GENERALIZED GAUSS-CHEBYSHEV INEQUALITIES FOR UNIMODAL DISTRIBUTIONS

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Abstract

Let g be an even function on $\mathbf{R}$ which is nondecreasing in $|x|$. Let k be a positive constant. Sharp inequalities relating $P(|X| \geq k)$ to $Eg(X)$ are obtained for random variables X which are unimodal with mode 0, and for random variables X which are unimodal with unspecified mode. The bounds in the mode 0 case generalize an inequality due to Gauss (1823), where $g(x) = x^2$. The bounds in the second case generalize inequalities of Vysochanskiĭ and Petunin (1980, 1983) and Dharmadhikari and Joag-dev (1985).

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Running head: Chebyshev Inequalities for Unimodal Distributions

1. INTRODUCTION

Let X be a random variable with a unimodal cdf F, meaning that for some mode m, F is concave on $[m, \infty)$ and convex on $(-\infty, m]$. We allow F to be "defective" with $P(X = -\infty) = \lim_{x \to -\infty} F(x)$ and $P(X = +\infty) = \lim_{x \to +\infty} 1 - F(x)$.

Gauss (1823) proved that

$$P(|X| \geq k) \begin{array}{ll} \leq \frac{4EX^2}{9k^2} & \text{if } k^2 \geq \frac{4}{3}EX^2 \\ \leq 1 - \left(\frac{k^2}{3EX^2}\right)^{1/2} & \text{if } k^2 \leq \frac{4}{3}EX^2 \end{array} \tag{1}$$

when X is unimodal with mode 0. When $k^2 \leq (4/3)EX^2$, the second bound $1 - \{k^2/\ (3EX^2)\}^{1/2}$ is less than or equal to $4EX^2/(9k^2)$. Thus, unimodality about 0 reduces the Chebyshev bound on $P(|X| \geq k)$ by a factor of at least 4/9. Winckler (1866) generalized (1) to r^{th} moments. For $r > 0$ and X unimodal with mode 0,

$$P(|X| \geq k) \begin{array}{ll} \leq \left(\frac{r}{r+1}\right)^r \frac{E|X|^r}{k^r} & \text{if } k^r \geq \frac{r^r}{(r+1)^{r-1}} E|X|^r \\ \leq 1 - \left\{\frac{k^r}{(r+1)E|X|^r}\right\}^{1/r} & \text{if } k^r \leq \frac{r^r}{(r+1)^{r-1}} E|X|^r. \end{array} \tag{2}$$

See also Beesack (1984), Pukelsheim (1994), and their references. Here, unimodality about 0 reduces the Chebyshev bound by a factor of at least $\{r/(r+1)\}^r$.

Vysochanskiĭ and Petunin (1980, 1983) proved that

$$P(|X| \geq k) \begin{array}{ll} \leq \max\left\{\frac{4EX^2}{9k^2},\ \frac{4EX^2}{3k^2} - \frac{1}{3}\right\} & \\ \leq \frac{4EX^2}{9k^2} & \text{if } k^2 \geq \frac{8}{3}EX^2 \\ \leq \frac{4EX^2}{3k^2} - \frac{1}{3} & \text{if } k^2 \leq \frac{8}{3}EX^2 \end{array} \tag{3}$$

when X is unimodal with any mode. Pukelsheim (1994) contains a proof of (3) using only elementary calculus, as well as three different proofs of the Gauss inequality (1).

Dharmadhikari and Joag-dev (1985) generalized (3) to r^{th} moments, showing that for X unimodal and $r > 0$,

$$P(|X| \geq k) \leq \max\left[\left\{\frac{r}{(r+1)k}\right\}^r E|X^r|, \frac{s}{(s-1)k^r}E|X^r| - \frac{1}{s-1}\right]. \tag{4}$$

Here, s is the constant satisfying $s > r + 1$ and $s(s - r - 1)^r = r^r$.

Inequalitites (1) through (4) are all "best possible," in the sense that equality can hold when EX^2 or $E|X|^r$ is small enough to make the upper bound on $P(|X| \geq k)$ less than or equal to 1.

Let $g : \mathbf{R} \to [0, \infty)$ be an even function which is nondecreasing on $[0, \infty)$ and satisfies $g(0) = 0$. To avoid certain trivialities, assume also that g is not constant on $(0, k)$, i.e.,

$$0 \leq g(0^+) < g(k^-), \tag{5}$$

where $g(0^+) = \lim_{x \downarrow 0} g(x)$, etc. This paper will derive generalizations of inequalities (1) through (4) with $Eg(X)$ in place of EX^2 or $E|X|^r$. These inequalities will also be sharp, with equality always attained by either a uniform distribution or a convex combination of a uniform distribution and a one-point distribution. Theorems 1, 2 and 3 give lower bounds on $Eg(X)$ which are functions of $P\{|X| \geq k\}$. This is "backwards" relative to the usual form of "Chebyshev type" inequalities like (1) through (4), which give upper bounds on $P\{|X| \geq k\}$ in terms of $Eg(X)$. These backwards inequalities are equivalent to inequalities of the standard form (see Corollaries 2 and 3 below), but stating the inequalities in the backwards form seems to make the proofs more transparent.

