PATH CONVERGENCE OF RANDOM WALK PARTLY REFLECTED AT EXTREMA

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Summary

We study the integer valued process $X_n, n \geq 0$, which behaves like fair nearest neighbor random walk, except that when one of its two nearest neighbors has been visited and the other has not, it jumps to the previously visited neighbor with probability $p > 1/2$. We show that $X_n/\sqrt{n}$ converges in distribution. Whether the analog for $p < 1/2$ holds is not resolved.

1. Introduction.

This paper studies the class of integer valued stochastic processes, parametrized by $p \in (1/2, 1)$, which behave like fair nearest neighbor random walk except when one neighbor has been visited and the other has not; then, the neighbor which has already been visited is jumped to with probability p. A precise definition is given below. These processes can be thought of as tracking idealized buy-low-sell-high stock markets, or as following the motion of creatures or impulses whose motion between adjacent integers changes the path between these integers in a way that makes it more hospitable for future crossings. This second viewpoint is essentially the (bond-) reinforced random walk approach, detailed in the next paragraph. Our processes are the simplest reinforced random walks, and are among the simplest walks which move in random environments of their own creation. They may also be thought of as vertex-reinforced random walks. More complex vertex-reinforced random walks have been used as models of learning. See Pemantle (1992).

Let $\delta > 0$. An integer valued process $X_0, X_1, X_2, \ldots$ which satisfies

$$P(X_{n+1} = X_n + 1 | X_i, i \leq n) = 1 - P(X_{n+1} = X_n - 1 | X_i, i \leq n)$$
$$= \frac{w(n, X_n)}{w(n, X_n) + w(n, X_n - 1)}, \quad n \geq 1,$$

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where $w(n,j) = 1+\delta$ if $j \in [\min_{i\leq n} X_i, \max_{i\leq n} X_i)$ and $w(n,j) = 1$ otherwise, is here designated as δ-reinforced random walk, δ-RRW for short. We call the intervals $(i, i+1)$ *bonds* and $w(n,j)$ the *weight* of the bond $(j, j+1)$ at time n. Thus δ-RRW jumps to nearest neighbors with probabilities proportional to the weights of the bonds connecting these neighbors to the current position, and the weight of a bond is initially one and is increased (reinforced) to $1+\delta$ the first time it is crossed. Reinforced random walks on various graphs and with a variety of reinforcement schemes, for instance, which add one to the weight of a bond each time it is crossed, are studied in Diaconis (1988), Pemantle (1988), Davis (1989, 1990), and Sellke (1994b). Reinforced random walks exactly analogous to δ-RRW, on graphs, are the subject of Sellke (1994a). The papers Durrett-Rogers (1992) and Cranston-Mountford (1994) are concerned with related continuous time processes. Path convergence in connection with other non-Markovian ways to construct nearest neighbor paths with self-attracting behavior is discussed in Bolthausen (1994), and Toth (1994) studies processes which can be interpreted as a "negatively reinforced" random walk, that is, the weights of bonds decrease when they are crossed.

This paper was immediately motivated by Nester (1993), which investigates various stopping times for δ-RRW, and also studies the mean square deviation EX_n^2 of δ-RRW, proving both $0 < \liminf_{n\to\infty} EX_n^2$, which follows almost immediately from Theorem 2.3 ii) of Nester (1994), and $EX_n^2/n < 1$, for any $\delta > 0$. Here we show that $\lim_{n\to\infty} EX_n^2/n$ exists, that $X_n/\sqrt{n}$ converges in distribution, and more generally prove path convergence for (scaled, of course) δ-RRW. This is Theorem 4.1, our main result. Many of Nester's results translate immediately to results about the limit process, and some of the resulting formulas are quite pretty. See, for example, the gambler's ruin probabilities given in the next to the last line of the text of this paper, which follows from Theorem 2.3 i) of Nester (1994).

As mentioned, δ-RRW behaves like fair random walk except at maxima and minima, where it goes up with probabilities $1-p$ and p respectively, $p = \frac{1+\delta}{2+\delta} > 1/2$. Harrison and Shepp (1981) show path convergence for the Markov process which behaves like fair random walk except at 0, where it goes up with a probability which is not one half.

Let $p > 1/2$ and consider the queueing-type process on the nonnegative integers which reflects completely at 0, which behaves like a fair random walk between 0 and its maxima, and at a maxima goes down one with probability p, and up one with probability $1-p$. Processes resembling this might result if new storage capacity for the queue is added only when existing capacity is full, and then not immediately. The proof of Theorem 4.1 extends without substantial changes to prove path convergence here, as well as for processes with any two different "reflection rates" $p_{\max} \geq 1/2$ and $p_{\min} \geq 1/2$, which behave like fair random walk away from extrema. We do not, and cannot, prove path convergence for the negatively reinforced analog of δ-RRW, defined the same way but with $-1 < \delta < 0$.

Now we sketch the proof of Theorem 4.1. First we study a one-sided version of δ-RRW which we call δ-PRMP, PRMP standing for partially reflecting at maxima process(es). A δ-PRMP $X_0, X_1, \ldots$ satisfies $P(X_{n+1} = X_n + 1 | X_i, i \leq n) = 1 - P(X_{n+1} = X_n - 1 | X_i, i \leq n) = 1/2$ unless $X_n = \max\{X_i : i \leq n\}$, in which case $1/(2+\delta)$ replaces $1/2$. For a continuous time process $F_t, t \geq 0$, F_t^* designates $\sup_{0 \leq s \leq t} F_s$. Let $W_t, t \geq 0$ be a standard Brownian motion started at 0, let $r \in (-1, 0)$, and put $Z_t = W_t + r W_t^*$. Note that $Z_t = Z_t^*$ if and only if $W_t = W_t^*$, and that if $Z_s < Z_a^*$, $a < s < b$, then $Z_b - Z_a = W_b - W_a$, while if $Z_b^* > Z_a^*$ then $Z_b - Z_a < W_b - W_a$. The processes Z behave like Brownian motion except at a maximum, where they are sub-Brownian, and they are shown to be the limits of scaled δ-PRMP. The proof is short. These processes Z have been intensively studied. See Yor (1992), Carmona, Petit, and Yor (1994), and the references therein.

The limit processes for δ-RRW are in fact two sided versions of the processes Z of the last paragraph, as constructed in Carmona, Petit, and Yor (preprint 12/1993). See also Le Gall (1986). We give a different construction in Section 3, closely connected with the rest of the proof of Theorem 4.1. The Carmona-Petit-Yor construction is more general than ours, in that processes which are in all likelihood the limits of δ-RRW for some negative δ, not too close to -1, are constructed. Knowledge of the existence of these processes does not enable us to extend the proof of Theorem 4.1 to any negative δ, however. The last section, Section 4, is devoted to showing that the Carmona-Petit-Yor processes are in fact the limits of δ-RRW. One of the easiest ways to prove that scaled fair random walk converges to Brownian motion is to start with Brownian motion and use the fact that the embedding scheme, based on successively stopping Brownian motion when it equals an integer different than the one it was at the previous stopping time, yields a fair random walk. We try to mimic this proof with our candidate process in place of Brownian motion, but the embedded integer valued process is not quite a δ-RRW. Close enough, though, which is shown in the following way. We start with a fair random walk, and alter it one way to get a δ-RRW and another way to get a copy of the embedded process. Then we show that these two alterations stay close enough to each other, with high enough probability, to guarantee that the δ-RRW converges to the candidate process, since the embedded process does.

