AN ELEMENTARY APPROACH TO NATURALITY, PREDICTABILITY, AND THE FUNDAMENTAL THEOREM OF LOCAL MARTINGALES

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An Elementary Approach to Naturality, Predictability, and the Fundamental Theorem of Local Martingales

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ABSTRACT

We give nontechnical proofs of the predictability of natural processes and the Fundamental Theorem of Local Martingales. Some arguments are shortened through a novel use of an elementary inequality known as "Chebyshev's Other Inequality."

1. Introduction

Stochastic integration has become an important tool in the applied sciences and engineering. Familiar examples of this are the famous Black and Scholes option pricing formula and the continuous-time version of the Kalman filter in control engineering. As the techniques of stochastic integration become more integrated into science and engineering curricula, it becomes important to develop approaches to the subject matter that encompass the most important ideas for applications with a minimum of mathematical background. Such an approach was set out in [12], where a deep and fine analysis of martingales in general was avoided. Our goal in this note is to further simplify the development presented in [12].

When P.A. Meyer proved what is now known as the Doob-Meyer decomposition [7, 8], he used the notion of a natural process in order to obtain uniqueness of the decomposition. Natural processes were indeed natural, as they arose intuitively from a limiting argument applied to Doob's discrete-time decomposition. Doléans-Dade showed in 1967 [3] that an increasing integrable càdlàg process is natural if and only if it is predictably measurable. This gave rise to elegant, but much less intuitive, proofs of the Doob-Meyer decomposition using the dual predictable projection (see, e.g., [2, 6]).

In [12], the idea of natural processes was resurrected as part of an attempt to keep proofs technically simple and intuitive. A simple proof that a bounded natural process is predictable ([12], p. 121) circumvented the need for some technical theorems of Meyer that were used in Doléans-Dade's proof [3]. There does not, however, appear to be a simple way to extend this proof from bounded to *integrable* natural processes.

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In this note we give a simple proof that natural processes are predictable. The key idea of our proof is an elementary inequality known as *Chebyshev's Other Inequality.* Our approach may be explained most easily in the context of the following theorem of Meyer:

Theorem 1 (from VII.T49 of [9]): Let A be a natural process. Then
(P1) $\Delta A_T = 0$ *a.s.* for any totally inaccessible stopping time T, and
(P2) ΔA_T is $\mathcal{F}(T-)$-measurable for any predictable stopping time T.

(See Section 2 for notation and terminology.) We take the following course. In Theorem 4 below, we show that if an adapted, càdlàg, finite-variation process A satisfies conditions (P1) and (P2) then it is predictable. Then we use "Chebyshev's Other Inequality" to prove that a natural process satisfies conditions (P1) and (P2). This proof, which may be seen as an alternative to Meyer's proof of Theorem 1 or to Doob's proof that "natural implies predictable" (Theorem 2.IV.7(a3), pp. 486–487), has two attractive features. First, it does not require A to be increasing. Second, it does not require an approximating sequence of continuous processes for A. The second point is what makes both Meyer's and Doob's proofs technical and unsuitable for the development in [12]. While such approximating sequences are needed in the proof of the Doob-Meyer decomposition, in an elementary development it is desirable to be able to set aside the details of the proofs of basic theorems as one progresses. Another feature of the approach presented here is that it leads naturally to a simple proof of the Fundamental Theorem of Local Martingales [11, 12].

2. Background

We presuppose the material of [12] up to Section 3 of Chapter 3. Here we give some standard setup, definitions, and results.

We are given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), P)$, satisfying the usual hypotheses. An adapted càdlàg process A with $A_0 = 0$ is said to be *natural* if it is of integrable variation and if for every bounded martingale M we have $E[A, M]_\infty = 0$, where $[A, M]$ denotes the quadratic covariation of A and M. By an FV process we mean a càdlàg process with finite variation on compacts. A process X is said to be *predictably measurable* (or simply *predictable*) if the function $X(t, \omega) : R^+ \times \Omega \to R$ is measurable with respect to the σ-field on $R^+ \times \Omega$ generated by the *a.s.* left-continuous adapted processes.