It is easy to modify the theorems and corollaries so that they apply even when (5) fails to hold, but the slight increase in generality does not seem to be worth the necessary cluttering of the exposition with caveats and special cases.

2. MODE 0

For g as above and $k > 0$, both fixed, define the function

$$L(t) = \frac{1}{t-k} \int_0^t g(x)dx, \quad k < t \leq \infty. \tag{6}$$

We take $L(\infty)$ to equal $g(\infty) = \lim_{x \uparrow \infty} g(x)$. Let a be the largest point in $(k, \infty]$ at which $L(\cdot)$ equals its minimum. Note that $L(\cdot)$ is continuous on $(k, \infty]$ and diverges to $+\infty$ as $t \downarrow k$. Thus, a is well defined. Note also that $g(\infty) \geq L(a) \geq g(k) > 0$. A convexity argument (see Lemma 1 below) will show that $L(\cdot)$ is nonincreasing on $(k, a]$ and strictly increasing on $[a, \infty]$. See Figure 1.

Let δ_x denote the unit point mass at $x \in \mathbf{R}$. Let $\mathcal{U}[c,d]$ be the uniform distribution on the interval $[c,d]$. Note that $L(t) = Eg(U_t)/P(U_t \geq k)$ for $U_t \sim \mathcal{U}[0,t],\ t > k$.

Theorem 1. *Let $k > 0$. Let the functions g and L and the constant a be as above. If X is unimodal about 0 and with $P(|X| \geq k) = \pi$, then*

$$\begin{aligned} Eg(X) \quad & \geq \pi L(a) && \text{if } \pi \leq (a-k)/a \\ & \geq \pi L\{k/(1-\pi)\} && \text{if } \pi \geq (a-k)/a \end{aligned} \tag{7}$$

Furthermore, if $\pi \leq (a-k)/a$, then

$$X \sim \frac{\pi a}{a-k}\mathcal{U}[0,a] + \left(1 - \frac{\pi a}{a-k}\right)\delta_0 \tag{8}$$

is unimodal about 0 and satisfies $P(|X| \geq k) = \pi$ and $Eg(X) = \pi L(a)$. (If $a = \infty$, interpret $a/(a-k)$ as 1 and $\mathcal{U}[0,a]$ as the unit point-mass at ∞.) If $\pi \geq (a-k)/a$, then

$$X \sim \mathcal{U}[0, k/(1-\pi)] \tag{9}$$

is unimodal about 0 and satisfies $P(|X|) \geq k) = \pi$ and $Eg(X) = \pi L\{k/(1-\pi)\}$. Hence, the lower bounds in (7) on $Eg(X)$ in terms of $P(|X| \geq k)$ are sharp.

The inequalities in (7) are equivalent to

$$Eg(X) \geq \min\{\pi/(a-k), (1-\pi)/k\} \int_0^{\max\{a,k/(1-\pi)\}} g(x)dx. \tag{7'}$$

If $g(x) = |x|^r$, then $L(t) = t^{r+1}/(t-k)(r+1)$, $a = (r+1)k/r$, and $L(a) = \{(r+1)k/r\}^r$. Plugging into (7) yields

Corollary 1. *Let $k > 0$ and $r > 0$. If X is unimodal about 0 and $P(|X| \geq k) = \pi$, then*

$$\begin{aligned} E|X|^r \quad & \geq \{(r+1)k/r\}^r \pi && \text{if } \pi \leq (r+1)^{-1} \\ & \geq k^r/\{(1-\pi)^r(r+1)\} && \text{if } \pi \geq (r+1)^{-1} \end{aligned} \tag{10}$$

The inequalities in (10) appear also in von Mises (1931) as formulas (76) and (77) on page 70. Von Mises (1931) shows that these inequalities imply the Winckler inequalities (2). Here is the argument. For X unimodal with mode 0, (10) says that the point $(\pi_X, \rho_X) \triangleq (P\{|X| \geq k\}, Eg\{X\})$ in $\mathbf{R}^2$ lies on or above the continuous, strictly increasing function

$c(\pi)$, $0 \le \pi \le 1$, appearing on the right side of (10). Hence, the point (ρ_X, π_X) will lie on or below the inverse function $c^{-1}(\rho)$, $0 \le \rho \le \infty$, and this is precisely the content of (2). If we define

$$R(\pi) = \pi L\{k/(1-\pi)\} = \frac{1-\pi}{k} \int_0^{k/(1-\pi)} g(x)dx \quad \text{for} \quad (a-k)/a \le \pi \le 1, \tag{11}$$

then the same argument applied to (7) yields Corollary 2.

