SUFFICIENTLY NONINFORMATIVE PRIORS FOR THE SECRETARY PROBLEM; THE CASE: $n = 3$

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Technical Report #94-20

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August 1994

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ABSTRACT. The premise of the so-called secretary problem is that "all we observe are the relative ranks" of a sequence of items. As pointed out by Ferguson, in his 1989 *Statistical Science* article, and Samuels, in his comments on that article, this begs the question of whether it is *possible* to have an exchangeable sequence, $X_1, X_2, \ldots, X_n$, for which a stopping rule based only on relative ranks is indeed optimal. For $n = 2$, the answer is no, by a well-known simple argument which is repeated in this paper. For $n \geq 3$, the answer is now known to be yes for the special case of the "best choice" problem; i.e., the problem of finding a stopping rule which maximizes the probability of selecting $\max(X_1, X_2, X_3)$. Silverman and Nádas (1992), for $n = 3$, and Gnedin (1995), for all $n \geq 3$, gave solutions with X_i's which are conditionally i.i.d., uniform on $(0, \Theta)$, where Θ has any of a class of appropriate prior distributions. Given their effect, these priors deserve to be called "sufficiently noninformative."

In this paper, a general payoff function for $n = 3$ is considered. Without loss of generality, we can take the payoff to be 1 for selecting the largest, 0 for selecting the smallest, and c, $0 \leq c \leq 1$, for selecting the middle value of $\{X_1, X_2, X_3\}$. Then $c = 0$ is the best choice problem, $c = 1$ is the problem of maximizing the probability of not getting the worst item, and $c = 1/2$ is the so-called "rank problem." A tantalizing extension of the argument for $n = 2$ seems to suggest that no such exchangeable distribution exists for $c > 0$; but the argument fails. Indeed, the sufficiently noninformative priors of Gnedin are shown to work also for any $c < 1/2$. For $c \geq 1/2$, the problem remains unsolved.

1. INTRODUCTION

Secretary problems are those sequential selection problems in which the payoff (or cost) depends on the observations only through their ranks. A subclass of such problems allows only selection rules based on relative ranks. A question which may be asked is this: Are there secretary problems in which a rule based only on relative ranks is optimal in a strictly larger class of all stopping rules adapted to the sequence of observations?

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 60G40, 62L15; Secondary 90A05.

Key words and phrases. Secretary problem, best choice problem, exchangeable sequence, Pareto prior, Googol.

For example, consider Martin Gardner's presentation of the following problem, called *Googol,* in his Mathematical Games column in the February, 1960 *Scientific American.*

> Ask someone to take as many slips of paper as he pleases, and on each slip write a different positive number. The numbers may range from small fractions of one to a number the size of a *googol* (1 followed by a hundred zeros) or even larger. These slips are turned face-down and shuffled over the top of a table. One at a time you turn the slips face up. The aim is to stop turning when you come to the number that you guess to be the largest of the series. You cannot go back and pick a previously turned slip. If you turn over all the slips, then of course you must pick the last one turned.

The "solution" in the March, 1960 column, took it for granted that only stopping rules based on the relative ranks of the numbers need be considered. (That was, after all, the point of calling it "Googol," wasn't it?)

But this begs the question of whether there is *any* exchangeable distribution for which it is optimal to consider only the relative ranks. (It is implicit in the statement of Googol that the numbers are exchangeable.) Samuels (1989) posed this question and called it Ferguson's Secretary Problem because Ferguson (1989) showed that for any $\varepsilon > 0$ and for any n, there is a two-parameter Pareto prior distribution on θ such that, when sequentially observing n uniform r.v.'s on $[0, \theta]$, the best rule based only on relative ranks comes within ϵ of being optimal. See also Samuels (1991).

For the case $n = 2$, the answer to the above question is NO; there is *no* exchangeable distribution for which it is optimal to consider only the relative ranks. This is easily seen by the following simple and well-known argument: Let X_1 and X_2 be the first and second numbers examined, respectively. Now let Y be any random variable, independent of the X's, with support on the whole real line, and choose X_1 if $X_1 > Y$; otherwise choose X_2. If both X_1 and X_2 turn out to be bigger than Y, or if both are smaller than Y, then (by exchangeability) this rule selects the larger of the two with probability $1/2$, while, if one random variable is larger than Y while the other is smaller, the larger one is sure to be chosen. Thus, setting the unknown $P\{\min(X_1, X_2) < Y < \max(X_1, X_2)\}$ equal to p, say, we have

$$P(X_\tau = \max(X_1, X_2)) = p + (1-p)/2 = (1/2)(1+p),$$

which is strictly greater than $1/2$. This beats rules based only on relative ranks, which, for $n = 2$, are necessarily constants, so have probability $1/2$ of success.