Recall that a càdlàg process X is called a *quasimartingale* if $E(X_0) < \infty$ and the supremum over all partitions $0 = t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_k$ of the quantity

$$E\left(\sum_{i=1}^{k} |E\left(X(t_{i+1}) - X(t_i) \mid \mathcal{F}(t_i)\right)|\right)$$

is finite. Here is Rao's theorem on the decomposition of quasimartingales:

Theorem 2 (Rao). A quasimartingale X has a unique decomposition $X = M + A$, where M is a local martingale and A is a natural process.

If X is also an FV process, the process A of Theorem 2 is called the *compensator* of X. We will use the following lemma on continuity of the compensator of certain simple processes that jump only at totally inaccessible stopping times.

Theorem 3. Let T be a totally inaccessible stopping time, and let V be a bounded $\mathcal{F}(T)$-measurable random variable. Let $U_t = VI_{(T\leq t)}$ and let $U = M + A$ be the decomposition of U given by Theorem 2. Then A is continuous.

This is a minor extension of Theorem III.11 of [12]. The proof is a bit technical, and involves energy inequalities.

3. Predictability of FV Processes

Theorem 4 below is an elementary condition for predictability of FV processes. It is a consequence of Proposition 7.7 of Métivier [6]. In the proof presented here, we will need a simple result on predictable stopping times.

Proposition 1. Let S and T be predictable stopping times, and let R be defined by

$$R = \begin{cases} S & \text{if } \; S \neq T \\ \infty & \text{if } \; S = T. \end{cases}$$

Then R is again a predictable stopping time.

Bass [1] gives an elementary proof of this, and points out the following useful consequence.

Proposition 2. Let (T_n) be a sequence of predictable stopping times. Then there exist another sequence of predictable stopping times (S_n) with disjoint graphs, such that the union of the graphs of the T_n's is the same as the union of the graphs of the S_n's.

Let us write $\Delta X(t)$ for the jump in a càdlàg process X at time t. Then we have the following theorem, which echoes the theorem of Meyer quoted in the introduction.

Theorem 4. Let A be a FV process. Then A is predictable if
(P1) $\Delta A_T = 0$ *a.s.* for any totally inaccessible stopping time T, and
(P2) ΔA_T is $\mathcal{F}(T-)$-measurable for any predictable stopping time T.

Proof: Let A be a FV process satisfying (P1) and (P2). In the proof, we carefully exploit the elementary fact that the pointwise limit (in t and ω, except for an evanescent set) of a sequence of predictable processes is predictable. We will not comment further on evanescent sets in the proof.

We write A^k for the process formed from the jumps of A of magnitude greater than $1/k$:

$$A_t^k = \sum_{0<s\leq t} \Delta A_s I_{(|\Delta A_s|>1/k)}.$$

Thus $A - A^k$ has no jumps of magnitude greater than $1/k$. Since A is FV, $A - A^k$ converges as $k \to \infty$ (for all t, ω) to an adapted continuous process, which is predictable by definition. (That A^k is adapted is elementary; see [9], IV.44.) Thus to prove that A is predictable, it is enough to prove that A^k is predictable. Let (T_n) be the sequence of jump times of A^k in increasing order. Since the jump of A at time T_n is nonzero on the the entire set $(T_n < \infty)$, (P1) implies that T_n is accessible. Let (T_{nk}) be an enveloping sequence of predictable stopping times for T_n. By Proposition 2 there is a sequence of predictable stopping times (S_n) with disjoint graphs such that the union of the graphs of the T_{nk}'s (over n and k) is the same as the union of the graphs of the S_n's. As A^k is then the sum of the processes $\Delta A^k(S_n)I_{(S_n\leq t)}$, to show that it is predictable it is enough by (P2) to show that the process $VI_{(S\leq t)}$ is predictable for S a predictable stopping time and $V \in \mathcal{F}(S-)$. Since such a process is a limit of a sequence of processes of the same form with V bounded, we can assume that V is bounded. Let (R_n) be a sequence of stopping times announcing S. Let $V_n = E(V \mid \mathcal{F}(R_n))$. Then $V_n \to V$ by martingale convergence and the fact that $\mathcal{F}(S-) = \bigvee \mathcal{F}(R_n)$. Thus $VI_{(S\leq t)}$ is the limit of the *left-continuous* adapted processes $V_n I_{(R_n<t)}$. It follows that $VI_{(S\leq t)}$ is predictable. This completes the proof. □

4. The Main Result

The main contribution of this note is a simple proof of the following theorem. The proof of Lemma 2 below is a little long, but the idea is very elementary and so perhaps a simpler proof along the same lines will be found.