Corollary 2. *Let $k > 0$. Let the functions g, L, and R and the constant a be as defined above. If X is unimodal with mode 0, then*

$$\begin{aligned} P(|X| \ge k) \quad & \le Eg(X)/L(a) \qquad \text{if } Eg(X) \le (a-k)L(a)/a \\ & \le R^{-1}\{Eg(X)\} \qquad \text{if } Eg(X) \ge (a-k)L(a)/a \end{aligned} \tag{12}$$

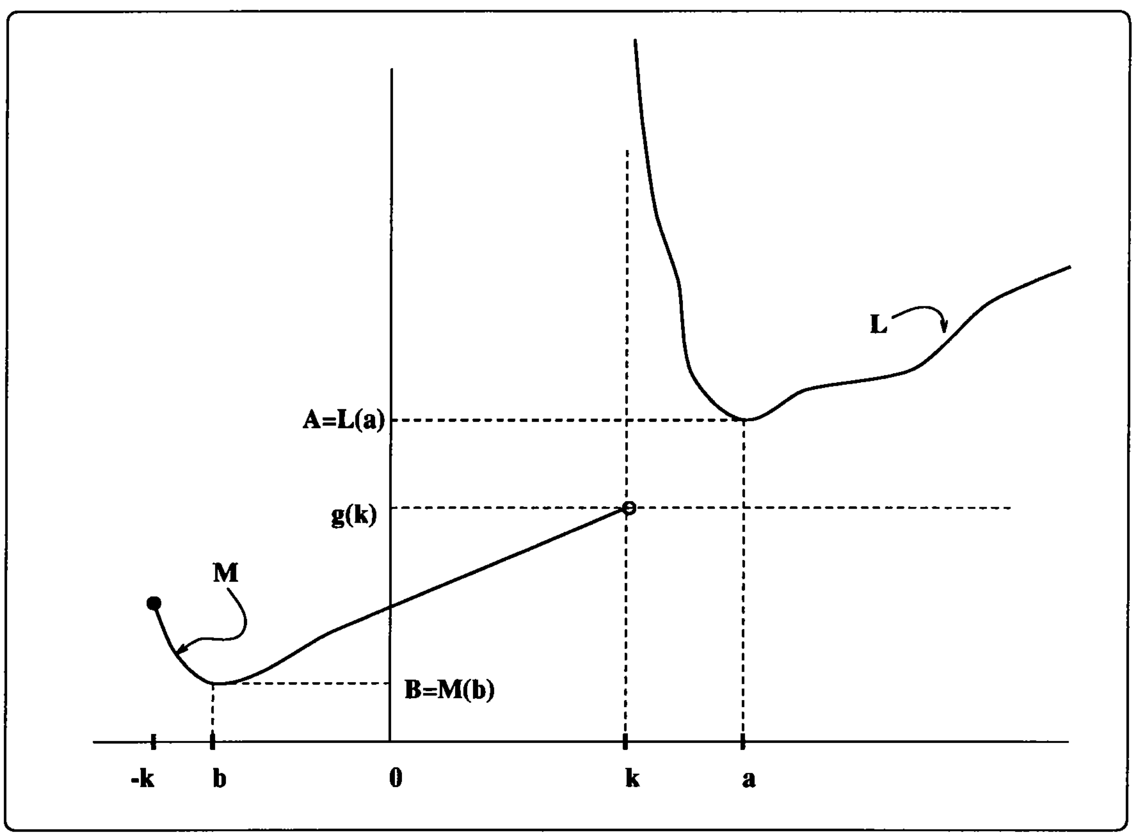

Figure 1.

Example 1. If $g(x) = \exp(|x|) - 1$ and $k = 1$, then $L(t) = \{\exp(t) - t - 1\}/(t-1)$, $a = 1.594$ and $L(a) = 3.922$. Suppose that X is unimodal with mode 0 and with $P(|X| \geq 1) = \pi$. By Theorem 1,

$$E\exp(|X|) \quad \begin{array}{ll} \geq 1 + 3.922\pi & \text{if } \pi \leq 0.3725 \\ \geq (1-\pi)\exp\{(1-\pi)^{-1}\} - (1-\pi) & \text{if } \pi \geq 0.3725, \end{array}$$

and these lower bounds on $E\exp(|X|)$ in terms of $P(|X| \geq 1)$ are sharp.