2. Notation and Reflection at Maxima.

From now on, δ is a positive number and $\theta(\delta) = \theta = -\delta/(1+\delta)$. Even when θ is used without any δ connected to it, it always stands for a number in $(-1,0)$. C and c are positive constants which may change from line to line, and may depend on δ or θ but nothing else. Dependence of constants on quantities other than δ and θ is shown by subscripts. A discrete time stochastic processes $X_n, n \geq 0$, is identified with its extension to a continuous time processes given by $X_t = X_{[t]}, t \geq 0$, where $[\ ]$ is the greatest integer function, and this process is denoted by $\boldsymbol{X}$. The process $\boldsymbol{X}^n$ is defined by $X_{t/n}/\sqrt{n} = X_t^n, t \geq 0$. $\boldsymbol{W} = W_t, t \geq 0$, is always standard Brownian motion started at 0, and $\mathcal{F}_t = \sigma(W_s, s \leq t)$. When we say a sequence of stochastic processes converges to a process we always mean convergence involving the usual metric connected with uniform convergence of functions on all compact intervals (see Chapter 5 of Pollard (1982)). The maximum of a and 0 is denoted by a^+.

For a function f on $[0,\infty)$ we denote $f^*(t) = \sup_{0\leq s\leq t} f(s)$ and $f^\#(t) = \inf_{0\leq s\leq t} f(s)$, and for a sequence $\boldsymbol{a} = a_0, a_1, \ldots$ we put $a_n^* = \max_{0\leq k\leq n} a_k$ and $a_n^\# = \min_{0\leq k\leq n} a_k$. A sequence $\boldsymbol{y}$ of integers is called a nearest neighbor path if $|y_i - y_{i-1}| = 1$, $i \geq 1$. If $\boldsymbol{z}$ is a nearest neighbor path, and if $\boldsymbol{e} = e_1, e_2, \ldots$ is a sequence of integers, each of which is either 0 or -2, we define the nearest neighbor path $\boldsymbol{r} = r_0, r_1, \ldots$, which we call $\boldsymbol{z}$ reduced by $\boldsymbol{e}$, by $r_0 = z_0$, $r_1 = z_1$ if $z_1 = z_0 - 1$, $r_1 = z_1 + e_1$ if $z_1 = z_0 + 1$, and for $n > 0$, $r_{n+1} - r_n = z_{n+1} - z_n$, if either $r_n < r_n^*$ or $z_{n+1} - z_n = -1$, and if $r_n = r_n^*$ and $z_{n+1} - z_n = 1$ by $r_{n+1} - r_n = z_{n+1} - z_n + e_{\Phi(n)+1}$, where $\Phi(n)$ is the number of k, $0 \leq k < n$, such that both $r_k = r_k^*$ and $z_{k+1} - z_k = 1$. Thus if all the e_i are 0, $\boldsymbol{r} = \boldsymbol{z}$, while if all e_i are -2, $r_i \leq r_0$, $i \geq 0$.

If $\boldsymbol{z}$ and $\boldsymbol{e}$ are as above, and if in addition $\boldsymbol{f} = f_1, f_2, \ldots$ is a sequence of integers each of which is either 0 or 2, we define the path $\boldsymbol{s}$, called $\boldsymbol{z}$ reduced by $\boldsymbol{e}$ and increased by $\boldsymbol{f}$, by $s_0 = z_0$, $s_1 = z_1$, and if $n \geq 1$, $s_{n+1} - s_n = z_{n+1} - z_n$ unless either both $s_n = s_n^*$ and $z_{n+1} - z_n = 1$, or both $s_n = s_n^{\#}$ and $z_{n-1} - z_n = -1$. If $s_n = s_n^*$ and $z_{n+1} - z_n = 1$, we define $s_{n+1} - s_n$ to be $z_{n+1} - z_n + e_{A(n)+1}$, where $A(n)$ is the number of $k = 1, \ldots, n-1$ such that $s_n = s_n^*$ and $z_{k+1} - z_k = 1$. If $s_n = s_n^{\#}$ and $z_{n+1} - z_n = -1$, we define $s_{n+1} - s_n = z_{n+1} - z_n + f_{B(n)+1}$ where $B(n)$ is the number of $k = 1, \ldots, n-1$ such that $s_k = s_k^{\#}$ and $z_{k+1} - z_k = -1$. Note that here, as opposed to the definition of the previous paragraph, s_1 is always z_1, regardless of $\boldsymbol{e}$ and $\boldsymbol{f}$.

If Γ_t, $t \geq 0$, is a process with continuous paths which starts at 0, we put $\tau_0 = \tau_0(\boldsymbol{\Gamma}) = 0$, $\tau_i = \tau_i(\boldsymbol{\Gamma}) = \inf\{t > \tau_{i-1}\colon\ |\Gamma_t - \Gamma_{\tau_{i-1}}| = 1\}$, $i \geq 1$.

We first prove a result about δ-PRMP, which were defined in the first section.

Proposition 2.1. *Let $\boldsymbol{X}$ be a δ-PRMP. Then $\boldsymbol{X}^n$ converges in distribution to $W_t + \theta W_t^*$, $t \geq 0$, where W_t, $t \geq 0$, is standard Brownian motion started at 0.*

To prove this proposition we use a lemma about (non-random) sequences of numbers. Let $\boldsymbol{r}$ be $\boldsymbol{z}$ reduced by $\boldsymbol{e}$, as defined above, and let Φ remain as it was defined in the same place. Then $z_n - r_n$ is a nonnegative even integer, which we denote by $2J(n)$. $J(n)$ is the number of those e_i, $1 \leq i \leq \Phi(n)$, which equal -2. Of course there are exactly z_n^* integers $0 \leq k < n$ such that $z_k = z_k^*$ and $z_{k+1} - z_k = 1$. $\Phi(n)$ is either $z_n^* - J(n)$ or $z_n^* - J(n) + 1$, since each of the negative twos added, except perhaps the last, lowers the resulting $\Phi(n)$ by one. The proof of the following lemma has a lot to do with the proof of Proposition 2 in Revesz (1981).

Lemma 2.2. *Let $-1 < x < 0$ and suppose $z_0 = 0$. Then*

$$\max_{1 \leq k \leq n} |r_k - (z_k + x z_k^*)| \leq \max_{1 \leq k \leq z_n^*} \Big|\sum_{i=1}^{k} e_i - k\frac{2x}{2+x}\Big| + 2. \tag{2.1}$$

Proof: We prove

$$|r_k - (z_k + x z_k^*)| \leq \max_{1 \leq \lambda \leq z_k^*} \Big|\sum_{i=1}^{\lambda} e_i - \lambda\frac{2x}{2+x}\Big| + 2,\ 1 \leq k \leq n. \tag{2.2}$$

Suppose first that $\Phi(k) = z_k^* - J(k)$. Put $\alpha = J(k)/z_k^*$, so that $r_k - (z_k + xz_k^*) = z_k^*(-2\alpha - x)$. Then

$$\begin{aligned}\sum_{i=1}^{z_k^*-J(k)} e_i - (z_k^* - J(k))(\frac{2x}{2+x})\\ &= -2J(k) - (z_k^* - J(k))(\frac{2x}{2+x})\\ &= \frac{2z_k^*}{2+x}(-2\alpha - x),\end{aligned}$$

and so, since of course $z_k^* - J(k) \in \{0, 1, \ldots, z_k^*\}$ and $0 < (2+x)/2 < 1$, the truth of (2.2), even without the final $+2$ follows.

