Reinforced Random Walk on the d-dimensional Integer Lattice

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Abstract

Let w_k, $k \geq 0$, be a sequence of positive constants. Consider nearest–neighbor reinforced random walk on the d-dimensional integer lattice starting at the origin with edge–weight sequence $\{w_k\}$. In this process, the weight of an edge connecting nearest neighbors in the lattice is w_k if the edge has previously been crossed exactly k times. At integer times, the process jumps to a nearest neighbor of the previous position, with probabilities being proportional to current edge-weights.

We show that the individual coordinates return to zero infinitely often if there is positive probability that the range (= set of visited sites) is infinite. If $\sum_{k=0}^{\infty} w_{2k}^{-1} = \infty$ and $\sum_{k=0}^{\infty} w_{2k+1}^{-1} = \infty$, then the range is infinite. If both sums are finite, then the process eventually gets stuck crossing the same edge over and over again. If one sum is finite and the other is infinite, then, with probability one, all edges not touching the origin are crossed at most finitely often. Hence, in this last case the process eventually crosses only edges touching the origin, or the distance from the origin diverges over time to infinity.

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1. Introduction

In 1987, Persi Diaconis posed the following problem. Put initial weight 1 on each edge of the graph with vertex set $\mathbb{Z}^d$ (the d-dimensional integer lattice) and edges between nearest neighbors. Then run a discrete-time, nearest-neighbor random walk on $\mathbb{Z}^d$ starting at the origin according to the following rules. (1) The (conditional, given the past) transition probabilities for the next jump are to be proportional to the current weights of the corresponding edges, and (2) the weight of an edge is increased by some positive constant δ each time the edge is crossed. Diaconis' problem is to determine for which values of the dimension d and the reinforcement parameter δ this reinforced random walk (RRW) is recurrent. Except for $d = 1$, no one yet proved recurrence or transience for *any* d and *any* $\delta > 0$.

Davis (1989, 1990) considered RRW on $\mathbb{Z}$ under a wide variety of different reinforcement conventions. Since no one had made progress on the original Diaconis problem for $d \geq 2$, Davis (1990) posed the (perhaps) simpler problem of establishing recurrence or transience for RRW on $\mathbb{Z}^d$ with one-time reinforcement, meaning that the current weight of an edge is $1 + \delta$ if it has been crossed at least once and 1 if it has never been crossed. There has been no progress on this problem either for $d \geq 2$.

Several years ago, Michael Keane posed an even simpler problem: show that RRW with one-time reinforcement on a doubly infinite ladder is recurrent. This problem was solved by Sellke (1995), but the techniques do not help on $\mathbb{Z}^d$, $d \geq 2$.

Pemantle (1988) has studied RRW on trees with Diaconis reinforcement (meaning "add δ for each crossing"), but the dependence of Pemantle's methods on the absence of cycles prevents them from applying to $\mathbb{Z}^d$. Coppersmith and Diaconis (see Diaconis (1988)) have shown that RRW with Diaconis reinforcement on a finite graph is a mixture of Markov random walks, and they even have an explicit formula for the mixing measure in terms of all the "loops" of the graph. Diaconis (personal communication) has suggested that the Coppersmith-Diaconis formula for finite graphs might be used to prove transience/recurrence results for Diaconis RRW on $\mathbb{Z}^d$, but to the author's knowledge no one has done this.

This paper will primarily study nearest-neighbor RRW on $\mathbb{Z}^d$ of the following type. Let w_k, $k = 0, 1, \ldots$ be a sequence of positive constants, to be referred to as the edge-weight sequence. For $\boldsymbol{x} \epsilon \mathbb{Z}^d$, define $N(\boldsymbol{x}) = \{\boldsymbol{y} \epsilon \mathbb{Z}^d : \quad ||\boldsymbol{y} - \boldsymbol{x}|| = 1\}$ to be the set of nearest-neighbors of $\boldsymbol{x}$ in $\mathbb{Z}^d$. Let $\boldsymbol{X}_0, \boldsymbol{X}_1, \ldots$ be a random sequence of points in $\mathbb{Z}^d$ with $\boldsymbol{X}_0 \equiv \mathbf{0}$ and $\boldsymbol{X}_{n+1} \epsilon N(\boldsymbol{X}_n)$ for all $n \geq 0$. For $\boldsymbol{y} \epsilon N(\boldsymbol{x})$, define the number of crossings of the edge between $\boldsymbol{x}$ and $\boldsymbol{y}$ before time n by

$$(1.1) \qquad C(n, \boldsymbol{x}, \boldsymbol{y}) = \sum_{i=0}^{n-1} \; I(\boldsymbol{X}_i = \boldsymbol{x}, X_{i+1} = \boldsymbol{y}) + I(\boldsymbol{X}_i = \boldsymbol{y}, \boldsymbol{X}_{i+1} = \boldsymbol{x}).$$

(Note that $C(n, \boldsymbol{x}, \boldsymbol{y}) = C(n, \boldsymbol{y}, \boldsymbol{x})$.) Let $\mathcal{F}_n$ be the σ-algebra generated by $(\boldsymbol{X}_0, \ldots, \boldsymbol{X}_n)$. The transition probabilities of the $\boldsymbol{X}_n$ sequence are assumed to be given by

$$(1.2) \qquad P\{\boldsymbol{X}_{n+1} = \boldsymbol{y} | \mathcal{F}_n\} = \frac{w_{C(n, \boldsymbol{X}_n, \boldsymbol{y})} \;\; I\{\boldsymbol{y} \epsilon N(\boldsymbol{X}_n)\}}{\sum\limits_{\boldsymbol{z} \epsilon N(\boldsymbol{X}_n)} \quad w_{C(n, \boldsymbol{X}_n, \boldsymbol{z})}}.$$

It is easy to show by induction that $\boldsymbol{X}_0 \equiv \mathbf{0}$ and (1.2) determine a unique probability distribution for $(\boldsymbol{X}_0, \ldots, \boldsymbol{X}_n)$ and hence for the entire stochastic process $\boldsymbol{X}_0, \boldsymbol{X}_1, \ldots$

The RRW described above, with the same deterministic sequence of edge-weights for all edges, is essentially what Davis (1990) calls an *initially fair sequence type* RRW, except that Davis (1990) assumes in addition that the w_k's are nondecreasing. We will expropriate this terminology and refer to the RRW's above on $\mathbb{Z}^d, d \geq 1$, as being of *sequence type.*

Theorem 1. *Let $\boldsymbol{X}_n, n \geq 0$, be a sequence type RRW on $\mathbb{Z}^d, d \geq 2$. If $P\{\sup \; ||\boldsymbol{X}_n|| = \infty\} > 0$, then the individual coordinates $X_n^{(i)}$, $i = 1, 2, \ldots, d$ of $\boldsymbol{X}_n$ visit zero infinitely often, a.s.*

The proof of Theorem 1 is based on the symmetry of the RRW with respect to translation, reflection, and coordinate interchange. Here is the heart of the argument. If it is possible for $X_n^{(1)}$ to visit zero only finitely often, then $|X_n^{(1)}|$ and $|X_n^{(2)}|$ will both diverge to infinity a.s. when $\boldsymbol{X}_n$ has infinite range. By the symmetry between $X_n^{(2)}$ and $-X_n^{(2)}$, $X_n^{(2)}$ is equally likely to diverge to $-\infty$ as to $+\infty$ on the event $\{|X_n^{(1)}| \to \infty\}$. However, $\{|X_n^{(1)}| \to \infty,\ X_n^{(2)} \to \infty\}$ and $\{|\boldsymbol{X}_n^{(1)}| \to \infty,\ X_n^{(2)} \to \infty\}$ are subsets of disjoint tail events for a filtration with a trivial tail field, so it is impossible for their common probability to be positive.

Theorem 4 below suggests that there will be sequence type RRW's on $\mathbb{Z}^d$ (d sufficiently large) for which $0 < P\{\sup \|X_n\| = \infty\} < 1$, but we have no proof.

Theorem 2. *If* $\sum_{k=0}^{\infty} w_{2k}^{-1} = \infty$ *and* $\sum_{k=0}^{\infty} w_{2k+1}^{-1} = \infty$, *then* $P\{\sup X_n^{(i)} = \infty\} = 1$ *and* $P\{\inf X_n^{(i)} = -\infty\} = 1$ *for each of the coordinates* $X_n^{(i)}, i = 1, 2, \ldots, d$ *of the corresponding sequence type RRW* $\boldsymbol{X}_n$ *on* $\mathbb{Z}^d, d \geq 2$.