3. UNSPECIFIED MODE

For g as above and $k > 0$, both fixed, define the function

$$M(v) = \frac{1}{k-v} \int_v^k g(x)dx, \qquad -k \leq v < k. \tag{13}$$

Let b be the smallest (or any) point in $[-k, k)$ at which $M(\cdot)$ equals its minimum. Since $M(v)$ is just the average value of g over the interval $[v, k]$, it follows easily from our assumptions on g that $M(\cdot)$ is nondecreasing on $[0, k]$. Note also that $M(\cdot)$ is continuous on $[-k, k)$. Thus, b is well defined and satisfies $-k \leq b \leq 0$. A convexity argument similar to that of Lemma 1 shows that $M(\cdot)$ is nonincreasing on $[-k, b]$ and nondecreasing on $[b, k)$, but we won't need this. Note that, by (5), we have $g(k) > M(b) > 0$.

Write $A = L(a)$ and $B = M(b)$ for the minimum values of $L(\cdot)$ and $M(\cdot)$, respectively. Then, $g(\infty) \geq A \geq g(k) > B > 0$. Also,

$$\int_v^k g(x)dx \geq (k-v)B, \quad -k \leq v \leq k. \tag{14}$$

Theorem 2. *Let $k > 0$. Let the function g and the constants $a, A, b,$ and B be as above. If X is unimodal with $P(|X| \geq k) = \pi$, then*

$$Eg(X) \quad \begin{array}{ll} \geq \pi A = \pi L(a) & \textit{if } \pi \leq B/\{A + B - g(k)\} \\ \geq \pi g(k) + (1-\pi)B & \textit{if } \pi \geq B/\{A + B - g(k)\}. \end{array} \tag{15}$$

Furthermore, if $0 \leq \pi \leq B/\{A + B - g(k)\}$, then

$$X \sim \frac{\pi a}{a-k}\mathcal{U}[0, a] + \left(1 - \frac{\pi a}{a-k}\right)\delta_0 \tag{16}$$

is unimodal with mode 0 *and satisfies* $P(|X| \geq k) = \pi$ *and* $Eg(X) = \pi A$. *(Again, if* $a = \infty$*, interpret* $a/(a-k)$ *as* 1 *and* $\mathcal{U}[0,a]$ *as the unit point-mass at* ∞*.) If* $B/\{A+B-g(k)\} \leq \pi \leq 1$*, then*

$$X \sim \pi\delta_k + (1-\pi)\mathcal{U}[b,k] \tag{17}$$

is unimodal with mode k *and satisfies* $P(|X| \geq k) = \pi$ *and* $Eg(X) = \pi g(k) + (1-\pi)B$. *Hence, the lower bounds in (15) on* $Eg(X)$ *in terms of* $P(|X| \geq k)$ *are sharp.*