Theorem 5. For A natural, the conditions (P1) and (P2) are satisfied. Thus A is predictable.

Our proof is based on Chebyshev's Other Inequality [5]. More precisely, it is the condition for equality that we need. One version of this inequality is

Lemma 1 (Chebyshev's Other Inequality). Let X be an integrable random variable, and let F be a bounded nondecreasing function. Then

$$E(XF(X)) \geq E(X)E(F(X)). \tag{1}$$

Moreover, equality holds if and only if $F(X)$ is *a.s.* constant.

Proof (following [5]): Let X_1, X_2 be i.i.d. with the same distribution as X. Because F is nondecreasing, $(X_1 - X_2)(F(X_1) - F(X_2)) \geq 0$. Take expectations on each side of the inequality and multiply out to get (1). Equality in (1) forces the product to be zero *a.s.* which forces $F(X_1) = F(X_2)$ *a.s.* since F is nondecreasing, which in turn implies that $F(X)$ is *a.s.* constant. $\square$

Inequality (1) says that X and $F(X)$ are nonnegatively correlated for F increasing. We need a version of Lemma 1 for conditional expectations. The extension of the inequality (1) to (2) below is immediate for finite σ-fields $\mathcal{G}$, and martingale convergence gives the general case (at least for $\mathcal{G}$ countably generated). The awkward part of the extension is in proving the condition for equality, and so we take a different approach to the proof.

Lemma 2 (Conditional expectation form of Lemma 1). Let X be an integrable random variable, and let F be a bounded nondecreasing function. Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Then

$$E(XF(X) \mid \mathcal{G}) \geq E(X \mid \mathcal{G})E(F(X) \mid \mathcal{G}) \quad a.s. \tag{2}$$

Moreover, equality holds if and only if $F(X)$ is $\mathcal{G}$-measurable (up to a null set).

Proof: Suppose F is right continuous. Since replacing F in (1) by $\alpha F + \beta$ with $\alpha > 0$ and $\beta \in R$ yields an equivalent inequality, we suppose w.l.o.g. that F is in fact a distribution function. Expand the probability space so that we can assume that there is a random variable Y with distribution function F that is independent of $\mathcal{F}$. Let us write $\mathcal{F}'$ to denote the expanded σ-field. Analogously let $\mathcal{G}'$ be the σ-field generated by $\mathcal{G}$ and Y (i.e., the smallest σ-field containing $\mathcal{G}$ with respect to which Y is measurable). Then we have

$$XI_{(X\geq Y)} \geq YI_{(X\geq Y)} \quad \text{and} \quad XI_{(X<Y)} \leq YI_{(X<Y)}.$$

Take conditional expectations with respect to $\mathcal{G}'$, writing W for $E(I_{(X\geq Y)} \mid \mathcal{G}')$ and noting that $E(X \mid \mathcal{G}') = E(X \mid \mathcal{G})$ *a.s.*, to get

$$E(XI_{(X\geq Y)} \mid \mathcal{G}') \geq YW \quad \text{and} \quad E(X \mid \mathcal{G}) - E(XI_{(X\geq Y)} \mid \mathcal{G}') \leq Y(1-W) \quad a.s. \tag{3}$$

Multiply the first of these inequalities by $1 - W$ and the second by W, and subtract to get

$$E(XI_{(X\geq Y)} \mid \mathcal{G}') \geq E(X \mid \mathcal{G})W \quad a.s.$$

Taking conditional expectations given $\mathcal{G}$, noting that $\mathcal{G} \subset \mathcal{G}'$, gives

$$E(XI_{(X\geq Y)} \mid \mathcal{G}) \geq E(X \mid \mathcal{G})E(I_{(X\geq Y)} \mid \mathcal{G}) \quad a.s.$$

Now

$$E(I_{(X\geq Y)} \mid \mathcal{G}) = E(E(I_{(X\geq Y)} \mid \mathcal{F}) \mid \mathcal{G}) = E(F(X) \mid \mathcal{G}) \quad a.s.,$$

and similarly

$$E(XI_{(X\geq Y)} \mid \mathcal{G}) = E(XE(I_{(X\geq Y)} \mid \mathcal{F}) \mid \mathcal{G}) = E(XF(X) \mid \mathcal{G}) \quad a.s.$$

Substituting these expressions into the previous inequality gives (2).