If $\Phi(k) = z_k^* - J(k) + 1$, $\sum_{i=1}^{z_n^*-J(k)} e_i$ equals either $-2J(k)$ or $-2J(k) + 2$, and so (2.2) follows, from the algebra just done, the $+2$ being necessary here. □

Now if f and $f_n, n \geq 1$, are functions on $[0, \infty)$, and $f_n \to f$ uniformly on compact intervals, then $f_n + \theta f_n^* \to f + \theta f^*$ uniformly on compact intervals. We also recall that if $\boldsymbol{Y}$ is fair nearest neighbor random walk then $\boldsymbol{Y}^n$ converges in distribution to the standard Wiener process $W_t, t \geq 0$, started at 0. Thus $(\boldsymbol{Y} + \theta \boldsymbol{Y}^*)^n$ converges to $W_t + \theta W_t^*$, $t \geq 0$. Let $\boldsymbol{E} = E_i$, $i \geq 0$, be iid random variables independent of $\boldsymbol{Y}$, with $P(E_i = 0) = 1 - P(E_i = -2) = 2/(2+\delta)$. Then $\boldsymbol{Y}$ reduced by $\boldsymbol{E}$, which we call $\boldsymbol{R}$, is a δ-PRMP. To complete the proof of Proposition 2.1 we will show that for any $t > 0$ and any $\varepsilon > 0$,

$$\lim_{n\to\infty} P(\sup_{0\leq s\leq t} |R_s^n - (Y_s^n + \theta Y_s^{*n})| > \varepsilon) = 0 \tag{2.3}$$

The proof is essentially the same for each t, so we just give it for $t = 1$. We have, for $n \geq 1$,

$$\begin{aligned}&P(\sup_{0\leq s\leq 1} |R_s^n - (Y_s^n + \theta Y_s^{*n})| > \varepsilon)\\ &= P(\max_{0\leq k\leq n} |R_k - (Y_k + \theta Y_k^*)| > \varepsilon\sqrt{n})\\ &\leq \frac{1}{\varepsilon\sqrt{n}} E \max_{0\leq k\leq n} |R_k - (Y_k + \theta Y_k^*)|\\ &\leq \frac{1}{\varepsilon\sqrt{n}}(E \max_{0\leq k\leq Y_n^*} |\sum_{i=1}^{k} E_i - \frac{2\theta}{2+\theta}k| + 2) \text{ by Lemma 2.2}\\ &\leq \frac{C}{\varepsilon\sqrt{n}} E(Y_n^*)^{1/2}\\ &= O(n^{-1/4}) \text{ as } n \to \infty.\end{aligned} \tag{2.4}$$

The last two lines need some justification.

First we note that $\sum_{i=1}^{k} E_i - \frac{2\theta}{2+\theta}k = \sum_{i=1}^{k}(E_i - \frac{2\theta}{2+\theta}) := \Gamma_k$ is a sum of iid random variables of mean 0 and variance not exceeding 1, and so Doob's inequality (see Doob (1951) p. 317) gives the second inequality in

$$E|\Gamma_n^*| \leq (E\Gamma_n^{*2})^{1/2} \leq (4E\Gamma_n^2)^{1/2} = Cn^{1/2}. \tag{2.5}$$

Since Y_n^* is independent of E, the last inequality in (2.4) is valid since it is valid conditioned on Y_n^* on $\{Y_n^* > 0\}$. To establish the last line of (2.4), we note that by the reflection principle, $P(Y_n^* \geq \alpha) \leq 2P(|Y_n| \geq \alpha)$, $\alpha > 0$, so $E(Y_n^*)^{1/2} \leq 2E|Y_n|^{1/2} \leq 2(EY_n^2)^{1/4} = 2n^{1/4}$.

Next we let $Z_t = W_t + \theta W_t^*$, for $t \geq 0$, recalling W is standard Brownian motion started at 0, and $-1 < \theta < 0$, and let $\tau_i = \tau_i(Z)$ be the canonical times defined before the statement of Proposition 2.1. Let $H_k = Z_{\tau_k}$, $k \geq 0$, and $D_k = H_k - H_{k-1}$. Let μ be the distribution of the first exit time of W_t from $[-1, 1]$, and let ν be the distribution of the first exit time of W from $(-1, (\theta+1)^{-1})$. Then $\tau_{k+1} - \tau_k$ has conditional distribution, given $\mathcal{F}_{\tau_k}$, equal to μ on $\{H_k < H_k^*\}$, and has conditional distribution on $\{H_k = H_k^*\}$ which puts smaller probability on $[y, \infty)$ than $\nu[y, \infty)$, since $\tau_{k+1} \leq \inf\{t > \tau_k : W_t - W_{\tau_k} \notin (-1, (\theta+1)^{-1})\}$.

Now let $\omega(y)$ be the probability that $W_t + \theta(W_t^* - y)^+$, $t \geq 0$, equals 1 before it equals -1. It is easily checked that $0 < \omega(0) < \omega(y)$, if $y > 0$. Let $N(k)$ be the number of $j < k$ such that $H_j = H_k^*$. The strong Markov property implies that the conditional distribution of $Z_{\tau_k}^* - H_k$, given $N(k) = m$ and $H_k = H_k^*$, depends only on m, and not on k or H_k^*, and furthermore that on $\{H_k = H_k^*\}$, $P(D_{k+1} = 1 | \mathcal{F}_{\tau_k}) = \omega(Z_{\tau_k}^* - H_k)$. Thus

$$P(D_{k+1} = 1 | H_k = H_k^*,\ N(k) = m) := \lambda(m) \geq \lambda(0) = \omega(0) > 0. \tag{2.6}$$

Clearly $\lambda(m) < 1/2$ since, roughly, something negative is added to the Brownian paths.

Now $\int_0^\infty x d\mu(x) = 1$ and $\int_0^\infty x d\nu(x) = r \in (1, \infty)$. Thus $E\tau_n \leq nr$, and so using the Burkholder-Gundy inequalities (see Burkholder (1973)),

$$EH_n^* \leq EZ_{\tau_n}^* \leq cE\sqrt{\tau_n} \leq c(E\tau_n)^{1/2} \leq C\sqrt{n}.$$

Thus

$$C\sqrt{n} \geq \sum_{k=1}^{n} E(H_k^* - H_{k-1}^*)$$
$$= \sum_{k=1}^{n} EP(D_k = 1|H_j, j \leq k-1)I(H_{k-1} = H_{k-1}^*)$$
$$\geq c\sum_{k=1}^{n} P(H_{k-1} = H_{k-1}^*).$$

Now if $\tilde{\tau}_{i+1} - \tilde{\tau}_i = (\tau_{i+1} - \tau_i)I(H_i < H_i^*) + V_i I(H_i = H_i^*)$, where $V_i, i \geq 0$ are iid and independent of $\boldsymbol{W}$ and have distribution μ, then $\tilde{\tau}_{i+1} - \tilde{\tau}_i$ are iid with distribution μ, and so $\tilde{\tau}_n/n \to 1$ in probability as $n \to \infty$. But $|\tilde{\tau}_n - \tau_n| = |\sum_{k=0}^{n-1}[V_k - (\tau_{k+1} - \tau_k)]I(H_k = H_k^*)|$ and thus $E|\tau_n - \tilde{\tau}_n| \leq \sum_{k=0}^{n-1} (1+r)P(H_k = H_k^*) \leq C\sqrt{n}$, and so τ_n/n approaches 1 in probability. Similarly $\tau_{[nt]}/n \to t$ in probability. From this it easily follows that $\boldsymbol{H}^n \to \boldsymbol{Z}$.

Now let Δ_i, $i \geq 0$, be iid positive integer valued random variables such that

$$P(\Delta_i = 1) = 2\lambda(0), P(\Delta_i = k|\Delta_i > k-1) = 2\lambda(k-1), \ k > 1. \tag{2.7}$$

Put $E\Delta_i = \eta$ and $M_k = \sum_{i=1}^{k} \Delta_i$. Let $G_i = 0$ if $i \in \{\bigcup_{k=1}^{\infty} M_k\}$, $i \geq 1$, and otherwise let $G_i = -2$. Let $Y_0, Y_1, Y_2, \ldots$ be fair random walk, independent of $\boldsymbol{G}$. Then $\boldsymbol{Y}$ reduced by $\boldsymbol{G}$ has the same distribution as $\boldsymbol{H}$.

In the long run, $(\eta - 1)/\eta := \beta$ of the G_i are -2, and the rest are 0. The following lemma is one way to state this more precisely.