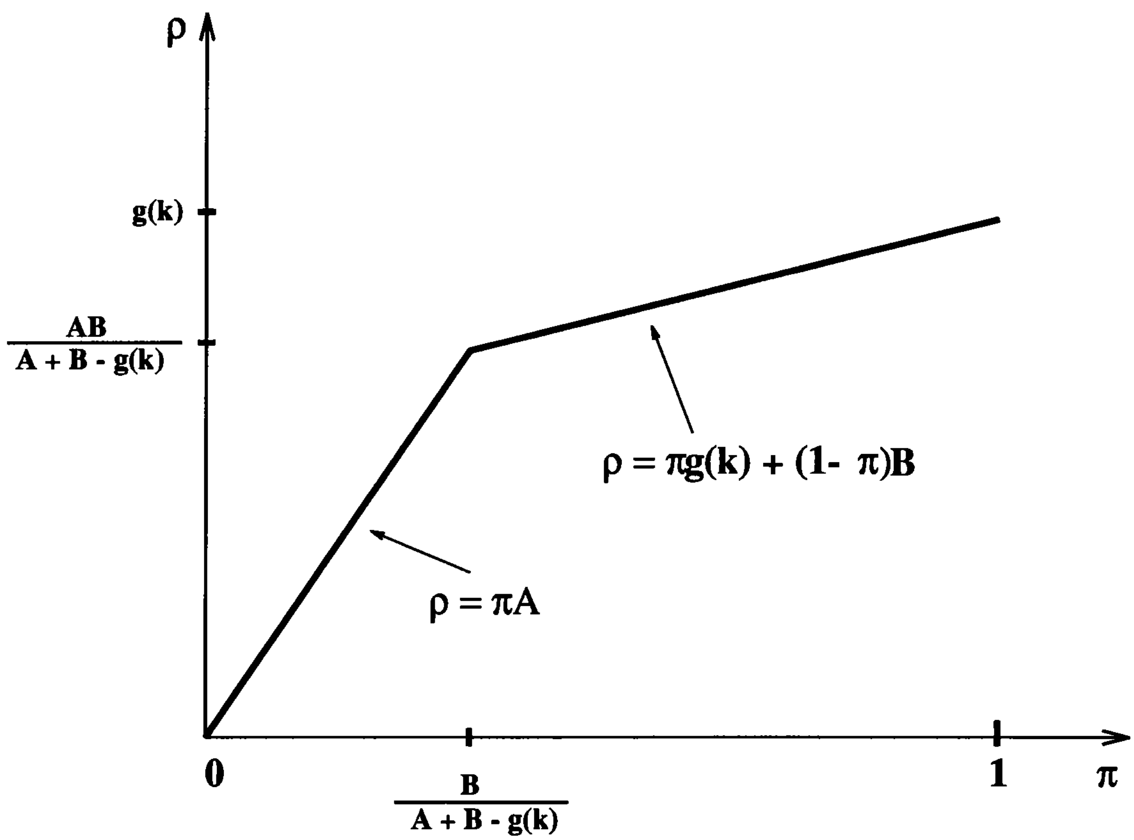

Figure 2.

Look at Figure 2. Theorem 2 says that the point $(\pi_X, \rho_X) \triangleq (P\{|X| \geq k\}, Eg\{X\})$ in $\mathbf{R}^2$ lies on or above the two bold line segments when X is unimodal. Furthermore, each point on the bold line segments equals (π_X, ρ_X) for some unimodal X.

Recall that $g(\infty) \geq A \geq g(k) > B > 0$. Thus, $0 < B/\{A+B-g(k)\} \leq 1$. This ratio equals one only when $A = g(k)$, which only happens when $g(\infty) = g(k)$. Also, $A > g(k) - B$, so that the slope of the second bold line segment in Figure 2 is less than the slope of the first.

Corollary 3. *Let $k > 0$. Let g, a, A, b, and B be as above. If X is unimodal, then*

$$\begin{aligned} P(|X| \geq k) \quad & \leq \max\left\{\tfrac{Eg(X)}{A}, \tfrac{Eg(X)-B}{g(k)-B}\right\} & \\ & \leq \tfrac{Eg(X)}{A} & \text{if } Eg(X) \leq \tfrac{AB}{A+B-g(k)} \\ & \leq \tfrac{Eg(X)-B}{g(k)-B} & \text{if } Eg(X) \geq \tfrac{AB}{A+B-g(k)}. \end{aligned} \tag{18}$$

Furthermore the upper bounds on $P(|X| \geq k)$ in terms of $Eg(X)$ are sharp for $Eg(X) \leq g(k)$.

It is easy to see from Figure 2 that Corollary 3 follows from Theorem 2.

If $g(x) = |x|^r$, then straightforward calculation shows that $b = -\beta k$ and $B = (\beta k)^r$, where $\beta > 0$ solves $(r + 1 + r\beta)\beta^r = 1$. If $s = r + 1 + r\beta$, then s solves $s(s - r - 1)^r = r^r$, and

$$\frac{E|X|^r - B}{k^r - B} = \frac{sE|X|^r - k^r}{(s-1)k^r}.$$

Also, $a = (r + 1)k/r$ and $A = \{(r + 1)k/r\}^r$. Hence

$$\frac{E|X|^r}{A} = \left\{\frac{r}{(r+1)k}\right\}^r E|X|^r.$$

Thus, (18) is equivalent to (4) when $g(x) = |x|^r$.

Example 1, continued. Again, let $g(x) = \exp(|x|) - 1$ and $k = 1$. Then $M(v) = (e^{-v} + e + v - 3)/(1 - v)$ for $-1 \leq v \leq 0$, and $b = -0.4555$ is the location of the minimum of $M(\cdot)$. Thus, $B = M(b) = 0.5769$. If X is unimodal, then by Corollary 3 we have

$$\begin{aligned} P(|X| \geq 1) \quad & \leq 0.2550\{E\exp(|X|) - 1\} & \text{if } E\exp(|X|) \leq 0.8138 \\ & \leq 0.8762E\exp(|X|) - 0.5055 & \text{if } E\exp(|X|) \geq 0.8138 \end{aligned}$$