Recently Silverman and Nádas (1992) have shown, to the surprise of many, that, for $n = 3$, there *are* such distributions, in the special case of the "best choice" problem; i.e., the problem of finding a stopping rule which maximizes the probability of selecting $\max(X_1, X_2, X_3)$. They conjectured that such distributions also exist for the best choice problem for all $n \geq 3$ and that they would be found within a specified family. That conjecture was indeed correct, as shown by Gnedin(1995). His Markov chain argument is quite elegant, but cannot be extended to arbitrary payoffs.

In the next section we will consider the $n = 3$ problem with arbitrary payoff. First we will examine a simple "extension" of the negative result for $n = 2$, which seems to suggest that the best choice problem is the only payoff for which such distributions can exist, but is fatally flawed. Then we will take a close look at the same family of distributions from which Gnedin found his exchangeable distributions as mixtures with respect to what we call "sufficiently noninformative" priors. As will be seen, the same family yields part but not all of the solution to the general $n = 3$ problem. (In an earlier version of this paper, Samuels (1992), before Gnedin's result was available, the Silverman and Nádas distributions were used, yielding a similar but somewhat weaker result.)

2. The case: $n = 3$

Let $X_{(1)} < X_{(2)} < X_{(3)}$ be the order statistics of X_1, X_2, X_3, assumed jointly continuous because we want the probability of ties to be zero. Let the payoffs for selecting $X_{(1)}$, $X_{(2)}$ and $X_{(3)}$ be c_1, c_2 and c_3, respectively, with $c_1 \leq c_2 \leq c_3$ and $c_1 < c_3$. Without loss of generality, we can (and will) normalize these payoffs to 0, $c = (c_2 - c_1)/(c_3 - c_1)$ and 1, with $0 \leq c \leq 1$.

Let $r(X_i)$ be the rank function: $r(X_i) = j$ if $X_i = X_{(j)}$; and $q(\cdot)$ be the payoff function: $q(1) = 0$, $q(2) = c$ and $q(3) = 1$.

Proposition 1. *For any c, $0 \leq c \leq 1$, the optimal stopping rule, τ, based only on relative ranks, is*

$$\tau = \begin{cases} 2 & \text{if } X_2 > X_1 \\ 3 & \text{otherwise.} \end{cases} \tag{1}$$

Proof. By exchangeability,

$$E[q(r(X_2))|X_2 < X_1] = \frac{2}{3} \cdot 0 + \frac{1}{3} \cdot c = \frac{c}{3} < \frac{0 + c + 1}{3} = E[q(r(X_3))|X_2 < X_1];$$

$$E[q(r(X_2))|X_2 > X_1] = \frac{2}{3} \cdot 1 + \frac{1}{3} \cdot c = \frac{c + 2}{3} > \frac{c + 1}{3} = E[q(r(X_3))|X_2 < X_1];$$

$$\begin{aligned} E[q(r(X_1))] = \frac{c+1}{3} < \frac{1}{2} \cdot \frac{c+1}{3} + \frac{1}{2} \cdot \frac{c+2}{3} \quad &= \quad P(X_2 < X_1)E[q(r(X_3))|X_2 < X_1] \\ &\quad + P(X_2 > X_1)E[q(r(X_2))|X_2 < X_1]. \end{aligned}$$

□

Here is a tantalizing argument which seems to suggest that the best relative ranks rule, τ, can be beaten for any $c > 0$: Choose any number, say y, and consider the stopping rule

$$\eta = \begin{cases} 2 & \text{if } X_2 > X_1 \text{ or if } y < X_2 < X_1 \\ 3 & \text{otherwise.} \end{cases} \tag{2}$$

By examining the six equally likely (by exchangeability) orderings of X_1, X_2, and X_3, one can show that the expected payoff using η is larger than using τ if and only if

$$\frac{P(X_{(1)} < y < X_{(2)})}{P(y < X_{(1)})} > \frac{1}{c}. \tag{3}$$

For i.i.d random variables, and any $c > 0$, such a y always exists, since the left side of (3) goes to ∞ as $y \to \infty$. (Here, the y depends on the distribution, unlike the $n = 2$ argument where there is a "universal" Y.) But, in the larger class of exchangeable random variables, there need not be such a y; for example, with the priors (6), the left side of (3) is identically α for all $y \geq 1$. So the tantalizing argument fails.