Lemma 2.3. *Let τ be a positive integer valued random variable which satisfies $\{\tau = n\} \subset \sigma(G_i, i \leq n) = \mathcal{G}_n, n \geq 1$. Then*

$$E \max_{1 \leq k \leq n} |\sum_{i=1}^{k} G_{\tau+i} + 2\beta k| \leq C\sqrt{n}, \quad n \geq 1. \tag{2.8}$$

Proof: Let $T_1 = \inf\{i \geq 1\colon\ M_i > \tau\}$ and $T_2 = \inf\{i \geq 1\colon\ M_i > \tau + n\}$.

Let $M_i' = M_i - \tau$ and let $\alpha_k = \sum_{i=1}^{k} G_{\tau+i} + 2\beta k$. Then $\alpha_{k+1} < \alpha_k$ if $M_i' \leq k < M_{i+1}' - 1$, $i \geq T_1$, and $\alpha_{k+1} < \alpha_k$ if $1 \leq k < M_{T_1}' - 1$. Furthermore $|\alpha_{k+1} - \alpha_k| < 2$ for all $k \geq 1$.

Thus

$$\max_{1\le k\le n}|\alpha_k| \le \max(\{|\alpha_1|\}\bigcup\{|\alpha_{M_i'}|: T_1\le i\le T_2\}\bigcup\{|\alpha_{M_i'-1}|: T_1\le i\le T_2\})$$

$$\le \max(\{|\alpha_1|\}\bigcup\{|\alpha_{M_i'}|:\ T_1\le i\le T_2\})+2 \tag{2.9}$$

$$\le |\alpha_{M'_{T_1}}-\alpha_1| + \max\{|\alpha_{M_i'}|: T_1\le i\le T_2\})+2$$

$$\le \max\{|\alpha_{M_i'}|: T_1\le i\le T_2\}+2(M_{T_i}-\tau)+2$$

$$= \max_{T_1\le k\le T_2}|\sum_{\tau+1}^{M_k} G_i + 2\beta(M_k-\tau)| + 2(M_{T_1}-\tau)+2$$

$$\le \max_{T_1\le k\le T_2}|\sum_{M_{T_1+1}}^{M_k} G_i + 2\beta(M_k-M_{T_1})| + 4(M_{T_1}-\tau)+2$$

$$= \max_{T_1\le k\le T_2}|-2[(M_k-M_{T_1})-(k-T_1)] + 2\beta(M_k-M_{T_1})| + 4(M_{T_1}-\tau)+2$$

$$= \max_{T_1\le k\le T_2}|2\beta-2||(M_k-M_{T_1})-\eta(k-T_1)| + 4(M_{T_1}-\tau)+2.$$

Now $P(G_{n+1}=0|\mathcal{G}_n)\ge 2\lambda(0)=2\omega(0)$. Thus $P(M_{T_1}-\tau>k)\le(1-2\lambda(0))^k$, $k>0$, and similarly $P(M_{T_2}-(\tau+n)>k)\le(1-2\lambda(0))^k$, so we have

$$E(M_{T_1}-\tau)<C \qquad \text{and} \qquad E(M_{T_2}-(\tau+n))<C. \tag{2.10}$$

If $M_{T_1+k}-M_{T_1}-\eta k=h_k$, $k\ge 0$, then given $\mathcal{G}_{M_{T_1+k}}=\sigma(\Delta_i, i\le T_1+k)$, $h_{k+1}-h_k$, $k\ge 0$, has the same distribution as $(\Delta_1-\eta)$. Thus h_k, $k\ge 0$, is a martingale with respect to $\mathcal{G}_{M_{T_1}+k}$ $k\ge 0$, and so its differences are orthogonal, and putting $\rho=E(\Delta_1-\eta)^2$, we get

$$Eh^2_{T_2-T_1}=E(T_2-T_1)\rho<\rho E(T_2-(\tau+n))+\rho n$$
$$< Cn,\ n\ge 1,\ \text{ using (2.10).}$$

Now by Doob's inequality, $E\max_{0\le k\le T_2-T_1} h_k^2<4Cn$, so $E\max_{0\le k\le T_2-T_1}|h_k|<2(Cn)^{1/2}$, and this together with (2.9) and the first inequality in (2.10), gives (2.8). □

We here include a paragraph which will not be referred to until much later, and then in a concrete case which may be easier for some to think about than the generality treated below. Note that in the proof above we can relax the requirement that the arrival time Δ_1 of the first 0 has the distribution given by (2.7), so long as the other Δ_i still do. It is enough to have a condition on the distribution of Δ_1 to control the overshoots $T_1-\tau$ and

$T_2 - (\tau + n)$. One such condition is $P(\Delta_1 = k | \Delta_1 > k-1) > c > 0$, $k \geq 1$. Furthermore, the proof of (2.8) shows that it holds conditioned on $\mathcal{G}_\tau$. Also, the σ-fields $\mathcal{G}_n$ could have been replaced by $\mathcal{D}_n := \sigma(\mathcal{G}_n \bigcup \varphi)$, where φ, is a σ-field independent of each $\mathcal{G}_n$. In fact, if Δ_1 satisfies the condition above, and τ is a stopping time with respect to $\mathcal{D}_n$, $n \geq 1$, then (2.7) holds conditioned on $\mathcal{D}_\tau$.

Now (2.8) holds also in the case $\tau \equiv 0$, by the same proof, and this together with Lemma 2.2 may be used, as (2.5) and Lemma 2.2 were used to prove Proposition 2.1, to prove that $\boldsymbol{H}^n$ converges to $W_t - (4\beta/(2+2\beta))W_t^*$, $t \geq 0$. (Note the inverse of $y = \frac{2x}{2+x}$ is $x = \frac{2y}{2-y}$, which gives $-4\beta/(2+2\beta)$ if $y = -2\beta$.) But we already know $\boldsymbol{H}^n$ converges to $W_t + \theta W_t^*$, $t \geq 0$. Thus

$$\theta = -4\beta/(2+2\beta). \tag{2.11}$$

3. Construction of the Limit Process.

We define class $\mathcal{B}$ to be those continuous functions f on $[0, \infty)$, such that $f(0) = 0$ and such that f is not constant on any nonempty open subintervals of $(0, \infty)$. We could extend the results of this section to functions that aren't in class $\mathcal{B}$, but since we are only going to apply them to Brownian paths, this is unnecessary.

Definition. *Let $-1 < \varepsilon < 0$ and let $f \in \mathcal{B}$. The function g on $[0, \infty)$ is called the ε contraction of f if $g \in \mathcal{B}$ and if both the following hold.*

i) If $[a, b] \subset (0, \infty)$ and $g(x) \geq g^\#(a)$, $x \in [a, b]$, then $g(b) - g(a) = f(b) - f(a) + \frac{\varepsilon}{1+\varepsilon}(g^*(b) - g^*(a))$.

ii) If $[a, b] \subset [0, \infty)$ and $g(x) \leq g^*(a)$, $x \in [a, b]$, then $g(b) - g(a) = f(b) - f(a) + \frac{\varepsilon}{1+\varepsilon}(g^\#(b) - g^\#(a))$.

Conditions i) and ii) are rephrased at the beginning of the proof of Theorem 3.1.

If h is a function on $[a, b]$ let $h^*([a, b]) = \sup_{a \leq x \leq b} h(x)$ and $h^\#([a, b]) = \inf_{a \leq x \leq b} h(x)$.

Theorem 3.1. *If $-1 < \varepsilon < 0$ and $f \in \mathcal{B}$, there is a unique ε contraction g of f. Furthermore, if $[a,b] \subset [0,\infty)$,*

$$g^*([a,b]) - g^\#([a,b]) \le f^*([a,b]) - f^\#([a,b]) \le (\frac{1}{1+\varepsilon})(g^*([a,b]) - g^\#([a,b])). \tag{3.1}$$

Proof: We note that condition i) of Definition 3.1 can be rephrased as

i′) If $[a,b] \subset (0,\infty)$ and if $g(x) \ge g^\#(a)$, $x \in [a,b]$, and if $s = \inf\{t \in [a,b]: g^*(a) = g(t)\} \le b$, then $g(t) - g(a) = f(t) - f(a)$, $a \le t \le s$, and $g(t) - g(s) = f(t) - f(s) + \epsilon\,(f^*(t) - f(s))$, $s \le t \le b$, while if $s \ge b$, $g(t) - g(a) = f(t) - f(a)$, $a \le t \le b$.