Theorem 3. *If* $\sum_{k=0}^{\infty} w_k^{-1} < \infty$, *then the corresponding sequence type RRW on* $\mathbb{Z}^d, d \geq 1$, *eventually gets stuck crossing the same edge over and over again. Thus,*

$$\sum_{\boldsymbol{x} \epsilon \mathbb{Z}^d} \sum_{\boldsymbol{y} \epsilon N(\boldsymbol{x})} P\{\boldsymbol{X}_{2n} = \boldsymbol{x} \ \ and \ \ \boldsymbol{X}_{2n+1} = \boldsymbol{y} \ \ for\ all\ sufficiently\ large \ \ n\} = 1.$$

Theorem 3 is almost certainly true on the two-dimensional triangular lattice, but we have no proof. The proof for Theorem 3 as stated is very dependent upon the fact that all finite cycles in the $\mathbb{Z}^d$ graph are of even length, and this is obviously false on the triangular graph. In fact, we can't even prove that sequence type RRW on a single triangle gets stuck on one edge when $\sum_{k=0}^{\infty} w_k^{-1} < \infty$.

Theorem 4. *If one of* $\sum_{k=0}^{\infty} w_{2k}^{-1}$ *and* $\sum_{k=0}^{\infty} w_{2k+1}^{-1}$ *is finite and one is infinite, then the corresponding sequence type RRW on* $\mathbb{Z}^d$, $d \geq 1$, *crosses each edge not touching the origin (at most) finitely often, a.s. Hence, with probability 1, either* $\boldsymbol{X}_{2n} = \boldsymbol{0}$ *for all sufficiently large* n *or* $\|\boldsymbol{X}_n\| \to \infty$ *as* $n \to \infty$. *If* $d = 1$, $\sum_{k=0}^{\infty} w_{2k}^{-1} < \infty$ *and* $\sum_{k=0}^{\infty} w_{2k+1}^{-1} = \infty$, *then* $P\{X_{2n} = 0$ *for all sufficiently large* $n\} = 1$. *If* $d = 1$, $\sum_{k=0}^{\infty} w_{2k}^{-1} = \infty$ *and* $\sum_{k=0}^{\infty} w_{2k+1}^{-1} < \infty$, *then* $P\{|X_{2n}| \to \infty\} = 1$. *If* $d \geq 2$, *then* $P\{\boldsymbol{X}_{2n} = \boldsymbol{0}$ *for all sufficiently large* $n\}$ *is strictly positive.*

We have no examples of RRW's as in Theorem 4 with $d \geq 2$ for which we can prove $P\{\|\boldsymbol{X}_n\| \to \infty\} > 0$. We conjecture that $P\{\boldsymbol{X}_{2n} = \boldsymbol{0}$ for all sufficiently large $n\} = 1$ for $d = 2$ under the conditions of Theorem 4, but we can only prove the partial result in Theorem 5.

Let $p_c(d)$ be the critical probability for bond percolation on $\mathbb{Z}^d$. (See for example Grimmett (1989).)

Theorem 5. *Suppose that* $\sum_{k=0}^{\infty} w_{2k}^{-1} = \infty$ *and* $\sum_{k=0}^{\infty} w_{2k+1}^{-1} < \infty$. *Let* Z_0 *be an exponential random variable with mean* w_0^{-1}. *For* $i = 1, 2, \ldots, 4d-2$ *and* $k = 0, 1, \ldots,$ *let* $Z_{2k+1}^{(i)}$ *be an exponential random variable with mean* $(4d-2)w_{2k+1}^{-1}$. *The* $Z_{2k+1}^{(i)}$*'s are to be independent of each other and of* Z_0. *If*

$$(1.3) \qquad P\{Z_0 \leq \sum_{i=1}^{4d-2} \sum_{k=0}^{\infty} Z_{2k+1}^{(i)}\} < p_c(d),$$

then $P\{\boldsymbol{X}_{2n} = \boldsymbol{0}$ *for all sufficiently large* $n\} = 1$ *for the corresponding sequence type RRW on* $\mathbb{Z}^d, d \geq 1$.

Remark One can use the Markov inequality to show that

$$(1.4) \qquad w_0 \sum_{k=0}^{\infty} w_{2k+1}^{-1} < \frac{p_c(d)}{(4d-2)^2(1-e^{-1})}$$

is enough to guarantee (1.3). However, it will be clear from the proof of Theorem 5 that (1.3) is far from being a necessary condition in Theorem 5. Theorem 5 shows that $\boldsymbol{X}_n$ eventually gets stuck jumping into and out of the origin if w_0 is sufficiently small compared to $(\sum_{k=0}^{\infty} w_{2k+1}^{-1})^{-1}$, but it only gives an extremely crude bound on what "sufficiently small" means.

The proofs of Theorems 2 through 5 use a construction of a continuous-time version of RRW described in Section 4. This construction was inspired by Herman Rubin's proof of his Generalized Polya Urn Theorem, as presented in the Appendix of Davis (1990). Davis used this theorem, in a different way, to prove the $d = 1$ case of Theorem 3 when the w_k's are increasing.

Theorems 1 through 5 are actually true in greater (but varying) generality. However, the main ideas in this paper all appear in the arguments concerning sequence type RRW on $\mathbb{Z}^d$. Rather than grubbing for maximum possible generality in the statements of Theorems 1 through 5 at the cost of complicating the exposition, we initially consider only the case of sequence type RRW on $\mathbb{Z}^d$. Remarks following the proofs of the stated theorems point out some other situations to which our methods apply and state some additional open problems.

2. Proof of Theorem 1

Suppose that $P(V_\infty) > 0$, where $V_\infty = \{\sup\ ||\boldsymbol{X}_n|| = \infty\}$ is the event that $\boldsymbol{X}_n$ visits infinitely many points in $\mathbb{Z}^d$. Since the coordinates of $\boldsymbol{X}_n$ are exchangeable, it will be enough to show that the first coordinate $X_n^{(1)}$ of $\boldsymbol{X}_n$ visits zero infinitely often, a.s. We will show that $P\{X_n^{(1)}$ visits zero finitely often $\} > 0$ and $P(V_\infty) > 0$ together lead to a contradiction.

Lemma 1. *If $P\{X_n^{(1)}$ visits 0 only finitely often $\} > 0$, then $P\{X_n^{(1)} > 0,\ \text{all } n \geq 1\} > 0$.*

Proof of Lemma 1. If $P\{X_n^{(1)}$ visits 0 finitely often $\} > 0$, then there exists an $n^* \geq 0$ for which $P\{X_n^{(1)} > 0,\ \text{all}\ \ n \geq n^*\} = \delta > 0$. But as $m \to \infty$,

$$P\{X_n^{(1)} > 0, n^* \leq n \leq m\} \downarrow P\{X_n^{(1)} > 0, n \geq n^*\} = \delta > 0. \tag{2.1}$$

Let $E_{n^*,m}$ be the set of all possible values $(\boldsymbol{x}_0 = \boldsymbol{0}, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_m)$ of the $(\boldsymbol{X}_0, \ldots, \boldsymbol{X}_m)$ vector for which $X_n^{(1)} > 0,\ n^* \leq n \leq m$, so that $P\{(\boldsymbol{X}_0, \ldots, \boldsymbol{X}_m)\ \epsilon E_{n^*,m}\} \geq \delta$. Let $\boldsymbol{u}^{(1)} = (1, 0, \ldots, 0)$ be the first unit vector in $\mathbb{Z}^d$. For each vector $(\boldsymbol{x}_0, \ldots, \boldsymbol{x}_m)$ in $E_{n^*,m}$, consider the vector $(0, \boldsymbol{u}^{(1)}, 2\boldsymbol{u}^{(1)}, \ldots,\ n^*\boldsymbol{u}^{(1)}, n^*\boldsymbol{u}^{(1)} + \boldsymbol{x}_1, \ldots,\ n^*\boldsymbol{u}^{(1)} + \boldsymbol{x}_m)$, and denote the set of such vectors by $F_{n^*,m}$. Then for each vector in $F_{n^*,m}$, the probability that $(\boldsymbol{X}_0, \ldots, \boldsymbol{X}_{n^*+m})$ takes on that value is at least

$$r_{n^*} = \left\{ \frac{\min(w_{0\ldots,} w_{2n^*})}{2d\ \max(w_{0,\ldots,} w_{2n^*})} \right\}^{2n^*} \tag{2.2}$$

times the probability that $(\boldsymbol{X}_0 \ldots, \boldsymbol{X}_m)$ equals the corresponding vector in $E_{n^*,m}$. (Note that the transition probabilities for the last $m - n^*$ steps are the same for the $F_{n^*,m}$ vector as for the corresponding $E_{n^*,m}$ vector. Also, transition probabilities for the first $2n^*$ steps cannot be smaller than the fraction in curly brackets in (2.2).) But if $(\boldsymbol{X}_0, \ldots, \boldsymbol{X}_{n^*+m})\epsilon F_{n^*,m}$, then $X_n^{(1)} > 0, 1 \leq n \leq n^* + m$. Hence, for all m,

$$P\{X_n^{(1)} > 0, 1 \leq n \leq n^* + m\} \geq r_{n^*}\delta,$$

and so

$$P\{X_n^{(1)} > 0, n \geq 1\} = \lim_{m \to \infty} P\{X_n^{(1)} > 0, 1 \leq n \leq n^* + m\} \geq r_{n^*}\delta. \qquad \square$$

Lemma 2. *If* $P\{X_n^{(1)} > 0,$ *all* $n \geq 1\} > 0$, *then*

$$P\{\sup\ |X_n^{(1)}| = \infty \ \textit{and} \ \liminf\ |X_n^{(1)}| < \infty\} = 0.$$

Proof of Lemma 2. This argument will use a construction of the RRW which causes the event $\{\sup\ |X_n^{(1)}| = \infty,\ \liminf\ |X_n^{(1)}| < \infty\}$ to be a subset of a probability-zero tail-field event.