Proposition 2. *A sufficient condition for the τ in Proposition 1 to be optimal among all stopping rules is that each of the following hold with probability one:*

$$(1-3c)P(X_1 = X_{(1)}|X_1) + (4-3c)P(X_1 = X_{(3)}|X_1) \leq 2(1-c); \tag{A}$$

$$2cP(X_3 = X_{(1)}|X_1, X_2) - (1-c)P(X_3 = X_{(3)}|X_1, X_2) \leq c; \tag{B-1}$$

$$2(1-c)P(X_3 = X_{(3)}|X_1, X_2) - cP(X_3 = X_{(1)}|X_1, X_2) \leq 1-c. \tag{B-2}$$

Conditions (B-1) and (B-2) are also necessary.

Proof. A sufficient condition for not stopping with X_1 is that its payoff be smaller than the payoff for using the τ, given by (1), which is the optimal rule based on relative ranks. This inequality is

$$\begin{aligned} P(X_1 = X_{(3)}|X_1) + cP(X_1 = X_{(2)}|X_1) \leq \\ cP(X_1 < X_2 < X_3|X_1) + 1P(X_1 < X_3 < X_2|X_1) \\ + 1P(X_2 < X_1 < X_3|X_1) + 1P(X_3 < X_1 < X_2|X_1) \\ + cP(X_2 < X_3 < X_1|X_1) + 0P(X_3 < X_2 < X_1|X_1) \\ = \frac{1+c}{2}P(X_1 = X_{(1)}|X_1) + 1P(X_1 = X_{(2)}|X_1) + \frac{c}{2}P(X_1 = X_{(3)}|X_1). \end{aligned}$$

The last equality follows from exchangeability. Substituting $1 - P(X_1 = X_{(1)}|X_1) - P(X_1 = X_{(3)}|X_1)$ for $P(X_1 = X_{(2)}|X_1)$ gives condition (A).

The necessary and sufficient condition for not stopping with $X_2 < X_1$ is that the payoff for stopping always be smaller than the payoff for continuing. By exchangeability, this inequality is

$$cP(X_3 = X_{(1)}|X_1, X_2) \leq cP(X_3 = X_{(2)}|X_1, X_2) + P(X_3 = X_{(3)}|X_1, X_2),$$

which, after rewriting, becomes condition (B-1). Similarly, the necessary and sufficient condition for stopping with $X_2 > X_1$ is

$$\begin{aligned} cP(X_3 = X_{(2)}|X_1, X_2) &+ P(X_3 = X_{(3)}|X_1, X_2) \\ &\leq cP(X_3 = X_{(3)}|X_1, X_2) + [1 - P(X_3 = X_{(3)}|X_1, X_2)], \end{aligned}$$

which becomes condition (B-2). □

Here is what conditions (A), (B-1) and (B-2) become in special cases:

Best Choice	(A)	$P(X_1 = X_{(3)}\|X_1) + \frac{1}{4}P(X_1 = X_{(1)}\|X_1) \leq \frac{1}{2}$
($c = 0$)	(B-1)	always true
	(B-2)	$P(X_3 = X_{(3)}\|X_1, X_2) \leq \frac{1}{2}$
Rank	(A)	$\frac{5}{4}P(X_1 = X_{(3)}\|X_1) - \frac{1}{4}P(X_1 = X_{(1)}\|X_1) \leq \frac{1}{2}$
($c = \frac{1}{2}$)	(B-1)	$P(X_3 = X_{(1)}\|X_1, X_2) - \frac{1}{2}P(X_3 = X_{(3)}\|X_1, X_2) \leq \frac{1}{2}$
	(B-2)	$P(X_3 = X_{(3)}\|X_1, X_2) - \frac{1}{2}P(X_3 = X_{(1)}\|X_1, X_2) \leq \frac{1}{2}$
Not Worst Choice	(A)	$P(X_1 = X_{(3)}\|X_1) \leq 2P(X_1 = X_{(1)}\|X_1)$
($c = 1$)	(B-1)	$P(X_3 = X_{(1)}\|X_1, X_2) \leq \frac{1}{2}$
	(B-2)	always true