4. VON MISES' INEQUALITY

Fix numbers $x_1 > x_0 \geq 0$. Suppose that F is a cdf on $\mathbf{R}$ which on $[x_0, \infty)$ stays below a line passing through the point $(x_1, F\{x_1\})$. That is, for some (necessarily nonnegative) constant d,

$$F(x) \leq F(x_1) + d(x - x_1), \qquad x_0 \leq x < \infty. \tag{19}$$

For $r > 0$, von Mises (1938) derived upper bounds on $P(X \geq x_1)$ in terms of $E|X|^r$ when X is a random variable with such a cdf. [See also Fréchet (1950) and Beesack (1984).] Let c be the unique solution ($c > x_1$) of

$$\frac{x_0^{r+1}}{c^r} + rc = (r+1)x_1. \tag{20}$$

Then

$$P(X \geq x_1) \leq c^{-r} E|X|^r \tag{21}$$

for any random variable whose cdf satisfies (19) for some $d \geq 0$. Now define $\alpha = \alpha(E|X|^r)$ to be the unique solution ($\alpha > x_0$) of

$$\alpha^{r+1} - x_0^{r+1} = (r+1)(\alpha - x_0)E|X|^r. \tag{22}$$

If $\alpha > c$, then

$$P(X \geq x_1) \leq \frac{\alpha - x_1}{\alpha - x_0} < c^{-r} E|X|^r \tag{23}$$

for such an X.

The formulation of von Mises' inequality in the previous paragraph follows that in Beesack (1984), except that Beesack (1984) has a different (but equivalent) condition than $\alpha > c$ for (23) to apply.

As before, let g be an even function on $\mathbf{R}$, nondecreasing in $|x|$ and with $g(0) = 0$. To avoid trivialities, assume $g(x_1^-) > 0$. One can easily generalize von Mises' inequality by replacing $E|X|^r$ with $Eg(X)$. For fixed $0 \leq x_0 < x_1 < \infty$ and g, define

$$\tilde{L}(t) = \frac{1}{t - x_1} \int_{x_0}^{t} g(x)dx, \qquad x_1 < t \leq \infty, \tag{24}$$

with $\tilde{L}(\infty) = g(\infty)$. Let $\tilde{a} \in (x_1, \infty]$ be the largest point where $\tilde{L}(\cdot)$ equals its minimum.

Theorem 3. *Let $0 \leq x_0 < x_1 < \infty$. Let the functions g and $\tilde{L}$ and the constant $\tilde{a}$ be as above. If the cdf of X satisfies (19) and if $P(|X| \geq x_1) = \pi$, then*

$$\begin{aligned} Eg(X) \quad & \geq \pi \tilde{L}(\tilde{a}) && \text{if } \pi \leq (\tilde{a} - x_1)/(\tilde{a} - x_0) \\ & \geq \pi \tilde{L}\{x_0 + (x_1 - x_0)/(1 - \pi)\} && \text{if } \pi \geq (\tilde{a} - x_1)/(\tilde{a} - x_0). \end{aligned} \tag{25}$$

Furthermore, if $\pi \le (\tilde{a}-x_1)/(\tilde{a}-x_0)$, *then*

$$X \sim \pi\frac{\tilde{a}-x_0}{\tilde{a}-x_1}\mathcal{U}[x_0,\tilde{a}] + \left(1-\pi\frac{\tilde{a}-x_0}{\tilde{a}-x_1}\right)\delta_0 \tag{26}$$

has a cdf satisfying (19) and satisfies $P(|X| \ge x_1) = \pi$ *and* $Eg(X) = \pi\tilde{L}(\tilde{a})$. *If* $\pi \ge (\tilde{a}-x_1)/(\tilde{a}-x_0)$, *then*

$$X \sim \mathcal{U}[x_0, x_0 + (x_1-x_0)/(1-\pi)] \tag{27}$$

has a cdf satisfying (19) and satisfies $P(|X| \ge x_1) = \pi$ *and* $Eg(X) = \pi\tilde{L}\{x_0 + (x_1 - x_0)/(1-\pi)\}$. *Hence, the lower bounds in (25) on* $Eg(X)$ *in terms of* $P(|X| \ge x_1)$ *are sharp.*

For purposes of bounding $Eg(|X|)$ in terms of $P(|X| \ge x_1)$, condition (19) gives the same result as assuming the cdf F to be concave on $[x_0,\infty)$. Also, there is no loss of generality in Theorems 1 and 3 in assuming $X \ge 0$, a.s. Modulo these observations, Theorem 1 is a special case of Theorem 3, with $x_0 = 0$. The proof of Theorem 3 is essentially the same as the proof of Theorem 1 given in the next section.

If one takes $g(x) = |x|^r$, one finds that the $\tilde{a}$ minimizing (24) is the c of (20). "Inverting" the inequality (25) in this special case yields von Mises' inequality (21), (23).

