}{2^q\Gamma(2\lambda)} \int_0^1 \frac{t^{2\lambda-1}}{1-t} \frac{(1-t^q)}{q} dt \tag{14}$$

where Γ is the Gamma function and q is a nonnegative integer.

Proof. Note that, by Lemma 2, $A_{-q,1}(\lambda) = EY_\lambda/X_\lambda^{(q+1)} < \infty$ for all q. For any arbitrary small $\epsilon > 0$, let

$$\begin{aligned} S_1 &= \int_0^\epsilon \exp(W_1(t) - \lambda t)\, dt, \\ S_2 &= \int_0^\epsilon t \exp(W_1(t) - \lambda t)\, dt, \\ T &= \exp(W_1(\epsilon) - \lambda\epsilon). \end{aligned}$$

then, we have

$$X_\lambda = S_1 + TX'_\lambda, \tag{15}$$

$$Y_\lambda = S_2 + T(Y'_\lambda + \epsilon X'_\lambda). \tag{16}$$

where X'_λ and Y'_λ are independent of T, S_1, and S_2, (X'_λ, Y'_λ) and (X_λ, Y_λ) have the same joint distribution. Let $g(s) = 1/[s + TX'_\lambda]^{(q+1)}$, Taylor expansion of $g(s)$ at

$s = 0$ yields:

$$\frac{Y_\lambda}{X_\lambda^{q+1}} = [S_2 + T(Y'_\lambda + \epsilon X'_\lambda)]\, g(S_1) \tag{17}$$

$$= \frac{Y'_\lambda}{T^q (X'_\lambda)^{q+1}} + \frac{\epsilon}{T^q (X'_\lambda)^q} - \frac{(q+1)S_1 Y'_\lambda}{T^{q+1}(X'_\lambda)^{q+2}} + R \tag{18}$$

where

$$\begin{aligned} R \;=\; & \frac{S_2}{(TX'_\lambda)^{q+1}} - \frac{(q+1)S_1S_2}{(TX'_\lambda)^{q+2}} - \frac{(q+1)\epsilon S_1}{(TX'_\lambda)^{q+1}} \\ & + \frac{(q+1)(q+2)S_1^2 S_2}{2(\theta + TX'_\lambda)^{q+3}} + \frac{(q+1)(q+2)TY'_\lambda S_1^2}{2(\theta+TX'_\lambda)^{q+3}} + \frac{(q+1)(q+2)\epsilon T X'_\lambda S_1^2}{2(\theta+TX'_\lambda)^{q+3}} \end{aligned}$$

and $0 < \theta < S_1$. Taking expectation on both sides of equation (18) gives

$A_{-q,1}(\lambda) = A_{-q,1}(\lambda)ET^{-q} + \epsilon ET^{-q}A_{-q,0}(\lambda) - (q+1)ES_1T^{-(q+1)}A_{-(q+1),1}(\lambda) + ER$

Therefore

$(ET^{-q} - 1)A_{-q,1}(\lambda) + \epsilon ET^{-q}A_{-q,0}(\lambda) - (q+1)ES_1T^{-(q+1)}A_{-(q+1),1}(\lambda) + ER = 0$

Note that $ET^{-q} = \exp(q(q+2\lambda)\epsilon/2)$, and $\lim_{\epsilon\to 0} ES_1T^{-(q+1)}/\epsilon = 1$, let $\epsilon \to 0$, we have from the above equation:

$$\frac{1}{2}q(q+2\lambda)A_{-q,1}(\lambda) + A_{-q,0}(\lambda) - (q+1)A_{-(q+1),1}(\lambda) + \lim_{\epsilon\to 0} E\frac{R}{\epsilon} = 0 \tag{19}$$

It remains to show that $\lim_{\epsilon\to 0} ER/\epsilon = 0$. But this is true by noting that $S_2 \le \epsilon S_1$, and $\lim_{\epsilon\to 0} E(T^{-1}S_1/\epsilon)^m = 1$, where m is any positive integer, and by an application of Cauchy-Schwarz inequality. That (14) solves (13) is easily checked. □

Lemma 4. *Let $A_{\alpha,n}(\lambda)$ be defined as in Lemma 3, then $A_{\alpha,1}(1/2) < \infty$ and it satisfies the following difference equation:*

$$\frac{1}{2}\alpha(\alpha-1)A_{\alpha,1}(1/2) + (\alpha-1)A_{\alpha-1,1}(1/2) + A_{\alpha,0}(1/2) = 0 \tag{20}$$

and the unique solution of the equation for $\alpha = -q$ is given by

$$A_{-q,1}(1/2) = \frac{2\Gamma(q+1)}{2^q}\int_0^1 t^{q-1}[-\ln(1-t)]dt \tag{21}$$

Proof. It is enough to prove that $A_{-q,1}(1/2) < \infty$ since we can repeat the same argument as given in Lemma 3 to derive the difference equation. By Fatou's Lemma, we find

$$A_{-q,1}(1/2) = E\liminf_{\lambda\to 1/2^+} X_\lambda^{-q}\left(\frac{Y_\lambda}{X_\lambda}\right) \le \liminf_{\lambda\to 1/2^+} A_{-q,1}(\lambda) = \frac{\Gamma(q+1)}{2^{q-1}}\int_0^1 \frac{1-t^q}{(1-t)q}dt$$

which shows that $A_{-q,1}(1/2)$ is finite. $\square$

Remark. The fact that $A_{-q,1}(1/2) < \infty$ can be obtained directly by a completely different method using the reflection principle for Brownian motion (defining appropriate stopping time) and Fubini's theorem. Instead we present the above proof in Lemma 3 and Lemma 4 because the truncation argument used in the proof is more intuitive and elementary.