Now suppose that the X_i's are conditionally i.i.d., uniform on $(0, \Theta)$, where Θ has some prior density. Then the posterior density of Θ, given X_1, X_2, depends only on $U = \max\{X_1, X_2\}$. Denote these densities by $h(\theta|U)$ and let $V = \min\{X_1, X_2\}$. Then

$$(4) \qquad P(X_3 = X_{(1)}|X_1, X_2) = \int_U^\infty \frac{V}{\theta} h(\theta|U) d\theta = \frac{V}{U}[1 - P(X_3 = X_{(3)}|X_1, X_2)].$$

Proposition 3. *If the X_i's are conditionally i.i.d., uniform on $(0, \Theta)$, where Θ has some prior density. Then a necessary condition for the τ in proposition 1 to be optimal among all stopping rules is that, with probability one,*

$$\text{(B)} \qquad \frac{c}{1+c} \leq P(X_3 = X_{(3)}|X_1, X_2) \leq \begin{cases} 1/2 & 0 < c < 1 \\ 1 & c = 1. \end{cases}$$

Proof. By (4) conditions (B-1) and (B-2) become, after substitution and rearrangement,

$$\frac{2c\left(\frac{V}{U}-\frac{1}{2}\right)}{1+2c\left(\frac{V}{U}-\frac{1}{2}\right)} \leq P(X_3 = X_{(3)}|X_1,X_2) \leq \frac{1+\frac{c}{1-c}\frac{V}{U}}{2+\frac{c}{1-c}\frac{V}{U}} I_{\{0\leq c<1\}} + 1 I_{\{c=1\}}. \tag{5}$$

But $P(X_3 = X_{(3)}|X_1, X_2)$ depends only on U, while V/U is independent of U with support on $(0,1]$, so the above inequalities hold with probability one if and only if they continue to hold when we take the supremum of the left side and the infimum of the right side over $\{V/U : 0 < V/U \leq 1\}$. The sup of the left side is $c/(1+c)$, attained at $V/U = 1$, and the inf of the right side, for $c < 1$, is $1/2$, attained as $V/U \to 0$. □

Silverman and Nádas (1992) found sufficiently noninformative priors for the best choice problem, with $n = 3$, within the following class of prior densities for Θ:

$$g(\theta) = tI_{\{0<\theta\leq 1\}} + (1-t)\frac{\alpha}{\theta^{1+\alpha}}I_{\{\theta>1\}} \qquad \alpha > 0,\ 0 \leq t \leq 1. \tag{6}$$

They conjectured a solution for all $n \geq 3$ within the larger class:

$$g(\theta) = t\frac{1-\beta}{\theta^{\beta}}I_{\{0<\theta\leq 1\}} + (1-t)\frac{\alpha}{\theta^{1+\alpha}}I_{\{\theta>1\}} \qquad \alpha > 0,\ 0 \leq \beta < 1,\ 0 \leq t \leq 1.$$

Gnedin(1995), in effect, verified this conjecture by showing that, for any $n \geq 3$, the prior

$$g(\theta) = \frac{\varepsilon(3-\varepsilon)}{6\theta^{1-\varepsilon}}I_{\{0<\theta\leq 1\}} + \frac{\varepsilon(3+\varepsilon)}{6\theta^{1+\varepsilon}}I_{\{\theta>1\}} \qquad \varepsilon > 0, \tag{7}$$

is sufficiently noninformative if ε is sufficiently close to zero. (Gnedin did not explicitly name his prior. He specified marginals

$$p(x_1,\ldots,x_n) = \begin{cases} \frac{\varepsilon}{2n}(x_1 \vee \cdots \vee x_n)^{-n+\varepsilon} & 0 < x_1 \vee \cdots \vee x_n < 1 \\ \frac{\varepsilon}{2n}(x_1 \vee \cdots \vee x_n)^{-n-\varepsilon} & x_1 \vee \cdots \vee x_n > 1, \end{cases}$$

which, for $n = 3$, are equivalent to i.i.d. uniforms with prior given by (7).)

By routine calculations, using (7), the posterior density of $\Theta|X_1 = x$ is

$$f(\theta|x) = \begin{cases} \left[\frac{(3-\varepsilon)(1-\varepsilon^2)}{(3-\varepsilon)(1+\varepsilon)x^{-1+\varepsilon}-4\varepsilon}\right]\theta^{-2+\varepsilon} & 0 \leq x \leq \theta \leq 1 \\ \left[\frac{(3+\varepsilon)(1-\varepsilon^2)}{(3-\varepsilon)(1+\varepsilon)x^{-1+\varepsilon}-4\varepsilon}\right]\theta^{-2-\varepsilon} & 0 \leq x \leq 1 \leq \theta \\ (1+\varepsilon)x^{1+\varepsilon}\theta^{-2-\varepsilon} & 1 \leq x \leq \theta. \end{cases}$$