For $i = 0, 1, \ldots$ and $j = 1, 2, \ldots$, let $U_j^{(i)}$ be iid $\mathcal{U}[0,1]$ random variables. Use the $U_j^{(i)}$'s as randomizers to construct sequence type RRW as follows. When $|X_n^{(1)}| = i$ for the first time, use $U_1^{(i)}$ to choose the next step with transition probabilities (1.2). Do this by partitioning $[0,1]$ into $2d$ intervals with lengths equal to the transition probabilities. The first interval should correspond to $|X_n^{(1)}|$ decreasing, the last interval should correspond to $|X_n^{(1)}|$ increasing, and the correspondence between the direction of movement and the order of the other intervals should always be the same. (When $X_n^{(1)} = 0$, have the first and last intervals correspond to $X_n^{(1)}$ decreasing and increasing, respectively.) When $|X_n^{(1)}| = i$ for the j^{th} time, do the same thing using $U_j^{(i)}$ as a randomizer. The crucial requirement of the construction is that the behavior of $(|X_n^{(1)}|, X_n^{(2)}, \ldots, X_n^{(d)})$ between the first time that $|X_n^{(1)}| = i$ and the first subsequent time that $|X_n^{(1)}| \leq i$ should be determined by the random variables $U_j^{(k)}, k \geq i, j \geq 1$.

Let A_i be the event that the $U_j^{(k)}$'s, $k \geq i > 0, j \geq 1$, are such that $|X_n^{(1)}|$ will (or would) stay strictly greater than i after the first time that $|X_n^{(1)}|$ equals i. If $P\{X_n^{(1)} > 0,$ all $n \geq 1\} = \epsilon > 0$, then $P(A_i) = \rho\epsilon > 0$, where $\rho = 2d\ w_0/\{w_1 + (2d-1)w_0\}$. The important thing to notice is just that $P(A_i)$ is some positive constant not depending on i if $P\{X_n^{(1)} > 0,$ all $n \geq 1\} > 0$.

Let $\mathcal{G}_i$ be the σ-algebra generated by $\{U_j^{(k)}, k \geq i, j \geq 1\}$. Then $\mathcal{G}_i \supset \mathcal{G}_{i+1} \supset \mathcal{G}_{i+2} \supset \ldots$, and the (proof of the) Kolmogorov Zero-One Law shows that $\mathcal{G}_\infty = \bigcap_i \mathcal{G}_i$ is trivial. Let $A_\infty = \limsup A_i = \bigcap_{l=0}^{\infty} \bigcup_{i \geq l} A_i$. Then $A_\infty \epsilon \mathcal{G}_\infty$, so $P(A_\infty)$ is either zero or one. However, $P(\bigcup_{i \geq l} A_i) \geq P(A_l) = \rho\epsilon$, so $P(A_\infty) \geq \rho\epsilon$, and so $P(A_\infty) = 1$.

On A_∞, there are infinitely many i's for which $|X_n^{(1)}|$ will stay greater than i after the first time that $|X_n^{(1)}|$ equals i. Thus, on A_∞, the event $\{\sup\ |X_n^{(1)}| = \infty\}$ implies the event $\{\liminf\ |X_n^{(1)}| = \infty\}$. □

Lemma 3. *$P\{\sup |X_n^{(1)}| = \infty$ and $\sup |X_n^{(2)}| < \infty\} = 0$.*

Proof of Lemma 3. It is enough to show that, for each integer $M > 0$, $P\{\sup |X_n^{(1)}| = \infty$ and $\sup\ |X_n^{(2)}| < M\} = 0$. But at each time n that $|X_n^{(1)}|$ reaches a new maximum, the (conditional, given the past) probability that $|X_{n+2M}^{(2)} - X_n^{(2)}| = 2M$ is $\epsilon_M = 2[w_0/\{w_1 + (2d-1)w_0\}]^{2M}$, since this ϵ_M is the conditional probability that the next $2M$ steps will all involve the second coordinate and will all be in the same direction. Thus,

$$P\{\sup |X_n^{(1)}| \geq i \text{ and } \sup |X_n^{(2)}| < M\} \leq (1-\epsilon_M)^i,$$

and so

$$P\{\sup |X_n^{(1)}| = \infty \text{ and } \sup |X_n^{(2)}| < M\} = 0. \qquad \square$$

Proof of Theorem 1. Again, suppose (in order to get a contradiction) that $P(V_\infty) > 0$ and $P\{X_n^{(1)}$ visits zero finitely often $\} > 0$. Lemmas 1, 2, and 3 together with the exchangeability of the coordinates of $\boldsymbol{X}_n$ imply that all coordinates diverge to ∞ in absolute value a.s. on V_∞. Since the $-\boldsymbol{X}_n$ process has the same distribution as the $\boldsymbol{X}_n$ process, $X_n^{(2)}$ must be equally likely to diverge to $-\infty$ as to $+\infty$ on V_∞. Hence,

$$(2.3) \quad P\{|X_n^{(1)}| \to \infty, X_n^{(2)} \to -\infty\} = P\{|X_n^{(1)}| \to \infty, X_n^{(2)} \to +\infty\} = (1/2)P(V_\infty) > 0.$$

Now recall the randomization scheme in the proof of Lemma 2, involving the independent $\mathcal{U}[0,1]$ r.v.'s $U_j^{(k)}$. Let A_i, $A_\infty = \limsup A_i, \mathcal{G}_i$ and $\mathcal{G}_\infty$ be as before. Recall that $P(A_\infty) = 1$, so that

$$P\{A_\infty, |X_n^{(1)}| \to \infty, X_n^{(2)} \to -\infty\} = P\{A_\infty, |X_n^{(1)}| \to \infty, X_n^{(2)} \to +\infty\} > 0 \qquad (2.4)$$

Let A_i^+ be the event { the $U_j^{(k)}$'s, $k \geq i, j \geq 1$, are such that $|X_n^{(1)}|$ will (or would) stay strictly greater than i after the first time that $|X_n^{(1)}|$ equals i *and* $X_n^{(2)} \to +\infty$ if $|X_n^{(1)}|$ ever equals i.} Define $A_\infty^+ = \limsup A_i^+ = \bigcap_{l=0}^{\infty} \bigcup_{i \geq l} A_i^+$, and let A_∞^- be defined analogously. Then $A_i^+ \epsilon \mathcal{G}_i$, so $A_\infty^+ \epsilon \mathcal{G}_\infty$, and likewise $A_\infty^- \epsilon \mathcal{G}_\infty$. Since $\mathcal{G}_\infty$ is trivial, the disjoint events A_∞^+ and A_∞^- cannot both have positive probability. However,

$$\{A_\infty, |X_n^{(1)}| \to \infty, X_n^{(2)} \to -\infty\} \subset A_\infty^- \text{ and } \{A_\infty, |X_n^{(2)}| \to \infty, X_n^{(2)} \to +\infty\} \subset A_\infty^+,$$

so that (2.4) implies that both A_∞^- and A_∞^+ have positive probability. Hence, we have a contradiction, and Theorem 1 is proved.□

3. Remarks on Theorem 1

It is easy to find other RRW's to which the symmetry arguments of Section 2 apply. For instance, suppose that each edge e on the $\mathbb{Z}^d$ nearest-neighbor graph has its own random edge-weight sequence $W_0^{(e)}, W_1^{(e)}, \ldots$ of positive random variables, with the edge-weight sequences of different edges being independent replications of each other. (The $W_k^{(e)}$'s for the same edge are not assumed independent.) Let $\mathcal{F}_n, n \geq 0$, be the σ-algebra generated by $\boldsymbol{X}_0, \ldots, \boldsymbol{X}_n$ and by all the $W_k^{(e)}$'s (or by all the weights "seen" by time n). Let (1.2) again define the transition probabilities. Call the resulting process RRW with iid weight sequences. If either (1) $W_1^{(e)}$ is bounded or (2) $W_0^{(e)}$ is independent of the remaining $W_k^{(e)}$'s, $k \geq 1$, for (the same) edge e, then the arguments of Section 2 carry over with very little change, except that one needs additional $\mathcal{U}[0,1]$ randomizers to generate the weights one "$x^{(1)}$-level" at a time.