Lemma 5. *Let $B_{\alpha,n}(\lambda) = EX_\lambda^\alpha(\frac{Z_\lambda}{X_\lambda})^n$, then $B_{\alpha,1}(1/2)$ satisfies the following difference equation:*

$$\frac{1}{2}\alpha(\alpha-1)B_{\alpha,1}(1/2) + (\alpha-1)B_{\alpha-1,1}(1/2) + 2A_{\alpha,1}(1/2) = 0 \tag{22}$$

and the unique solution of the equation for $\alpha = -q$ is given by

$$B_{-q,1}(1/2) = \frac{8\Gamma(q+1)}{2^q}\int_0^1 t^{q-1}\int_0^t \frac{1}{1-s}\int_s^1 \frac{-\ln(1-u)}{u^2}dudsdt \tag{23}$$

The proof of this Lemma 5 is similar to the proofs of Lemma 3 and Lemma 4. We omit the proof.

For notational simplicity, we omit the subscripts on X, Y, and Z throughout the rest of the paper with the understanding that $\lambda = 1/2$. Observe that the random variable ξ_2 can be written as

$$\xi_2 = \frac{Y^{(2)} - Y^{(1)}}{X^{(1)} + X^{(2)}} \tag{24}$$

where $X^{(i)} = \int_0^\infty \exp(W_i(t) - t/2)dt$, and $Y^{(i)} = \int_0^\infty t\exp(W_i(t) - t/2)dt, i = 1, 2$

Lemma 6.

$$E(\frac{Y^{(2)} - Y^{(1)}}{X^{(1)} + X^{(2)}})^2 = E\frac{Z}{X^{(1)} + X^{(2)}} \tag{25}$$

Proof. A direct application of Theorem 3 of Golubev (1979). $\square$

Now we shall prove Theorem 1.

Proof. Let $\psi(X) = E(\frac{Z}{X}|X)$, then by Lemma 5, $EX^{-q}\psi(X) = B_{-q,1}(1/2)$. Observe that the growth of $B_{-q,1}(1/2)$ as a function of q is sufficient slow, approximately in the order of $2^{-q}q!$, so that the above moment problem has a unique solution which is

$$\psi(x) = 8\int_0^1\int_0^t \frac{1}{1-s}\int_s^1 [\frac{-\ln(1-u)}{u^2}]\,du\,dst^{-2}\exp(-\frac{2(1-t)}{xt})dt$$

Observe that the joint density of $(X^{(1)}, X^{(2)})$ is

$$f(x_1, x_2) = 4(x_1x_2)^{-2}\exp(-2(\frac{1}{x_1} + \frac{1}{x_2}))$$

Using Lemma 6, we find

$$E\xi_2^2 = E[\frac{X^{(1)}}{X^{(1)}+X^{(2)}}E(\frac{Z}{X^{(1)}}|X^{(1)})] = \int_0^\infty \int_0^\infty \frac{x_1}{x_1+x_2} f(x_1,x_2)\psi(x_1)dx_1dx_2$$

Writing $x_1 = 1/twz, x_2 = 1/(1-w)z$, we obtain

$$\begin{aligned} &= 32\int_0^1\int_0^t\int_s^1[\frac{-\ln(1-u)}{u^2}\frac{1}{1-s}dudst^2\int_0^\infty\int_0^1[\frac{1-w}{1+(t-1)w}]tze^{-2z}dwdzdt \\ &= 8\int_0^1 \frac{1-t+t\ln(t)}{t(1-t)^2}\int_0^t\int_s^1 \frac{-\ln(1-u)}{u^2(1-s)}dudsdt \end{aligned}$$

Note that $d[1-t+\ln(t)]/(1-t) = [1-t+t\ln(t)]/[t(1-t)^2]dt$, integrating by parts gives

$$= -8\int_0^1 \frac{1-t+\ln(t)}{(1-t)^2}\int_t^1 \frac{-\ln(1-u)}{u^2}dudt$$

Writing $v = 1-t$, we obtain

$$= -8\int_0^1 \frac{v+\ln(1-v)}{v^2}\int_0^v \frac{-\ln(u)}{(1-u)^2}dudv$$

Observe again that $d[(1-v)\ln(1-v)/v] = -[v+\ln(1-v)]/v^2$, integrating by parts yields

$$\begin{aligned} &= 8\int_0^1 \frac{\ln(v)\ln(1-v)}{v(1-v)}dv \\ &= 16\int_0^1 \frac{\ln(1-v)\ln(v)}{v}dv \\ &= 16\zeta(3) \end{aligned}$$

□

Thus it follows from Theorem 1 that the asymptotic efficiency of maximum likelihood estimators of discontinuous signal in a Gaussian white noise is

$$\kappa = \frac{E\xi_2^2}{E\xi_1^2} = \frac{8}{13}\zeta(3)$$

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REFERENCES

Golubev, G.K. (1979) On the efficiency of the maximum likelihood estimator for discontinuous observed signal in white noise. *Probl. Pered. Inform.* **15** 61-69.

Ibragimov, I.A. and Has'minskii, R.Z. (1981) Statistical Estimation. Springer, Berlin.

Paranjape, S.R. and Rubin, H. (1975) Special case of the distribution of the median. *Ann. Statist.* **3** 251-256.

Pitman, E.J.G. (1938) The estimation of the location and scale parameters of a continuous population of any given form. *Biometrika* **30** 391-421

Rubin, H. (1961) The estimation of discontinuities in multivariate densities and related problems in stochastic processes. *Proc. 4th Berkeley Symp.* **1** 563-574