Condition ii) can also be similarly rephrased as ii′). It is easy to see i′) and ii′) imply i) and ii), and while more difficult to show i) and ii) imply i′) and ii′), this is not too hard. We omit this argument, but note that the verification of i′) is based on the easy cases when either $g^*(a) = g^*([a,b])$ or $g^*(b) = g^*([a,b])$: if $u = \inf\{t > a\colon\ g(t) = g^*(a)\} \le b$, put $v = \sup\{t \in [u,b]\colon\ g(t) = g^*(b)\}$ and use i) and ii) on $[a,u]$, $[u,v]$, and $[v,b]$, to verify that if i) and ii) hold then $g(b)$ is as described in i′).

For $\delta > 0$, let g_δ be the continuous function on $[0,\infty)$ which equals f on $[0,\delta]$ and which, when substituted for g in i) and ii) (or i′) and ii′)), satisfies these conditions not for all intervals $[a,b] \subset (0,\infty)$, but only for $[a,b] \subset [\delta,\infty)$. It is easy to show g_δ exists and is unique, since i′) and ii′) for $[a,b] \subset [\delta,\infty)$ provide a recipe for constructing $g_\delta(t)$, $t \ge \delta$: If, for example, $g_\delta(\delta) \in (f^\#(\delta), f^*(\delta))$, either i′) or ii′) guarantees $g_\delta(s) - g_\delta(\delta) = f_\delta(s) - f_\delta(\delta)$ until $g_\delta(s)$ equals either $f^\#(\delta)$ or $f^*(\delta)$, after which either ii′) or i′), respectively, determines the increments of g_δ for a while, and so on.

We have, for any $\delta > 0$, if $[a,b] \subset [\delta,\infty)$,

$$g_\delta^*([a,b]) - g_\delta^\#([a,b]) \le f^*([a,b]) - f^\#([a,b]) \le (\frac{1}{1+\varepsilon})(g_\delta^*([a,b]) - g_\delta^\#([a,b])). \tag{3.2}$$

The left hand inequality is almost immediate: Suppose a max of g_δ in $[a,b]$ occurs before a min, that is there exist $a \le s_1 \le s_2 \le b$ such that $g_\delta(s_1) = g_\delta^*([a,b])$. and $g_\delta(s_2) = g_\delta^\#([a,b])$. Then $f(t) - f(s_1) = g_\delta(t) - g_\delta(s_1)$, $s_1 \le t \le s_2$, so $f^*([a,b]) - f^\#([a,b]) \ge |f(s_1) - f(s_2)| = g_\delta^*([a,b]) - g_\delta^\#([a,b])$. If a min occurs before a max the argument is similar. To prove the right hand inequality, assume with no loss of generality that there exist $a \le t_1 \le t_2 \le b$ such that $f(t_1) = f^*([a,b])$ and $f(t_2) = f^\#([a,b])$. Let $t_1 = c_0 \le c_1 \le \ldots \le c_n = t_2$, where the c_i are picked so that each pair c_i, c_{i+1} satisfies one of the following conditions

(a) $g(s) \in [g^\#(c_i), g^*(c_i)]$, $c_i \le s \le c_{i+1}$,

(b) $g(c_i) = g^\#(c_i)$ and $g(c_{i+1}) = g^\#(c_{i+1})$, and $g(s) \le g^*(c_i)$, $c_i \le s \le c_{i+1}$,

(c) $g(c_i) = g^*(c_i)$ and $g(c_{i+1}) = g^*(c_{i+1})$, and $g(s) \ge g^\#(c_i)$, $c_i \le s \le c_{i+1}$.

It is easy to show such c_i exist. We may take $c_1 = \inf\{t\colon\ g(t) \notin (g^\#(c_0), g^*(c_0))\}$ or b, whichever is smaller. If $c_1 < b$ and, say, $g(c_1) = g^*(c_1)$ then if there is a number in $(c_1, b]$ such that $g(t) = g^\#(t)$, we may take c_3 to be the smallest such number, and $c_2 = \sup\{s \in [c_1, t]\colon g^*(s) = g(s)\}$, while if there is no $t \in (c_1, b]$ such that $g(t) = g^\#(t)$, we may take $c_2 = \sup\{s \in [c_1, b]\colon\ g^*(s) = g(s)\}$ and $c_3 = b$. In case $c_3 < b$, we continue in this manner. Now the g_δ versions of i) and ii) imply $g_\delta(c_i) - g_\delta(c_{i+1}) \ge (1+\varepsilon)(f(c_i) - f(c_{i+1}))$, for all $i = 0, 1, \ldots, n-1$. Adding these n inequalities gives the right hand inequality in (3.2).

To prove the existence of the function g of Theorem 3.1, we note that the left hand side of (3.2) guarantees that for any $t > 0$, the functions $g_{n^{-1}}$, $n \ge 1$, are equicontinuous on $[0, t]$, and the usual diagonalization argument gives a subsequence uniformly convergent on compact intervals. The limit function is g. That g is not constant on any interval follows from the right hand side of (3.2). That i) and ii) are preserved under the uniform convergence of $g_{n^{-1}}$ is most easily seen by decomposing, for each n, the interval $[a, b]$ into subintervals in a manner similar to the decomposition of $[t_1, t_2]$ in the preceeding paragraph, using $g_{n^{-1}}$ in place of g, and showing that if $g(s) \ge g^\#(a)$, $a \le s \le b$, then $g_{n^{-1}}(b) - g_{n^{-1}}(a)$ is almost $f(b) - f(a) + \frac{\varepsilon}{1+\varepsilon}(g^*_{n^{-1}}(b) - g^*_{n^{-1}}(a))$ if n is large, by using i) and ii) for $g_{n^{-1}}$ on each of the subintervals, and adding.

To prove uniqueness, suppose both g_1 and g_2 are ε contractions of f. That $g_1 = g_2$ follows from the fact that $\varphi(s) = |g_1(s) - g_2(s)|^*$ $(= \sup_{0 \le t \le s} |g_1(t) - g_2(t)|)$ is a monotone decreasing (i.e. non-increasing) function on $(0, \infty)$, and so since it is also continuous and vanishes at 0, it must be identically zero. To prove φ is monotone decreasing, suppose by way of contradiction that there is $t > 0$, such that $\varphi(t + \delta) > \varphi(t)$, $\delta > 0$. Suppose without loss of generality that $g_1(t) \geq g_2(t)$. Clearly then $\varphi(t) = g_1(t) - g_2(t)$, and i) and ii) and the assumption we are trying to contradict imply that either both $g_1(t) < g_1^*(t)$ and $g_2(t) = g_2^*(t)$ or both $g_1(t) = g_1^\#(t)$ and $g_2(t) < g_2^\#(t)$. In the first case, however, if $t_0 < t$ satisfies $g_1(t_0) = g_1^*(t)$, we have

$$\varphi(t) \geq \varphi(t_0) \geq g_1(t_0) - g_2(t_0) \geq g_1^*(t) - g_2^*(t) > g_1(t) - g_2(t) = \varphi(t).$$

A similar contradiction holds in the second case. $\square$

4. Convergence of δ-RRW.

For the rest of this paper, $\psi_t, t \geq 0$, will stand for the θ contraction of standard Brownian motion $W_t, t \geq 0$, (recall $\theta = -\delta/(1+\delta)$), and τ_n will stand for $\tau_n(\boldsymbol{\Psi})$, as defined after the proof of Proposition 2.1. We prove the following theorem.