One can also adopt the arguments of Section 2 to other sufficiently symmetric graphs. For example, consider the "triangular" graph in two dimensions. (See Figure 1.) If $P\{\sup ||\boldsymbol{X}_n|| = \infty\} > 0$ for a sequence type RRW starting at the origin on this triangular graph, then $\boldsymbol{X}_n$ visits each of the "symmetry axes" L_1, L_2, and L_3 infinitely often, a.s. Here is a sketch of the argument. Again, if $P\{\boldsymbol{X}_n$ visits L_1 finitely often $\} > 0$, then $P\{\boldsymbol{X}_n \notin L_1$, all $n \geq 1\} > 0$. Also, the distance from $\boldsymbol{X}_n$ to each L_i diverges to infinity a.s. on the event $\{\sup ||\boldsymbol{X}_n|| = \infty\}$. This implies that $\boldsymbol{X}_n$ eventually stays in one of the six 60° angles I, ..., VI between L_i's on the events $\{\sup ||\boldsymbol{X}_n|| = \infty\}$, and the conditional

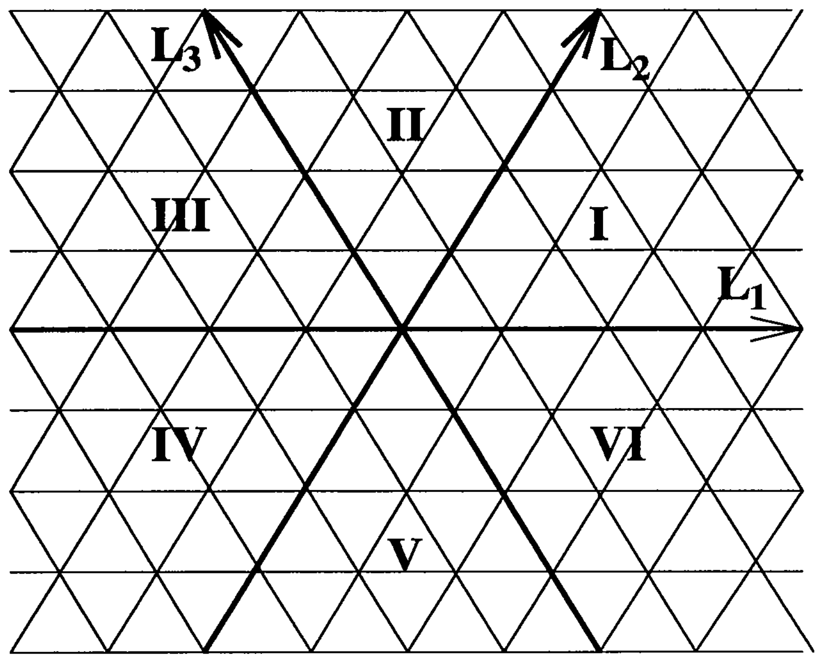

Figure 1.

probabilities for the angles must be equal, by symmetry. However, if we use independent $\mathcal{U}[0,1]$ randomizers $U_1^{(i)}, U_2^{(i)}$ in succession when the (absolute) distance of $\boldsymbol{X}_n$ to L_1 is i, with $\mathcal{G}_i$ and the trivial $\mathcal{G}_\infty$ as before, then the events { distances to L_i's $\to \infty$, $\boldsymbol{X}_n$ eventually in I or VI }, { distances to L_i's $\to \infty, \boldsymbol{X}_n$ eventually in II or V }, and { distances to L_i's $\to \infty, \boldsymbol{X}_n$ eventually in III or IV } are disjoint $\mathcal{G}_\infty$ events and so cannot all have positive probability.

The argument sketched in the previous paragraph also works on the "honeycomb" graph in two dimensions.

4. The Rubin Construction

Fix $d \geq 1$ and a weight sequence $w_k, k \geq 0$. For each edge $\boldsymbol{e}$ connecting nearest neighbors in $\mathbb{Z}^d$, suppose we have a sequence of independent exponential random variables $Y_0^{(\boldsymbol{e})}, Y_1^{(\boldsymbol{e})}, \ldots$, with $E\ Y_k^{(\boldsymbol{e})} = w_k^{-1}$. The sequences for different edges are to be independent. Now construct a continuous-time random walk $\tilde{\boldsymbol{X}}_t, t \geq 0$, on $\mathbb{Z}^d$ as follows. Each edge $\boldsymbol{e}$ has a clock which keeps track of how long the process has been in contact with edge $\boldsymbol{e}$ since the last previous crossing of edge $\boldsymbol{e}$. (The process $\tilde{\boldsymbol{X}}_t$ is "in contact" with edge $\boldsymbol{e}$

whenever $\tilde{\boldsymbol{X}}_t$ is at one of the endpoints of $\boldsymbol{e}$.) After an edge $\boldsymbol{e}$ has been crossed exactly k times, its clock "sounds an alarm" (for the $k+1^{st}$ time) as soon as it equals $Y_k^{(e)}$, at which time $\tilde{\boldsymbol{X}}_t$ jumps instantaneously across edge $\boldsymbol{e}$ and the clock for $\boldsymbol{e}$ is reset to zero. Start with $\tilde{\boldsymbol{X}}_0 = \boldsymbol{0}$, and with all edge-clocks almost surely set at zero. Since the $Y_k^{(e)}$'s are continuous and independent, different edge-clocks almost surely do not sound their alarms simultaneously. It can happen that $\tilde{\boldsymbol{X}}_t$ jumps infinitely often in finite time. Set $\tilde{\boldsymbol{X}}_t = \infty$ if there are infinitely many jumps before time t. For the sake of being specific, make the paths of $\tilde{\boldsymbol{X}}_t$ right-continuous in t.

Let $\boldsymbol{X}_n, n \geq 0$, equal the position of $\tilde{\boldsymbol{X}}_t$ after the n^{th} jump. It should be obvious from the independence of the $Y_k^{(e)}$'s and the fact that $Y_k^{(e)}$ has a constant hazard rate equal to w_k that the conditional transition probabilities of $\boldsymbol{X}_n$ are given by (1.2). (Cf. Section 5 of Davis (1990).)

Finally, note that $Y_0^{(e)}, Y_2^{(e)}, \ldots$ are the contact times for the first, third, ... crossings of $\boldsymbol{e}$. Thus, the "even" $Y_k^{(e)}$'s are the contact times for the "odd" crossings of $\boldsymbol{e}$.

5. Proof of Theorem 2

Theorem 1 implies $P\{\limsup X_n^{(1)} = -\infty\} = 0$. Hence, $P\{\sup X_n^{(1)} = \infty\} = 1$ will follow if we can show $P\{\limsup X_n^{(1)} = m\} = 0$ for all $m \in \mathbb{Z}$. Theorem 2 will then follow from the symmetry of $\boldsymbol{X}_n$ with respect to sign-change and coordinate interchange.

It is easy to see that $P\{\limsup X_n^{(1)} = m$, and $\boldsymbol{X}_n$ visits infinitely many points in $\mathbb{Z}^d$ with first coordinate $m\} = 0$. The argument is similar to that of Lemma 3 above. If this probability were positive, then $P\{\sup_{n \geq n^*} X_n^{(1)} = m$, and $\boldsymbol{X}_n$ visits infinitely many points in $\mathbb{Z}^d$ with first coordinate $m\}$ would be positive for some n^*. But this last probability is obviously smaller than $P\{\sup_{n \geq n^*} X_n^{(1)} = m$, and $\boldsymbol{X}_n$ visits at least i different points of $\mathbb{Z}^d$ with first coordinate m for the first time after time $n^*\}$, which is less than or equal to $[1 - w_0/\{w_1 + (2d-1)w_0]^i$. (There is conditional probability $w_0/\{w_1 + (2d-1)\}$ that the first coordinate of $\boldsymbol{X}_{n+1}$ will equal $m+1$ each time $\boldsymbol{X}_n$ visits a new point with first coordinate m.) Since this holds for all i, the claim in the first sentence of this paragraph follows.

Theorem 2 will now follow if we can prove

$$P\{\limsup X_n^{(1)} = m, \boldsymbol{X}_n = \boldsymbol{x}^{**} \text{ infinitely often }\} = 0 \tag{5.1}$$

for all points $\boldsymbol{x}^{**}$ in $\mathbb{Z}^d$ with first coordinate m. (5.1) can be proven without reference to the continuous-time RRW described above, but it seems easier to use $\tilde{\boldsymbol{X}}_t$. So, suppose that $\boldsymbol{X}_n$ and $\tilde{\boldsymbol{X}}_t$ are constructed as in Section 4 using independent exponential $Y_k^{(e)}$ random variables. Since $\sum_{k=0}^{\infty} w_{2k}^{-1} = \infty$ and $\sum_{k=0}^{\infty} w_{2k+1}^{-1} = \infty$, it is easy to see (from the Kolmogerov "three series" theorem, say) that $P\{\sum_{k=0}^{\infty} Y_{2k}^{(e)} = \infty\} = 1$ and $P\{\sum_{k=0}^{\infty} Y_{2k+1}^{(e)} = \infty\} = 1$ for every edge $\boldsymbol{e}$.