Also, the posterior density of $\Theta|(X_1 = x_1, X_2 = x_2,\, u = \max(x_1, x_2))$ is

$$h(\theta|u) = \begin{cases} \left[\frac{(3-\varepsilon)(4-\varepsilon^2)}{(3-\varepsilon)(2+\varepsilon)u^{-2+\varepsilon}-2\varepsilon}\right]\theta^{-3+\varepsilon} & 0 \le u \le \theta \le 1 \\ \left[\frac{(3+\varepsilon)(4-\varepsilon^2)}{(3-\varepsilon)(2+\varepsilon)u^{-2+\varepsilon}-2\varepsilon}\right]\theta^{-3-\varepsilon} & 0 \le u \le 1 \le \theta \\ (2+\varepsilon)u^{2+\varepsilon}\theta^{-3-\varepsilon} & 1 \le u \le \theta. \end{cases}$$

Then, for this family,

$$\text{(8)}\quad P(X_1 = X_{(3)}|X_1 = x) = \int_0^\infty \frac{x^2}{\theta^2} f(\theta|x)d\theta$$
$$= \frac{1-\varepsilon^2}{(3-\varepsilon)(1+\varepsilon)-4\varepsilon x^{1-\varepsilon}} I_{\{0<x\le 1\}} + \frac{1+\varepsilon}{3+\varepsilon} I_{\{x\ge 1\}},$$

$$\text{(9)}\quad 1 - P(X_1 = X_{(1)}|X_1 = x) = \int_0^\infty \left[1-(1-\frac{x}{\theta})^2\right] f(\theta|x)d\theta$$
$$= \frac{1-\varepsilon^2}{(3-\varepsilon)(1+\varepsilon)-4\varepsilon x^{1-\varepsilon}}\left[\frac{4-\varepsilon}{2-\varepsilon} - \frac{4\varepsilon}{4-\varepsilon^2}x^{2-\varepsilon}\right] I_{\{0<x\le 1\}} + \frac{(1+\varepsilon)(4+\varepsilon)}{(2+\varepsilon)(3+\varepsilon)} I_{\{x\ge 1\}},$$

and

$$\text{(10)}\quad 1 - P(X_3 = X_{(3)}|X_1 = x_1, X_2 = x_2,\, u = \max(x_1,x_2)) = \int_0^\infty \frac{u}{\theta}h(\theta|u)d\theta$$
$$= \frac{4-\varepsilon^2}{(3-\varepsilon)(2+\varepsilon)-2\varepsilon u^{2-\varepsilon}} I_{\{0<u\le 1\}} + \frac{2+\varepsilon}{3+\varepsilon} I_{\{u\ge 1\}}.$$

We can now extend to $0 \le c < 1/2$ what Silverman and Nádas (1992) and Gnedin (1995) did for the case $c = 0$.

Proposition 4. *If $0 \le c < 1/2$, then conditions (A) and (B) are satisfied for prior densities of the form (7) if ε is sufficiently close to zero.*

Proof. When $\varepsilon = 0$, the right sides of (8) and (9) become identically $1/3$ and $1 - 1/3$, respectively. When these values are substituted into the left side of condition (A), it becomes $(5 - 6c)/3$, so condition (A) is satisfied for all c for sufficiently small ε.

¿From (10), we see that $P(X_3 = X_{(3)}|X_1, X_2)$ would be identically $1/3$ if ε could be taken to be zero, and can be kept, with probability one, as close as we like to $1/3$ by choosing a sufficiently small ε. Hence condition (B) is satisfied almost surely if and only if $c < 1/2$. □

The failure of these priors for $c \ge 1/2$ should not surprise us. As formula (4) shows, if $c > 1/2$, then, when $1 < X_2 < X_1$, but X_2/X_1 is close enough to 1, it would be wise to stop with X_2 since X_3 is nearly twice as likely to be smallest as to be largest. Could some other mixture of i.i.d. uniforms on $(0, \Theta)$ do better? The answer has to be no, because for any mixture, $P(X_3 = X_{(3)}|X_1, X_2)$ must be non-constant with mean 1/3. Gnedin(1995) points out that the class of all such mixtures of i.i.d. uniforms is equivalent to the class of all densities of the form $p(x_1, \ldots, x_n) = g(x_1 \vee, \cdots \vee, x_n), \ x_i > 0$; hence, obviously, the same applies to the latter.

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