Theorem 4.1 *If $\boldsymbol{X}$ is δ-RRW then $\boldsymbol{X}^n$ converges to $\boldsymbol{\Psi}$.*

Let $\Gamma_i = \psi_{\tau_i}$. We will prove

$$\boldsymbol{\Gamma}^n \to \boldsymbol{\Psi} \text{ as } n \to \infty. \tag{4.1}$$

We will be brief, as this argument is closely related to the proof of Section 2 that $\boldsymbol{H}^n \to \boldsymbol{Z}$. The definitions of μ, r, and λ, symbols which appear below, are given in the vicinity of (2.6). We have $P(\Gamma_{i+1} = \Gamma_i + 1 | \mathcal{F}_{\tau_i}) = 1/2$ on $\{\Gamma_i^\# < \Gamma_i < \Gamma_i^*\}$. Furthermore, if $N^+(i)$ is the number of $j, 0 < j < i$, such that $\Gamma_j = \Gamma_i^*$, and if $N^-(i)$ is the number of $j, 0 < j < i$, such that $\Gamma_j = \Gamma_i^\#$, then

$$P(\Gamma_{i+1} = \Gamma_i + 1 | \Gamma_k, k \leq i) = \lambda(N^+(i)) \text{ on } \{\Gamma_i = \Gamma_i^* > 0\}. \tag{4.2}$$

and

$$P(\Gamma_{i+1} = \Gamma_i - 1 | \Gamma_k, k \leq i) = \lambda(N^-(i)) \text{ on } \{\Gamma_i = \Gamma_i^\# < 0\}. \tag{4.3}$$

On $\{\Gamma_i = \Gamma_i^* = 0\}$, the conditional probability in (4.2) is bounded below by $\omega(0) = \lambda(0)$, which is also a lower bound for the conditional probability in (4.3) on $\{\Gamma_i = \Gamma_i^\# = 0\}$. On $\{\Gamma_i^\# < \Gamma_i < \Gamma_i^*\}$, the conditional distribution of $\tau_{i+1} - \tau_i$ is μ, while on $\{\Gamma_i = \Gamma_i^*\}$ $\cup \{\Gamma_i = \Gamma_i^\#\}$, $i > 0$, $E(\tau_{i+1} - \tau_i | \mathcal{F}_{\tau_i}) \leq r$.

The right side of (3.1) with $a = 0, b = t$ gives

$$\psi_t^* - \psi_t^\# \geq (1+\varepsilon)(W_t^* - W_t^\#)$$

This implies $\tau_1 \leq \inf\{t : W_t \notin (-2/(1+\varepsilon),\ 2/(1+\varepsilon))\}$, so $E\tau_1 < \infty$. Also, using (4.2) and the Burkholder–Gundy inequalities as in Section 2, we have

$$E(\Gamma_n^* - \Gamma_n^\#) \leq C\sqrt{n},$$

and using this to control $\sum_{k=0}^{n-1} P(\Gamma_k = \Gamma_k^* \text{ or } \Gamma_k^\#)$ in a manner very similar to the way $\sum_{k=0}^{n-1} P(H_k = H_k^*)$ was controlled in Section 2, (4.1) follows by a minor modification of the argument in Section 2.

Now let $\boldsymbol{Y} = Y_0, Y_n \ldots$ be the fair random walk of Section 2. Let $\boldsymbol{T} = T_1, T_2, \ldots$ and $\boldsymbol{B} = B_1, B_2 \ldots$ have the distributions of $\boldsymbol{E}$ and $-\mathbf{E}$, respectively, where $\boldsymbol{E}$ is the sequence defined in Section 2, and let $\boldsymbol{Y}, \boldsymbol{T}$, and $\boldsymbol{B}$ be independent. Let $\boldsymbol{M}$ be $\boldsymbol{Y}$ reduced by $\boldsymbol{T}$ and increased by $\boldsymbol{B}$. Then $\boldsymbol{M}$ is δ-RRW. To help in remembering the notation, T is for top and B is for bottom, since that is where, on the path of $\boldsymbol{M}$, they act.

Next we discuss how to make a copy of $\boldsymbol{\Gamma}$ by reducing and increasing $\boldsymbol{Y}$. $\boldsymbol{\Gamma}$ behaves like a fair random walk unless at a max or min. At a max or min which is not 0, (4.2) and (4.3) govern its behavior. We also have $P(\Gamma_1 = 1) = 1/2$, and for $k > 0$ we define $\tilde{\lambda}(i)$, $i \geq 0$ by

$$P(\Gamma_{k+1} = 1 | \Gamma_j, j \leq k) = \tilde{\lambda}(N^+(k)), \text{ on } \{\Gamma_k = 0, \Gamma_k^* = 0\}$$

and so, by symmetry,

$$P(\Gamma_{k+1} = -1 | \Gamma_j, j \leq k) = \tilde{\lambda}(N^-(k)), \text{ on } \{\Gamma_k = 0,\ \Gamma_k^\# = 0\}.$$

Here, by the same argument that gave (2.6), we have

$$\omega(0) \leq \tilde{\lambda}(j),\ j \geq 1. \tag{4.4}$$

Now let $\boldsymbol{\Delta} = \Delta_i,\ i \geq 1$ be as in Section 2. Let $\boldsymbol{\Delta\prime}$ have the distribution of $\boldsymbol{\Delta}$ and let $\boldsymbol{Y}, \boldsymbol{\Delta}$, and $\boldsymbol{\Delta\prime}$ be independent. Let X be positive integer valued and independent of $(\boldsymbol{Y}, \boldsymbol{\Delta}, \boldsymbol{\Delta}')$, and have distribution given by $P(X = 1) = 2\tilde{\lambda}(1)$, $P(X = k|X > k-1) = 2\tilde{\lambda}(k)$, $k \geq 2$. Let $\hat{\boldsymbol{T}} = \hat{T}_i, i \geq 1$, be defined on $\{Y_i = -1\}$ by $\hat{T}_k = -2$ unless $k \in \{X + \sum_{i=1}^{n} \Delta_i, n \geq 1\} \bigcup \{X\}$, in which case $\hat{T}_i = 0$. Let $\hat{\boldsymbol{T}}$ be defined on $\{Y_1 = +1\}$ by $\hat{T}_k = -2$ unless $k \in \{\sum_{i=1}^{n} \Delta_i,\ n \geq 1\}$, in which case $\hat{T}_k = -2$. Let $\hat{\boldsymbol{B}} = \hat{B}_i$, $i \geq 1$, be defined on $\{Y_1 = +1\}$ by $\hat{B}_k = +2$ unless $k \in \{X + \sum_{i=1}^{n} \Delta'_i,\ n \geq 1\} \bigcup \{X\}$, in which case $\hat{B}_k = 0$, and be defined on $\{Y_1 = -1\}$ by $\hat{B}_k = 2$ unless $k \in \{\sum_{i=1}^{n} \Delta'_i,\ i \geq 1\}$, in which case $\hat{B}_k = 0$. Then $\boldsymbol{Y}$ reduced by $\hat{\boldsymbol{T}}$ and increased by $\hat{\boldsymbol{B}}$, which we denote by $\hat{\boldsymbol{M}}$, has the same distribution as $\boldsymbol{\Gamma}$. (Recall that the definition of a sequence $\boldsymbol{y}$ reduced and increased left the first jump of $\boldsymbol{y}$ alone, as opposed to the definition of a sequence reduced only.) We assume that $\boldsymbol{Y}, \boldsymbol{T}, \boldsymbol{B}, \boldsymbol{\Delta}, \boldsymbol{\Delta}'$, and X are independent.

Let $J_n = |M_n - \hat{M}_n|^*$. We will prove

$$EJ_n/\sqrt{n} \to 0 \text{ as } n \to \infty, \tag{4.5}$$

which together with (4.1) will complete the proof of Theorem 4.1. The following lemma is concerned with (non-random) sequences of numbers, and its proof recalls the proof of the uniqueness part of Theorem 3.1.