On the event that a particular edge $\boldsymbol{e}$ is crossed infinitely often by $\boldsymbol{X}_n$ (or by $\tilde{\boldsymbol{X}}_t$), the total time that $\tilde{\boldsymbol{X}}_t$ is in contact with $\boldsymbol{e}$ will be $\sum_{k=0}^{\infty} Y_k^{(e)} \overset{a.s.}{=} \infty$. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are the endpoints of $\boldsymbol{e}$, the total time in contact with $\boldsymbol{e}$ equals the total time at $\boldsymbol{x}$ plus the total time at $\boldsymbol{y}$, so that the total time at $\boldsymbol{x}$ and/or the total time at $\boldsymbol{y}$ must be infinite. If the total time spent at $\boldsymbol{x}$ is infinite, then *all* edges with endpoint $\boldsymbol{x}$ must be crossed infinitely often, since the total contact time for each such edge is infinity.

Again, let $\boldsymbol{e}^*$ be a particular edge, with endpoints $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$. The probability is zero that $\boldsymbol{X}_n$ (or $\tilde{\boldsymbol{X}}_t$) visits $\boldsymbol{x}^*$ infinitely often and eventually arrives at and leaves $\boldsymbol{x}^*$ only by jumping across $\boldsymbol{e}^*$ from and to $\boldsymbol{y}^*$. Here's why. On this event there would be some positive integer-valued random variable J for which the total time at $\boldsymbol{x}^*$ is at least either $\sum_{k=J}^{\infty} Y_{2k}^{(e^*)}$ or $\sum_{k=J}^{\infty} Y_{2k+1}^{(e^*)}$. (For instance, if all but the first $J-1$ departures from $\boldsymbol{x}^*$ are across $\boldsymbol{e}^*$ to $\boldsymbol{y}^*$, with all of these subsequent departures being "odd" crossings of $\boldsymbol{e}^*$, then the total time at $\boldsymbol{x}$ is at least $\sum_{k=J}^{\infty} Y_{2k}^{(e^*)}$.) But both of these sums are almost surely infinite whatever the value of J, and it was pointed out in the last paragraph that all edges touching $\boldsymbol{x}^*$ are crossed infinitely often if the total time at $\boldsymbol{x}^*$ is infinite.

Now back to (5.1). The previous paragraph shows that $\boldsymbol{X}_n$ cannot with positive probability visit $\boldsymbol{x}^{**}$ infinitely often by eventually arriving and departing only across the edge $\boldsymbol{e}^{**}$ between $\boldsymbol{x}^{**}$ and its unique neighbor $\boldsymbol{y}^{**}$ with first coordinate $m-1$. But the second to last paragraph shows that $\limsup X_n^{(1)} \geq (m+1)$ almost surely when an edge with both endpoints having first coordinate m is crossed infinitely often. (5.1) follows. □

6. Remarks related to Theorem 2

In this section, let $\mathbb{G}$ be a connected graph for which each vertex is on finitely many edges, and suppose that for every edge e there is an edge-weight sequence $w_k^{(e)}, k \geq 0$, of positive constants, with $\sum_{k=0}^{\infty}(w_{2k}^{(e)})^{-1}=\infty$ and $\sum_{k=0}^{\infty}(w_{2k+1}^{(e)})^{-1}=\infty$ for every e.

For some $\mathbb{G}$'s, the arguments of the previous section show that the corresponding RRW will almost surely have infinite range if $\mathbb{G}$ is infinite and will visit all vertices of $\mathbb{G}$ infinitely often in the case that $\mathbb{G}$ is finite. Let G be a finite proper subgraph of $\mathbb{G}$, and consider whether the edges of G could be precisely those edges which our RRW on $\mathbb{G}$ crosses infinitely often. If G contains an edge with both endpoints on non-G edges, then the edges crossed infinitely often cannot coincide with those of G (See the paragraph third from the end of Section 5.), and likewise if G contains a vertex which is not part of any G-loop and which is also on at least one non-G edge. In this second case, eventually all returns to the designated vertex are across the same edge as the previous departure. An argument like that of the second-to-last paragraph shows that the total time at the designated vertex is ∞ if this vertex is visited infinitely often, which in turn implies that all adjoining edges are crossed infinitely often. If these observations exclude all finite proper subgraphs of $\mathbb{G}$, then the RRW has infinite range for infinite $\mathbb{G}$ and visits all vertices infinitely often for finite $\mathbb{G}$. Examples of such $\mathbb{G}$'s are (a) a triangle, a square, or any finite "cycle" (b) any tree and (c) the nearest-neighbor graph on $\mathbb{Z}^d$. However, these arguments do not apply when $\mathbb{G}$ is a triangle with a "hanging edge" attached to one corner. In this last case, the author has not succeeded in proving that the hanging edge must be crossed infinitely often.

We conjecture that RRW on any $\mathbb{G}$ with edge-weights as described in the first paragraph is either a.s. recurrent or a.s. transient. Here, recurrent means that every vertex of $\mathbb{G}$ is visited infinitely often, and transient means that no vertex of $\mathbb{G}$ is visited infinitely often. This result is easy to prove on $\mathbb{Z}$. To wit,

Theorem 6. *For each edge e connecting nearest neighbors in $\mathbb{Z}$, let $w_k^{(e)}, k = 0, 1, \ldots$ be a sequence of positive edge-weights. If $\sum_{k=0}^{\infty}(w_{2k}^{(e)})^{-1}=\infty$ and $\sum_{k=0}^{\infty}(w_{2k+1}^{(e)})^{-1}=\infty$ for every edge e, then either $P\{X_n \text{ is recurrent }\} = 1$ or $P\{X_n \text{ is transient }\} = 1$ for the corresponding RRW on $\mathbb{Z}$*

Proof of Theorem 6. Use the Rubin construction to get a continuous-time RRW $\tilde{X}_t$. Let $x \epsilon \mathbb{Z}, x > 0$ say, with edge e^- between x and $x-1$ and edge e^+ between x and $x+1$. If $\tilde{X}_t$ visits x infinitely often, then $\tilde{X}_t$ must either jump across e^- from x to $x-1$ infinitely often, or across e^+ from x to $x+1$ infinitely often, or both. The total waiting time at x for infinitely many jumps down to $x-1$ is $\sum_{k=0}^{\infty} Y_{2k+1}^{(e^-)}$, and the total waiting time at x for infinitely many jumps up to $x+1$ is $\sum_{k=0}^{\infty} Y_{2k}^{(e^+)}$. Since both sums are a.s. infinite, $\tilde{X}_t$ must spend an infinite amount of time at x if it visits x infinitely often. But it follows that $\tilde{X}_t$ must exit from x infinitely often in *both* directions if $\tilde{X}_t$ visits x infinitely often, entailing that both $x-1$ and $x+1$ are visited infinitely often if x is. Thus, $P\{X_n \text{ is recurrent }\}+ P\{X_n \text{ is transient }\} = 1$.

Let $\mathcal{G}_i$ be the σ-algebra generated by the $Y_k^{(e)}$'s for edges to the right of i or to the left of $-i$. Let $\mathcal{G}_\infty = \bigcap_i \mathcal{G}i$. The event $\{X_n \text{ is transient }\}$ is in the completion of the trivial σ-algebra $\mathcal{G}_\infty$, so $P\{X_n \text{ is transient }\}$ must be 0 or 1.□

If $\{\boldsymbol{X}_n \text{ visits } \boldsymbol{x} \text{ infinitely often }\}$ implies $\{X_n \text{ visits } \boldsymbol{y} \text{ infinitely often }\}$ whenever $\boldsymbol{y}$ is a nearest neighbor of $\boldsymbol{x}$ in the graph $\mathbb{G}$, then obviously

$$P\{\boldsymbol{X}_n \text{ is recurrent }\} + P\{\boldsymbol{X}_n \text{ is transient }\} = 1. \tag{6.1}$$