5. PROOF OF THEOREM 1.

Lemma 1. *$L(\cdot)$ is nonincreasing on $(k,a]$ and strictly increasing on $[a,\infty]$.*

Proof of Lemma 1. Let $\varphi(y) = \int_0^{k+y} g(x)dx$. Note that φ is convex on $[0,\infty)$, since g is nondecreasing on $[k,\infty)$. Since $L(k+y) = \varphi(y)/y$, $0 < y < \infty$, and since $L(\cdot)$ is continuous at ∞, it will be enough to show that

$$\varphi(y_1)/y_1 \le \max\{\varphi(y_0)/y_0,\ \varphi(y_2)/y_2\} \ \text{ for } \ 0 < y_0 < y_1 < y_2 < \infty,$$

with strict inequality if $\varphi(y_0)/y_0 \ne \varphi(y_2)/y_2$. For fixed $0 < y_0 < y_1 < y_2 < \infty$, let

$\alpha \in (0,1)$ be such that $y_1 = \alpha y_0 + (1-\alpha)y_2$. Then by the convexity of φ,

$$\begin{aligned}\frac{\varphi(y_1)}{y_1} &\le \frac{\alpha\varphi(y_0) + (1-\alpha)\varphi(y_2)}{y_1} \\ &\le \frac{\alpha y_0\{\varphi(y_0)/y_0\} + (1-\alpha)y_2\{\varphi(y_2)/y_2\}}{y_1} \\ &\le \frac{\alpha y_0 + (1-\alpha)y_2}{y_1}\max\{\varphi(y_0)/y_0,\ \varphi(y_2)/y_2)\} \\ &\le \max\{\varphi(y_0/y_0,\ \varphi(y_2)/y_2\},\end{aligned}$$

and the third inequality is strict if $\varphi(y_0)/y_0 \neq \varphi(y_2)/y_2$. □

Lemma 2. *If X is unimodal with mode 0 and $P(|X| \ge k) = \pi$, then there exists a random variable X_1, unimodal with mode 0, and satisfying $P(|X_1| \ge k) = \pi$, $Eg(X_1) \le Eg(X)$, and*

$$X_1 \sim \gamma\mathcal{U}[0,t] + (1-\gamma)\delta_0 \tag{28}$$

for some constants $0 \le \gamma \le 1$ and $k < t \le \infty$.

Proof of Lemma 2. Without loss of generality, $P(X \ge 0) = 1$. Otherwise just replace X by $|X|$, which is also unimodal with mode 0.

Let F be the cdf of X. Let f be the right-hand derivative of F. Define the cdf F_1 by

$$F_1(x) = \begin{cases} 0 & \text{if } x < 0 \\ \min\{1, F(k) + f(k)(x-k)\} & \text{if } x \ge 0. \end{cases}$$

Since $F(k) + f(k)(x-k)$ is a tangent line to F at k, and since F is concave on $[0,\infty)$, we have $F_1(x) \ge F(x)$ for $x \ge 0$. Hence, $Eg(X_1) \le Eg(X)$ if $X_1 \sim F_1$. Since $F_1(k^-) = F(k^-) = 1 - \pi$, we have $P(|X_1| \ge k) = \pi$. Finally, (28) holds for $\gamma = 1 - F_1(0)$ and $t = k + \{1 - F(k)\}/f(k)$. (If $f(k) = 0$, take $t = \infty$.) □

Lemma 3. *Suppose that $X \sim \gamma\mathcal{U}[0,t] + (1-\gamma)\delta_0$ for some constants $0 \le \gamma \le 1$ and $k < t \le \infty$. Let $\pi = P(|X| \ge k)$. Then $Eg(X) \ge \pi L(a) = \pi A$.*

Proof of Lemma 3. If $t = \infty$, then $Eg(X) = \gamma g(\infty) = \pi g(\infty) \ge \pi L(a)$. If $k < t < \infty$, then $\pi = \gamma(t-k)/t$, and

$$\begin{aligned} Eg(X) &= \gamma\frac{1}{t}\int_0^t g(x)dx \qquad (\text{since } g(0)=0) \\ &= \gamma\frac{t-k}{t}\left(\frac{1}{t-k}\int_0^t g(x)dx\right) \\ &= \pi L(t) \geq \pi L(a), \text{ by the definition of } a. \ \square \end{aligned} \tag{29}$$

Lemma 4. *Suppose that* $X \sim \gamma\mathcal{U}[0,t] + (1-\gamma)\delta_0$ *for some constants* $0 \leq \gamma \leq 1$ *and* $k < t \leq \infty$. *If* $\pi = P(|X| \geq k) \geq (a-k)/a$, *then* $Eg(X) \geq \pi L\{k/(1-\pi)\}$.