Lemma 4.2. *Let $\boldsymbol{r}$ be $\boldsymbol{z}$ decreased by $\mathbf{e}$ and increased by $\boldsymbol{f}$. Let $\boldsymbol{w}$ be $\boldsymbol{z}$ decreased by $\mathbf{e}$ and increased by $\boldsymbol{f}$. Let $\lambda_n = |r_n - z_n|^*$. Then $\lambda_n = \lambda_{n+1}$ unless either both $r_n \geq r_n^* - 1$ and $w_n \geq w_n^* - 1$, or both $w_n \leq w_n^\# + 1$ and $r_n \leq r_n^\# + 1$.*

Proof: Clearly, $r_{n+1} - z_{n+1} = r_n - z_n$ unless at least one of $r_n = r_n^*$, $r_n = r_n^\#$, $w_n = w_n^*$, or $w_n = w_n^\#$ holds. Suppose that $r_n = r_n^*$. (The other three cases are similar.) We will show that if $w_n \neq r_n$ and $w_n < r_n^* - 1$, then $\lambda_n = \lambda_{n+1}$, while if $w_n = r_n$ then either $w_n = w_n^*$ or $\lambda_n = \lambda_{n+1}$ or both. If $w_n < r_n$ but w_n is not w_n^*, then it can happen that $r_{n+1} - r_n \leq z_{n+1} - z_n$ but not that $w_{n+1} - w_n \leq z_{n+1} - z_n$, so $\lambda_{n+1} = \lambda_n$. If $w_n > r_n$ but $w_n < w_n^* - 1$ then $w_{n+1} - w_n$ could be $z_{n+1} - z_n$ while $r_{n+1} - r_n$ could be $z_{n+1} - z_n - 2$, so $|w_{n+1} - r_{n+1}|$ could be as big as $(w_n - r_n) + 2$. This still would not result in $\lambda_{n+1} > \lambda_n$ however, since $\lambda_n \geq w_n^* - r_n^* \geq (w_n + 2) - r_n$.

If $w_n = r_n$ and $\lambda_n = 0$ then $w_n^* = r_n^*, w_n = w_n^*$, while if $w_n = r_n$ and $\lambda_n > 0$ then λ_n is at least 2, so $\lambda_{n+1} = \max(\lambda_n, |w_{n+1} - r_{n+1}|) \le \max(\lambda_n, 2) = \lambda_n$. □

Now let $L_n = M_n^* - M_n^{\#}$, and $\hat{L}_n = \hat{M}_n^* - \hat{M}_n^{\#}$. Fix the integer $m \ge 3$, and define $\eta_i(m) = \eta_i,\ i \ge 1$, by

$$\begin{aligned}
\eta_1 &= \inf\{k\colon\ L_k \ge 2m \text{ and } \hat{L}_k \ge 2m\},\\
\eta_i &= \min(\inf\{n \ge \eta_{i-1}\colon M_n^* - M_n \le 1 \text{ and } \hat{M}_n^* - \hat{M}_n \le 1\},\\
&\qquad \inf\{n \ge \eta_{i-1}\colon\ M_n - M_n^{\#} \le 1 \text{ and } \hat{M}_n - \hat{M}_n^{\#} \le 1\}),\\
&\qquad \text{if } i \text{ is even and positive, and}\\
\eta_i &= \eta_{i-1} + m, \text{ if } i \text{ is odd and exceeds 1.}
\end{aligned}$$

In view of the last lemma,

$$\begin{aligned}
J_n &= J_{\min(\eta_1,n)} + \sum_{i=1}^{\infty}(J_{\min(\eta_{2i+1},n)} - J_{\eta_{2i}})I(\eta_{2i} < n) \\
&= \sum_{i=1}^{\infty}(J_{\eta_{2i+1}} - J_{\eta_{2i}})I(\eta_{2i} \le n - m) + X_n,
\end{aligned} \tag{4.6}$$

where $0 \le X_n \le J_{\eta_1} + 2m$.

It is not difficult to show $E\eta_1 < \infty$, one way being to show $P(\eta_1 > k \cdot 2m) \le (1 - \lambda(0)^{2m})^k$ and so $EJ_{\eta_1} < 2E\eta_1 < \infty$. We will prove that there exist absolute constants c, C, not depending on m, such that

$$E(J_{\eta_{2i+1}} - J_{\eta_{2i}})I(\eta_{2i} \le n - m) \le cm^{1/4}P(\eta_{2i} \le n - m), i \ge 1, \tag{4.7}$$

and

$$E(L_{\eta_{2i+1}} - L_{\eta_{2i}})I(\eta_{2i} \le n - m) \ge Cm^{1/2}P(\eta_{2i} \le n - m), i \ge 1. \tag{4.8}$$

Before we prove (4.7) and (4.8), we show that they imply (4.5). That $EL_n \to \infty$ as $n \to \infty$ follows from $\eta_1(m) < \infty$ a.s. (for each m), since $L_n \ge 2m$ on $\eta_1(m) \le n$. Thus, using (4.6), and the fact that

$$EL_n \ge \sum_{i=1}^{\infty} E(L_{\eta_{2i+1}} - L_{\eta_{2i}})I(\eta_{2i} \le n - m),$$

we get from (4.7), (4.8), and $\sup_n EX_n < \infty$, that

$$\limsup_{n\to\infty} EJ_n/EL_n \le c/Cm^{1/4}.$$

Since c and C do not depend on m,

$$\lim_{n\to\infty} EJ_n/EL_n = 0. \tag{4.9}$$

We have $L_n \le Y_n^* - Y_n^\# + 1$, by essentially the argument which yielded (3.1), so $EL_n < C\sqrt{n}$, which together with (4.9) gives (4.5).

Now define $\Phi_T(n)$ to be the number of T_i, $i \ge 1$, by which $Y_0, Y_1, \ldots, Y_n$ is reduced to get $M_0, M_1, \ldots, M_n$. Rephrased, $\Phi_T(n)$ is the number of i, $1 \le i < n$, such that $M_i = M_i^*$ and $Y_{i+1} - Y_i = 1$. Let $\Phi_B(n)$ be the number of B_i by which $Y_0, Y_1, \ldots, Y_n$ is increased to get M, and similarly define $\Phi_{\hat{T}}(n)$ and $\Phi_{\hat{B}}(n)$. It is immediate that if τ is a stopping time with respect to $\mathcal{H}_n = \sigma(Y_i, i \le n,\ M_i,\ i \le n,\ \hat{M}_i,\ i \le n)$ then $T_{\Phi_T(\tau)+k}$, $k \ge 1$, has conditional distribution, conditioned on any of the atoms of $\mathcal{H}_\tau$, which is exactly the distribution of T_i, $i \ge 1$, and a similar statement holds for B in place of T.

Thus (2.5) gives both

$$E\Big(\max_{1\le k\le n} \Big|\sum_{i=1}^{k} T_{\Phi_T(\tau)+i} - k\frac{2\theta}{2+\theta}\Big| \,\Big|\, \mathcal{H}_\tau\Big) < C\sqrt{n},\ n \ge 1, \tag{4.10}$$

and

$$E\Big(\max_{1\le k\le n} \Big|\sum_{i=1}^{k} B_{\Phi_B(\tau)+i} + k\frac{2\theta}{2+\theta}\Big| \,\Big|\, \mathcal{H}_\tau\Big) < C\sqrt{n},\ n \ge 1. \tag{4.11}$$

We also claim

$$E\Big(\max_{1\le k\le n} \Big|\sum_{i=1}^{k} \hat{T}_{\Phi_{\hat{T}}(\tau)+i} - k\frac{2\theta}{2+\theta}\Big| \,\Big|\, \mathcal{H}_\tau\Big) < C\sqrt{n},\ n \ge 1. \tag{4.12}$$

and

$$E\Big(\max_{1\le k\le n} \Big|\sum_{i=1}^{k} \hat{B}_{\Phi_{\hat{B}}(\tau)+i} + k\frac{2\theta}{2+\theta}\Big| \,\Big|\, \mathcal{H}_\tau\Big) < C\sqrt{n},\ n \ge 1. \tag{4.13}$$