This implication holds if the edge-weight sequences $w_k^{(e)}$ are the same for all edges and nondecreasing in k, with $\sum_{k=0}^{\infty} w_k^{-1} = \infty$. A weaker condition implying (6.1) is the following. Define

$$v_k(\boldsymbol{x}) = \max_{e \ni \boldsymbol{x}} w_k^{(e)}$$

where the maximum is over all edges $\boldsymbol{e}$ touching $\boldsymbol{x}$. Thus, $v_k(\boldsymbol{x})$ is the maximum k^{th} edge-weight for the edges touching $\boldsymbol{x}$. Define $u_k(\boldsymbol{x}) = \max_{j \le k} v_j(\boldsymbol{x})$. Then $\sum_{k=0}^{\infty} \{u_k(\boldsymbol{x})\}^{-1} = \infty$ is a sufficient condition for $\{\boldsymbol{X}_n \text{ visits } \boldsymbol{x} \text{ infinitely often}\}$ to imply $\{\boldsymbol{X}_n \text{ visits } \boldsymbol{y} \text{ infinitely often }\}$ for $\boldsymbol{y}$ a neighbor of $\boldsymbol{x}$ (and so (6.1) must be true if $\sum_{k=0}^{\infty} \{u_k(\boldsymbol{x})\}^{-1} = \infty$ for all $\boldsymbol{x}$.) Here is why. When the continuous-time RRW $\tilde{\boldsymbol{X}}_t$ visits $\boldsymbol{x}$ for the k^{th} time, the crossing numbers for all edges touching $\boldsymbol{x}$ are $2k-1$ or less, and therefore all the edge-weights of these edges are $u_{2k-1}(\boldsymbol{x})$ or less. It follows that the sojourn time of $\boldsymbol{X}_t$ at $\boldsymbol{x}$ between the k^{th} arrival and the k^{th} departure is (conditionally, given the past) greater or equal to an exponential random variable Z_k with mean $\{u_{2k-1}(\boldsymbol{x})\}^{-1}/\{\#\boldsymbol{x}\text{-edges}\}$, where the denominator is the number of edges touching $\boldsymbol{x}$. But $\sum_{k=1}^{\infty}\{u_{2k-1}(\boldsymbol{x})\}^{-1} = \infty$, so $\sum_{k=1}^{\infty} Z_k = \infty$, a.s. Hence if $\boldsymbol{X}_t$ visits $\boldsymbol{x}$ infinitely often, then $\boldsymbol{X}_t$ spends an infinite amount of time at $\boldsymbol{x}$, and therefore every edge touching $\boldsymbol{x}$ must be crossed infinitely often, since the total contact time for each such edge is infinity.

7. Proof of Theorem 3

Suppose that a sequence type RRW $\boldsymbol{X}_n$, $n \geq 0$, with $\sum_{k=0}^{\infty} w_k^{-1} < \infty$ is constructed as described in Section 4 using independent exponential $Y_k^{(e)}$ random variables. In this case it is easy to see that $P\{\sum_{k=0}^{\infty} Y_k^{(e)} < \infty\} = 1$ for every edge $\boldsymbol{e}$. Furthermore, $T^{(e)} \triangleq \sum_{k=0}^{\infty} Y_k^{(e)}$ is a continuous random variable whose density is continuous and positive on $(0, \infty)$, and the different $T^{(e)}$'s are independent.

Lemma 4. *Under the conditions of Theorem 3,* $P\{\sup ||\boldsymbol{X}_n|| < \infty\} = 1$.

Proof of Lemma 4. Whenever $|X_n^{(1)}|$ hits a new maximum, there is some positive conditional probability $\varepsilon > 0$ that $\boldsymbol{X}_n$ will get stuck crossing one of the edges that it has just come into contact with for the first time. This is because a generic $T = \sum_{k=0}^{\infty} Y_k$ has positive probability of being less than the minimum of $Y_1^*, Y_0^{(1)}, \ldots, Y_0^{(4d-3)}$, where Y_1^*, and the $Y_0^{(j)}$'s are independent of T and of each other, $Y_1^* \stackrel{D}{=} Y_1^{(e)}$, $Y_0^{(j)} \stackrel{D}{=} Y_0^{(e)}$. The T corresponds to the "total contact time" needed for $\tilde{\boldsymbol{X}}_t$ to cross a particular "new" edge infinitely often. The Y_1^* corresponds to the $Y_1^{(e)}$ for the edge just crossed when $|X_n^{(1)}|$ hits its new maximum. The $Y_0^{(j)}$'s correspond to the $Y_0^{(e)}$'s for the other edges adjacent to the one corresponding to T. Hence, $P\{\sup |X_n^{(1)}| > m\} \leq \varepsilon^{m-1}$, and so $P\{\sup |X_n^{(1)}| = \infty\} = 0$. Since the same is true of the other coordinates, the Lemma follows. □

By Lemma 4, the number of edges that $\tilde{\boldsymbol{X}}_t$ comes into contact with is a.s. finite. But the "total contact time" $T^{(e)}$ needed to cross any edge $\boldsymbol{e}$ infinitely often is also a.s. finite. Hence, $\tilde{\boldsymbol{X}}_t$ will have infinitely many jumps in finite time, a.s. The idea behind Theorem 3 is that the "jump explosion" cannot with positive probability occur across several edges simultaneously.

Suppose Theorem 3 is false. By Lemma 4 there must be a finite connected graph G consisting of 2 or more of the nearest-neighbor edges of $\mathbb{Z}^d$ such that $P\{$the edges that $\tilde{\boldsymbol{X}}_t$ crosses infinitely often are precisely those of $G\} > 0$.

We show first that G cannot contain any cycles. Suppose first that G contains a square with right side labelled as edge 1, top side as edge 2, left side as edge 3, and bottom as edge 4. Whenever $\tilde{\boldsymbol{X}}_t$ is on a corner of the square, it is in contact with one odd edge and one even edge. If a "jump explosion" occurs simultaneously across all edges of the square, it must be the case that $T^{(1)} + T^{(3)} = T^{(2)} + T^{(4)}$, where $T^{(j)}$ is the T for edge j. But this equality occurs with probability zero, since the $T^{(j)}$'s are independent continuous random variables. Likewise, any other cycle in G must be of even length, so it can be decomposed into alternating "odd" and "even" edges. Whenever $\tilde{\boldsymbol{X}}_t$ is in contact with the cycle, it is in contact with one "odd" edge and one "even" edge. If a jump explosion occurs simultaneously across all edges of the cycle, the "total time in contact with the cycle" at the time of the explosion must simultaneously equal the sum of the "odd" T's and the sum of the "even" T's. But these two sums are independent continuous random variables, so they have probability zero of being equal.

We next show that G cannot be acyclic with 2 or more edges. If this were possible, then there would exist an n^* and a path $(\boldsymbol{x}_0 \equiv 0,\ \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n^*)$ with positive probability for $(\boldsymbol{X}_0, \ldots, \boldsymbol{X}_{n^*})$ with the following two properties. First, $\boldsymbol{x}_{n^*}$ is an "interior" point of G with (at least) two G-edges $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ touching $\boldsymbol{x}_{n^*}$. Second, the conditional probability of the event $E_{n^*,G} \triangleq \{\boldsymbol{X}_n$ crosses all edges of G infinitely often and no other edges whatsoever after time $n^*, \}$ given that $(\boldsymbol{X}_0, \ldots, \boldsymbol{X}_{n^*}) = (\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{n^*})$, is strictly positive. Note that on $E_{n^*,G}$, the subsequent RRW X_n, $n \geq n^*$, must visit $\boldsymbol{x}_{n^*}$ infinitely often, with all returns to $\boldsymbol{x}_{n^*}$ being across the same G-edge as the previous departure, since G is acyclic.

Let k_1 be the number of times that the path $(\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{n^*})$ crosses $\boldsymbol{e}_1$, and likewise for k_2 and $\boldsymbol{e}_2$. Take the situation after $(\boldsymbol{X}_0, \ldots, \boldsymbol{X}_{n^*}) = (\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{n^*})$ as the initial situation for a Rubin construction as in Section 4 for the subsequent RRW $\boldsymbol{X}_n$, $n \geq n^*$. Call the resulting continuous-time process $\tilde{\boldsymbol{X}}_t^*$, $t \geq 0$. (Note that $\tilde{\boldsymbol{X}}_0^* \equiv \boldsymbol{x}_{n^*}$.) In particular, let $Y_{k_1}^{(1)}, Y_{k_1+1}^{(1)}, Y_{k_1+2}^{(1)}, \ldots$ be the independent exponential "contact times" between jumps for edge $\boldsymbol{e}_1$. If $\tilde{\boldsymbol{X}}_t^*$ crosses only G-edges, then $\tilde{\boldsymbol{X}}_t^*$ must be at $\boldsymbol{x}_{n^*}$ for $Y_{k_1}^{(1)}$ units of time before jumping across e_1 for the first time, $\tilde{X}_t^*$ must then be at x_{n^*} for $Y_{k_1+2}^{(1)}$ additional units of time before the *third* jump across e_1, and in general $\tilde{X}_t^*$ must be at x_{n^*} for exactly $Y_{k_1+2j}^{(1)}$ time units between the $(2j-1)^{th}$ and the $(2j+1)^{th}$ jumps across e_1. At the time of the jump explosion for $\tilde{X}_t^*$, $\tilde{X}_t^*$ must have been at x_{n^*} for precisely $\sum_{j=0}^{\infty} Y_{k_1+2j}^{(1)}$ units of time if $\tilde{X}_t^*$ is to cross e_1 infinitely often without ever crossing any non-G edges. Likewise, $\tilde{X}_t^*$ must have been at x_{n^*} for precisely $\sum_{j=0}^{\infty} Y_{k_2+2j}^{(2)}$ units of time if $\tilde{X}_t^*$ is to cross e_2 infinitely often without ever crossing any non-G edges. But these two sums are independent continuous random variables, and so the probability that they are equal is zero. Furthermore, the behavior of the successive positions of $\tilde{X}_t^*$ is by construction the same as the behavior of X_n, $n \geq n^*$, given $(X_0, \ldots, X_{n^*}) = (x_0, \ldots, x_{n^*})$. Thus, the conditional probability of $E_{n^*,G}$, given $X_0, \ldots, X_{n^*}) = (x_0, \ldots, x_{n^*})$, *cannot* be positive, and so G cannot be acyclic (with 2 or more edges) either. The only remaining possibility is that X_n eventually crosses a single edge over and over again, so Theorem 3 is proved. □