Proof of Lemma 4. If $t = \infty$, then $Eg(X) = \pi g(\infty) = \pi L(\infty) \geq \pi L\{k/(1-\pi)\}$, since $L(\infty) \geq L\{k/(1-\pi)\}$ by Lemma 1. If $k < t < \infty$, then

$$\frac{a-k}{a} \leq \pi = P(|X| \geq k) = \gamma\frac{t-k}{t} \leq \frac{t-k}{t}$$

implies $t \geq k/(1-\pi) \geq a$. Thus,

$$Eg(X) = \pi L(t) \geq \pi L\{k/(1-\pi)\},$$

where the equality is from (29) and the inequality follows from Lemma 1. □

Proof of Theorem 1. The inequalities in (7) follow from Lemmas 2, 3, and 4. It is easy to check that the random variables in (8) and (9) satisfy $P(|X| \geq k) = \pi$ and have the claimed values of $Eg(X)$. □

6. PROOF OF THEOREM 2.

Lemma 5. *If* X *is unimodal with mode* $m \in [0, k)$ *and* $P(|X| \geq k) = \pi$, *then there exists a random variable* X_2, *satisfying* $P(|X_2| \geq k) = \pi$ *and* $Eg(X_2) \leq Eg(X)$, *which is unimodal with mode either* 0 *or* k.

Proof of Lemma 5. Let F be the cdf of X, and let f be its right-hand derivative. Define the cdf F_2 by

$$F_2(x) = \begin{cases} \max\{F(x), F(k) + f(k)(x-k)\}, & 0 \leq x \leq k \\ F(x) & \text{otherwise.} \end{cases}$$

Note that $F_2(x) = F(k) + f(k)(x-k) \geq F(x)$ for $m \leq x < k$, since F is concave on $[m, \infty)$ and $F(k) + f(k)(x-k)$ is a tangent to F at k. If $F_2(0) = F(0)$, then F_2 is convex on $(-\infty, k]$, so that F_2 is unimodal with mode k. If $F_2(0) = F(k) - kf(k)$, then F_2 is concave on $[0, \infty)$, so that F_2 is unimodal with mode 0. If $X_2 \sim F_2$, then $|X_2|$ is stochastically smaller than $|X|$ in either case, so that $Eg(X_2) \leq Eg(X)$. Since F_2 and F are equal and continuous at $\pm k$, $P(|X_2| \geq k) = P(|X| \geq k) = \pi$. □

Lemma 6. *If X is unimodal with mode $m \geq k$ and $P(|X| \geq k) = \pi$, then*

$$Eg(X) \geq \pi g(k) + (1-\pi)B. \tag{29}$$

Proof of Lemma 6. Define X_3 by $X_3 = XI\{|X| < k\} + kI\{|X| \geq k\}$ Then X_3 is unimodal with mode k, and $Eg(X) \geq Eg(X_3)$, since $|X| \geq |X_3|$. The distribution of X_3 assigns probability π to the point k but otherwise has a nondecreasing density $f_3(\cdot)$ on $(-k, k)$. (See the discussion on page 2 of Dharmadhikari and Joag-dev (1988).) Thus, for any $u > 0$, the set $\{x : f_3(x) > u\}$ is an interval of the form (v, k) (or $[v, k)$, it doesn't matter), with $-k < v < k$. Hence,

$$\begin{aligned}
Eg(X_3) &= \pi g(k) + \int_{-k}^{k} g(x) f_3(x) dx \\
&= \pi g(k) + \int_{-k}^{k} g(x) \int_{0}^{\infty} I\{f_3(x) > u\} du dx \\
&= \pi g(k) + \int_{0}^{\infty}\int_{-k}^{k} g(x) I\{f_3(x) > u\} dx du \\
&\geq \pi g(k) + \int_{0}^{\infty}\int_{-k}^{k} B I\{f_3(x) > u\} dx du \\
&= \pi g(k) + (1-\pi)B,
\end{aligned}$$

where the inequality follows from (14). □

Lemma 7.

$$\frac{a}{a-k} \cdot \frac{B}{A+B-g(k)} \leq 1.$$

Proof of Lemma 7. Since we interpret $a/(a-k)$ as 1 if $a = \infty$, the Lemma follows from $B/\{A+B-g(k)\} \leq 1$ in this case. Hence, assume $a < \infty$.

The definition of B implies $kB \leq kM(0) = \int_0^k g(x)dx$. The fact that g is nondecreasing on $[0,\infty)$ implies $(a-k)g(k) \leq \int_k^a g(x)dx$. Thus,

$$kB + ag(k) \leq \int_0^a g(x)dx + kg(k) = (a-k)A + kg(k),$$

which implies the Lemma. □

Proof of Theorem 2. Applying Lemmas 5, 2, 3, and 6 to X or $-X$ (depending on the sign of the mode) shows that

$$Eg(X) \geq \min\{\pi A, \pi g(k) + (1-\pi)B\}. \tag{30}$$

One can see from Figure 2 that (30) is equivalent to (15).

Lemma 7 implies that the right side of (16) really is a probability distribution when $0 \leq \pi \leq B/\{A+B-g(k)\}$. That the random variables in (16) and (17) satisfy $P(|X| \geq k) = \pi$ and have the claimed values of $Eg(X)$ is easy to check. □

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