The proofs of these two inequalities are similar. We prove (4.12), which follows by slightly different considerations on $\{Y_1 = 1\}$ and on $\{Y_1 = -1\}$. Now $\Phi_{\hat{T}}(\tau)$ is a stopping time with respect to $\sigma(\hat{T}_i,\ i \le n,\ \boldsymbol{Y},\ \boldsymbol{B},\ \boldsymbol{T},\ \boldsymbol{\Delta}')$, $n \ge 1$, and the last four objects are conditionally independent of $\hat{\boldsymbol{T}}$ given Y_1. On $\{Y_1 = 1\}$, $\hat{\boldsymbol{T}} = \boldsymbol{G}$, $\boldsymbol{G}$ as defined in Section 2. Thus the truth of (4.12) on $\{Y_1 = 1\}$ follows from (2.11), Lemma 2.3, and the version of Lemma 2.3 discussed just after its proof, although in this case the part of the discussion concerning the distribution of the arrival time of the first zero is not relevant. The truth of (4.12) on $\{Y_1 = -1\}$ follows similarly; here the arrival time of the first zero is X, which probably does have a different distribution than the other interarrival times. The necessary control on the distribution of X so that the discussion applies, that is, the fact that $P(X = k | X > k-1) > c > 0$, $k > 1$, is provided by (4.4).

Next we prove (4.8), which will be shown to hold conditioned on $\mathcal{H}_{\eta_{2i}}$. If A is an atom of $\mathcal{H}_{\eta_{2i}}$, then on A either $M_{\eta_{2i}} \ge M^*_{\eta_{2i}} - 1$ or $M_{\eta_{2i}} \le M^{\#}_{\eta_{2i}} + 1$. Suppose the first holds and that the inequality holds, so that $M_{\eta_{2i}} = M^*_{\eta_{2i}}$. Then $M_{\eta_{2i}+k} - M_{\eta_{2i}}$, $0 \le k \le m$, has exactly the distribution of the first k steps of a δ-PRMP: since $\eta_1 < \eta_{2i}$, $M^*_{\eta_{2i}} - M^{\#}_{\eta_{2i}} \ge 2m$, so $M_{\eta_{2i}+k} > M^{\#}_{\eta_{2i}}$, $0 \le k \le m$, and thus none of $\boldsymbol{B}$ increases $\boldsymbol{Y}$ between η_{2i} and $\eta_{2i} + m$. Thus, by Proposition 2.1, if m is large enough, and if E_A denotes conditional expectation on A,

$$\frac{1}{\sqrt{m}} E_A(L_{\eta_{2i}+m} - L_{\eta_{2i}}) =$$

$$\frac{1}{\sqrt{m}} E_A(M^*_{\eta_{2i}+m} - M^*_{\eta_{2i}}) \ge \frac{1}{\sqrt{m}} E_A(M_{\eta_{2i}+m} - M_{\eta_{2i}})^+ \ge E(W_1 + \theta W_1^*)^+/2 > 0.$$

This proves (4.8) for all large enough m, in this case, which implies (4.8) for all $m \ge 3$. The case $M_{\eta_{2i}} = M^*_{\eta_{2i}} - 1$ requires only minor modifications to this argument ($M_{\eta_{2i+k}} - M_{\eta_{2i}}, 0 \le k \le m$, looks like a δ-PRMP except that jumps from 0 are fair), and the other cases are similar.

Finally we turn to the proof of (4.9). Again, (4.9) holds conditioned on $\mathcal{H}_{\eta_{2i}}$. Let A be an atom of $M_{\eta_{2i}}$. On A, either both $M_{\eta_{2i}} \ge M^*_{\eta_{2i}} - 1$ and $\hat{M}_{\eta_{2i}} \ge \hat{M}^*_{\eta_{2i}} - 1$, or both $M_{\eta_{2i}} \le M^{\#}_{\eta_{2i}} + 1$ and $\hat{M}_{\eta_{2i}} \le \hat{M}^{\#}_{\eta_{2i}} + 1$. Suppose the former holds and suppose both inequalities hold. Then on A, $M_{\eta_{2i}+k} - M_{\eta_{2i}}$, $0 \le k \le m$ is $Y_{\eta_{2i}+k} - Y_{\eta_{2i}}$, $0 \le k \le m$, reduced by the sequence $T_{\Phi_T(\eta_{2i})+k}$, $k \ge 1$ and $\hat{M}_{\eta_{2k}+k} - \hat{M}_{\eta_{2k}}$, $0 \le k \le m$, is $Y_{\eta_{2i}+k} - Y_{\eta_{2i}}$, $0 \le k \le m$, reduced by $\hat{T}_{\Phi_{\hat{T}}(\eta_{2i})+k}$. Thus, using Lemma 2.2, (4.10), and (4.12), and the fact that given A, $\psi_k = Y_{\eta_{2i}+k} - Y_{\eta_{2i}}$, $0 \le k \le m$, has the distribution of $Y_0, Y_1, \ldots, Y_m$, and recalling that $E\sqrt{Y_m^*} \le 2\sqrt[4]{m}$ (proved after (2.5)), we have,

$$
\begin{aligned}
&E_A(J_{\eta_{2i+1}} - J_{\eta_{2i}}) \le E_A \max_{1\le k\le m} |(M_{\eta_{2i}+k} - M_{\eta_{2i}}) - (\hat{M}_{\eta_{2i}+k} - \hat{M}_{\eta_{2i}})| \\
&\le E_A \max_{1\le k\le m} |(M_{\eta_{2i}+k} - M_{\eta_{2i}}) - (\psi_k + \theta\psi_k^*)| + E_A \max_{1\le k\le m} |(\hat{M}_{\eta_{2i}+k} - \hat{M}_{\eta_{2i}}) - (\psi_k + \theta\psi_k^*)| \\
&\le E_A \max_{1\le k\le \psi_m^*} |\sum_{j=1}^{k} T_{\Phi_T(\eta_{2i})+j} - k\frac{2\theta}{2+\theta}| + E_A \max_{1\le k\le \psi_m^*} |\sum_{j=1}^{k} \hat{T}_{\phi_{\hat{T}}(\eta_{2i})+j} - k\cdot\frac{2\theta}{2+\theta}| + 4 \\
&\le E_A(C\sqrt{\psi_m^*} + C\sqrt{\psi_m^*} + 4) \\
&\le c\sqrt[4]{m} + 4 \\
&< C\sqrt[4]{m}. \qquad \square
\end{aligned}
$$

We note that Theorem 4.1 together with the sentence after (4.9), or alternatively using Nester's result (Nester (1993)) that if $\boldsymbol{X}$ is δ-RRW and $\boldsymbol{Y}$ is fair random walk then $P(|X_n| > \alpha) \le P(|Y_n| > \alpha)$, $\alpha > 0$, imply that $\lim_{n\to\infty} E|X_n|^p/n^{p/2} = E|\Psi_1|^p$, $p > 0$, where $\boldsymbol{\Psi}$ is the θ-contraction of $\boldsymbol{W}$, especially, they imply that this limit exists. A number of Nester's theorems translate immediately to results about the θ contraction of Brownian motion. The following is an example.

Theorem 4.2. *Let $\boldsymbol{\Psi}$ be the θ contraction of standard Brownian motion, $-1 < \theta < 0$. Let a, $b > 0$ and put $\tau = \inf\{t > 0 \colon Z_t \notin (-b, a)\}$. Then*

$$
P(\Psi_\tau = a) = \int_0^{b/(a+b)} t^\delta (1-t)^\delta dt \Big/ \int_0^1 t^\delta(1-t)^\delta dt.
$$

Note that the limit as $\theta \to 0$ (so $\delta \to 0$) is of course $b/(a+b)$, the Brownian probability.

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