8. Remarks on Theorem 3

Let $\mathbb{G}$ be any connected graph for which each vertex is on finitely many edges. Suppose that for each edge e there is an edge-weight sequence $w_k^{(e)}$, $k \geq 0$, of positive constants, with $\sum_{k=0}^{\infty} (w_k^{(e)})^{-1} < \infty$ for each edge e. We conjecture that $P\{X_n$ is transient$\} + P\{X_n$ eventually gets stuck crossing one edge$\} = 1$ for the corresponding RRW on $\mathbb{G}$. Again, transience here means that no vertex is visited infinitely often. For finite $\mathbb{G}$, the conjecture reduces to $P\{X_n$ eventually gets stuck crossing one edge$\} = 1$. For infinite $\mathbb{G}$, if the number of edges per vertex is bounded and if all edges have the same edge-weight sequence, then it is easy to show that X_n will a.s. visit finitely many vertices. (The argument is essentially that of Lemma 4 in the previous section.) Hence, in this case as well the conjecture becomes $P\{X_n$ eventually gets stuck crossing one edge$\} = 1$.

Here is a very simple unresolved special case of the above conjecture. Let $\mathbb{G}$ be a triangle, and suppose that the three edges have a common edge-weight sequence w_k, $k \geq 0$, with $\sum_{k=0}^{\infty} w_k^{-1} < \infty$. Show that the corresponding RRW X_n eventually gets stuck on one edge, a.s. The apparent difficulty of this problem is an indication of how crucial the "even cycle length" property of the $\mathbb{Z}^d$ graph was in the proof of Theorem 3.

A situation where the arguments of the previous section do apply is the following. Consider RRW $\boldsymbol{X}_n$ on $\mathbb{Z}^d$, with different edges having different edge-weight sequences $w_k^{(e)}$, $k \geq 0$. If $\sum_{k=0}^{\infty} (w_k^{(e)})^{-1} < \infty$ for all $\boldsymbol{e}$, then the proof of Theorem 3 shows that $\boldsymbol{X}_n$ gets stuck on one edge a.s. on the event $\{\sup ||\boldsymbol{X}_n|| < \infty\}$.

9. Proof of Theorem 4

Use independent exponential $Y_k^{(e)}$ random variables to construct the corresponding continuous-time RRW $\tilde{\boldsymbol{X}}_t$, $t \geq 0$, as described in Section 4.

Lemma 5. *Let $\boldsymbol{x} \in \mathbb{Z}^d$, $\boldsymbol{x} \neq 0$. If one of $\sum_{k=1}^{\infty} w_{2k}^{-1}$ and $\sum_{k=0}^{\infty} w_{2k+1}^{-1}$ is finite and one is infinite, then $\tilde{\boldsymbol{X}}_t$, $t \geq 0$, will a.s. spend (at most) a finite amount of time at $\boldsymbol{x}$. Thus, $\int_0^{\infty} I(\tilde{\boldsymbol{X}}_t = \boldsymbol{x})\, dt < \infty$, a.s.*

Proof of Lemma 5. Suppose that $\sum_{k=0}^{\infty} w_{2k}^{-1} = \infty$, $\sum_{k=0}^{\infty} w_{2k+1}^{-1} < \infty$. Then for every edge $\boldsymbol{e}$, $P\{\sum_{k=0}^{\infty} Y_{2k}^{(e)} = \infty\} = 1$ and $P\{\sum_{k=0}^{\infty} Y_{2k+1}^{(e)} < \infty\} = 1$. Since $\tilde{\boldsymbol{X}}_0 = \boldsymbol{0}$, and since the number of edges touching $\boldsymbol{x}$ is $2d$ and therefore even, at least one of the edges touching $\boldsymbol{x}$ will have been crossed an odd number of times whenever $\tilde{\boldsymbol{X}}_t$ is at $\boldsymbol{x}$. Hence, the total time at $\boldsymbol{x}$ is bounded by the sum of all the "even crossing" waiting times $Y_{2k+1}^{(e)}$ for all $2d$ edges touching $\boldsymbol{x}$. But this sum is finite, a.s.

The argument for the other case $\sum_{k=0}^{\infty} w_{2k}^{-1} < \infty$, $\sum_{k=0}^{\infty} w_{2k+1}^{-1} = \infty$ is exactly the same, except that one uses the fact that some edge touching $\boldsymbol{x}$ must have been crossed an even number of times whenever $\tilde{\boldsymbol{X}}_t$ is at $\boldsymbol{x}$. □

Proof of Theorem 4. If one of $\sum_{k=0}^{\infty} w_{2k}^{-1}$ and $\sum_{k=0}^{\infty} w_{2k+1}^{-1}$ is finite and one is infinite, then the total time $\sum_{k=0}^{\infty} Y_k^{(e)}$ that $\tilde{\boldsymbol{X}}_t$ must be in contact with an edge $\boldsymbol{e}$ to cross it infinitely often is a.s. infinite. But Lemma 5 implies that the total time that $\tilde{\boldsymbol{X}}_t$ is in contact with an edge not touching the origin is a.s. finite. Hence, any edge not touching the origin will be crossed only finitely often, a.s.

If $d = 1$, $\sum_{k=0}^{\infty} w_{2k}^{-1} = \infty$ and $\sum_{k=0}^{\infty} w_{2k+1}^{-1} < \infty$, then there will be some $x > 0$ for which $\sum_{k=0}^{\infty} Y_{2k+1}^{(e^-)} < Y_0^{(e^+)}$. Here e^- is the edge between $x-1$ and x, and e^+ is the edge between x and $x+1$. In fact, the events that different positive x's have this property are independent, with a common positive probability. But the RRW can never jump across e^+ to $x+1$ for such an x, so $X_n \to \infty$ is impossible. Likewise $X_n \to -\infty$ is also impossible, so $P\{X_{2n} = 0 \text{ for all sufficiently large } n\} = 1$.

Suppose $d = 1$, $\sum_{k=0}^{\infty} w_{2k}^{-1} < \infty$ and $\sum_{k=0}^{\infty} w_{2k+1}^{-1} = \infty$. Let e_0 be the edge between 0 and 1, and let e_1 be the edge between 1 and 2. Then $\sum_{k=0}^{\infty} Y_{2k+1}^{(e_0)} = \infty$ and $\sum_{k=0}^{\infty} Y_{2k}^{(e_1)} < \infty$. But then X_n cannot jump from 1 to 0 across e_0 infinitely often, since this would entail $\tilde{X}_t$ being 1 for an infinite amount of time, while the total waiting time for infinitely many jumps from 1 to 2 across e_2 is finite. Likewise, X_n cannot jump from -1 to 0 infinitely often either, so $P\{|X_n| \to \infty\} = 1$.

Suppose now that $d \geq 2$, $\sum_{k=0}^{\infty} w_{2k}^{-1} = \infty$ and $\sum_{k=0}^{\infty} w_{2k+1}^{-1} < \infty$. Consider the random variables $\sum_{k=0}^{\infty} Y_{2k+1}^{(e_0)}$ for all the edges e_0 which touch the origin. Recall that these sum random variables are iid with a density strictly positive on $(0, \infty)$. Thus, there is positive probability that every edge e_0 touching the origin has a $\sum_{k=0}^{\infty} Y_{2k+1}^{(e_0)}$ value which is smaller than the $Y_0^{(e^*)}$ values for all other edges e^* sharing a non-origin endpoint with e_0. This would mean that the total "contact time" for all even crossings of e_0 is less than the shortest "first crossing" contact time of any other edge e^* with which e_0 shares a non-origin endpoint. If this holds for all edges e_0 touching the origin, then $\tilde{X}_t$ will visit each of the nearest neighbors of the origin for a finite but strictly positive amount of time and will be at the origin otherwise.

Now suppose that $d \geq 2$, $\sum_{k=0}^{\infty} w_{2k}^{-1} < \infty$ and $\sum_{k=0}^{\infty} w_{2k+1}^{-1} = \infty$. There is positive probability that after $4d$ steps $\tilde{X}_t$ will be back at the origin with every edge touching the origin having been crossed exactly once. If this happens the situation is similar to that in the preceding paragraph. There is positive conditional probability that the "total contact" time for all remaining "odd crossings" of edges touching the origin is less than the smallest "next crossing" contact time for all other adjacent edges. If this happens, $\tilde{X}_t$ will never again cross any edge not touching the origin.

Here is another and perhaps more interesting argument for why $\tilde{\boldsymbol{X}}_t$ has strictly positive probability of eventually crossing only edges touching the origin for $d \geq 2$. Consider sequence type RRW on a finite graph for which each vertex is on an even number of edges. Under the "one sum finite, one sum infinite" condition on the weights, the same arguments as above apply to show that *with probability one* eventually only edges touching the starting point will be crossed. Now consider a finite subgraph of the $\mathbb{Z}^d$ graph which (1) contains all edges touching the origin and all adjacent edges and (2) has an even number of edges touching each vertex. For some positive integer n^*, there is positive probability δ that RRW $\boldsymbol{X}_n$ restricted to this subgraph will have $\boldsymbol{X}_{2n} = \boldsymbol{0}$ for all $n \geq n^*$. But then a likelihood ratio argument similar to the proof of Lemma 1 shows that unrestricted RRW on $\mathbb{Z}^d$ has probability at least $r_{n^*}\delta$ that $\boldsymbol{X}_{2n} = \boldsymbol{0}$, $n \geq n^*$, where r_{n^*} is given in (2.2) □

10. Proof of Theorem 5

Let $Y_0^{(e)}, Y_2^{(e)}, Y_4^{(e)}, \ldots$ be as in the Rubin construction of Section 4. The "odd" $Y_{2k+1}^{(e)}$'s will also end up being as in Section 4, but they will be constructed in terms of yet other exponential random variables.

For each ordered pair of adjacent edges $(\boldsymbol{e}, \boldsymbol{e}')$ and each $k = 0, 1, \ldots$, let $Z_{2k+1}(\boldsymbol{e}, \boldsymbol{e}')$ be an exponential random variable with mean $(4d-2)w_{2k+1}^{-1}$. The different $Z_{2k+1}(\boldsymbol{e}, \boldsymbol{e}')$ random variables should be independent of each other and of the $Y_{2k}^{(e)}$'s. (In particular, $Z_{2k+1}(\boldsymbol{e}, \boldsymbol{e}')$ and $Z_{2k+1}(\boldsymbol{e}', \boldsymbol{e})$ are iid, not equal.) Define $Y_{2k+1}^{(e)}$ to be the minimum of the $4d-2$ different $Z_{2k+1}(\boldsymbol{e}, \boldsymbol{e}')$'s with $\boldsymbol{e}$ as the first element of the ordered pair of edges. Then the $Y_{2k+1}^{(e)}$'s are independent exponential random variables with $Y_{2k+1}^{(e)}$ having mean w_{2k+1}^{-1}, agreeing with the set-up in Section 4. Let $\tilde{\boldsymbol{X}}_t$, $t \geq 0$, be the resulting continuous-time RRW.

Now, call an edge $\boldsymbol{e}'$ not touching the origin "closed" if

$$Y_0^{(e')} > \sum_{\boldsymbol{e}} \sum_{k=0}^{\infty} Z_{2k+1}(\boldsymbol{e}, \boldsymbol{e}'), \tag{10.1}$$

where the first sum is over the edges $\boldsymbol{e}$ adjacent to $\boldsymbol{e}'$. Otherwise, call $\boldsymbol{e}'$ "open". (This usage of "open" and "closed" corresponds to the lingo of percolation theory.) Note that different edges $\boldsymbol{e}'$ are independent when it comes to being open and closed. Condition

(1.3) says that the probability of an edge being open is less than the critical probability for dimension d, so with probability one there is no infinite open path.

Since $Y_{2k+1}^{(e)} \leq Z_{2k+1}(\boldsymbol{e}, \boldsymbol{e}')$, condition (10.1) for an edge $\boldsymbol{e}'$ not touching the origin to be closed implies

$$Y_0^{(e')} > \sum_{e} \sum_{k=0}^{\infty} Y_{2k+1}^{(e)}, \tag{10.2}$$

where the first sum is again over all edges $\boldsymbol{e}$ adjacent to $\boldsymbol{e}'$. Recall from the last section that whenever $\tilde{\boldsymbol{X}}_t$ is in contact with an endpoint of $\boldsymbol{e}'$, there is an edge adjacent to this endpoint which has been crossed an odd number of times. If $\boldsymbol{e}'$ has never been crossed, this "odd" edge must of course be one adjacent to $\boldsymbol{e}'$, not $\boldsymbol{e}'$ itself. But (10.2) says that the sum of the total contact times for all even crossings of these adjacent edges is less than the contact time for the first crossing of $\boldsymbol{e}'$. It follows that $\tilde{X}_t$ can never cross a closed edge $\boldsymbol{e}'$, since it cannot stay in contact with $\boldsymbol{e}'$ long enough. Since there are no infinite open paths a.s., there are only finitely many edges which $\tilde{\boldsymbol{X}}_t$ will ever cross at all. Theorem 4 tells us that any edge not touching the origin will only be crossed finitely often, so eventually only edges touching the origin will be crossed. □

11. Remarks on Theorems 4 and 5

We conjecture that $P\{\boldsymbol{X}_{2n} = \boldsymbol{0} \text{ for all sufficiently large } n\} = 1$ for $d = 2$ in Theorem 4. This conjecture may be more plausible in the $\sum_{k=0}^{\infty} w_{2k}^{-1} = \infty$, $\sum_{k=0}^{\infty} w_{2k+1}^{-1} < \infty$ case than in the other.

A fascinating RRW of sorts on $\mathbb{Z}^2$ related to the $\sum_{k=0}^{\infty} w_{2k}^{-1} < \infty$, $\sum_{k=0}^{\infty} w_{2k+1}^{-1} = \infty$ case of Theorem 4 has apparently been considered before. Suppose that $w_0 = 1$ and $w_1 = 0$. The resulting RRW $\boldsymbol{X}_n$ might be called "erasing random walk", since edges are "erased" as they are crossed. According to Greg Lawler (personal communication), it has been conjectured that this $\boldsymbol{X}_n$ process on $\mathbb{Z}^2$ always ends up at the starting point (without loss of generality the origin) with all edges out erased. Here are two simple observations which make this conjecture more plausible than it might seem initially. First, the process cannot "get stuck" without a way out at any point of $\mathbb{Z}^2$ except for the origin. This is because each point of $\mathbb{Z}^2$ is on four edges, and for $\boldsymbol{x} \neq \boldsymbol{0}$, the process must have crossed either one or three of the edges touching $\boldsymbol{x}$ whenever $\boldsymbol{X}_n$ is at $\boldsymbol{x}$. (Either "in" or "in-out-in".) Second, there is always an "open" path back to the origin whenever the $\boldsymbol{X}_n \neq \mathbf{0}$. This is because any finite path for this $\boldsymbol{X}_n$ can be embedded in a large finite subgraph G of the $\mathbb{Z}^2$ graph, with G having the property that all vertices of G lie on an even number of G-edges. (You can take G to be a "box" with a "fringe" of loops added to the edge.) But erasing random walk on a finite graph must get stuck somewhere, and by the first observation it can't get stuck anywhere but at the starting point if all vertices are on an even number of edges.

An obvious conjecture for $d \geq 3$ is that "erasing random walk" has a strictly positive probability of never returning to the origin.

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References

Davis, B. (1989). Loss of recurrence in reinforced random walk. In *Almost Everywhere Convergence Proc. of the International Conference or Almost Everywhere Convergence in Probability and Ergodic Theory* (G.A. Edgar and L. Sucheston, eds.) 179–188. Academic Press, San Diego.

Davis, B. (1990). Reinforced random walk. *Probab. Th. Rel. Fields* **84**, 203–229.

Diaconis, P. (1988). Recent progress on de Finetti's notions of exhangeability. *Bayesian Statistics* **3**, 111–125, J.M. Bernardo, M.H. DeGroot, D.V. Lindley, and A.F.M. Smith, Eds. Oxford University Press.

Grimmett, G. (1989). Percolation. Springer, New York.

Pemantle, R. (1988). Phase transition in reinforced random walk and RWRE on trees. *Ann. Probab.* **16**, 1229–1241.

Sellke, T. (1995). Recurrence of reinforced random walk on a ladder. To appear in *Probability Theory and Related Fields.*