Suppose now that we have equality in (2). On the set $(W > 0)$ we must then have *a.s.* equality in the second inequality of (3), but this implies that

$$E((X - Y)I_{(X<Y)} \mid \mathcal{G}') = 0 \ \ a.s. \ \text{ on } \ W > 0.$$

This in turn implies that on the set $(W > 0)$ we must have $P(X < Y \mid \mathcal{G}') = 0$, or equivalently $W = 1$, *a.s.* We conclude that W is an indicator function, and since $W = E(I_{(X \geq Y)} \mid \mathcal{G}')$, we must have $W = I_{(X \geq Y)}$ *a.s.* But since W is $\mathcal{G}'$-measurable, it follows that the event $X \geq Y$ differs by only a null set from some element C of $\mathcal{G}'$. For this C we have

$$P(C \mid \mathcal{F}) = P(X \geq Y \mid \mathcal{F}) = F(X) \ \ a.s.$$

But since C is in $\mathcal{G}'$ and Y is independent of $\mathcal{F}$, the function $P(C \mid \mathcal{F})$ is $\mathcal{G}$-measurable, and it follows that $F(X)$ is $\mathcal{G}$-measurable as required.

If F is not right continuous then it may be expressed as the sum of a right-continuous function and a left-continuous function. The general result follows through a similar treatment of the left-continuous part. □

One final lemma concerning martingales and stopping times will be needed in the proof of Theorem 5. It is well known (see [10], for example), but we provide a proof for the reader's convenience.

Lemma 3. Let T be a stopping time, and let Y be a bounded $\mathcal{F}(T)$-measurable random variable. Let Z denote $Y - E(Y \mid \mathcal{F}(T-))$, and let $M_t = ZI_{(T \leq t)}$. Then M is a càdlàg bounded martingale.

Proof: That M is càdlàg and bounded is obvious. To see that it is a martingale, let $B \in \mathcal{F}_s$, and suppose that $t > s$. We must prove that $E(M_t I_B) = E(M_s I_B)$. We have

$$E(M_t I_B) = E(ZI_{(T \leq t)} I_B) = E(ZI_{(T \leq s)} I_B) + E(ZI_{(s<T \leq t)} I_B) = E(ZI_{(T \leq s)} I_B) = E(M_s I_B).$$

The next-to-last equality is because $E(Z \mid \mathcal{F}(T-)) = 0$ *a.s.*, and the event

$$(s < T \leq t) \cap B = (T > s) \cap B - (T > t) \cap B$$

is in $\mathcal{F}(T-)$. □

Proof of Theorem 5. Let A be a natural process. We must prove that A is predictable. Since a natural process is an FV process, we may use Theorem 4. We treat the two conditions in turn.

Proof of Condition (P1): Let T be a totally inaccessible stopping time. Let F be a bounded nondecreasing function on the reals, and let $V = F(\Delta A_T)$. Let U and M be as defined in Theorem 3 for this T and V, but denote the compensator of U by B here to distinguish it from the given natural process A. Thus $U = M + B$. By Theorem 3 the martingale M is continuous except for a jump of size V at T. Let M^n denote the martingale M stopped when its magnitude first reaches or exceeds n. Then, like M, M^n is a martingale and is continuous except for a jump at time T of magnitude $V_n = VI_n$, where I_n is the inducator function of the event that the magnitude of M has not reached n before time T. Naturality of A then gives $E[A, M^n] = 0$ since M^n is bounded. But by elementary calculation (see [12], Section II.3)

$$[A, M^n] = \Delta A_T \Delta M_T^n = (\Delta A_T)V_n.$$

Now we take the limit as $n \to \infty$, observing that $V_n \to V$ *a.s.*, to conclude by dominated convergence that $E[(\Delta A_T)V] = E[\Delta A_T F(\Delta A_T)] = 0$. The choice $F(x) = \text{sign}(x)$ implies that $E(|\Delta A_T|) = 0$, and it follows that $\Delta A_T = 0$ *a.s.*

Proof of Condition (P2): Let T be a predictable stopping time. Let F be a bounded, strictly increasing function, and let X denote ΔA_T. Let $Y = F(X)$, and let Z and M be as in Lemma 3 with this choice of T and Y. As M is a bounded martingale and A is natural, we have $E[A, M]_\infty = 0$. Since M has only one jump, at time T, it follows that $[A, M]_\infty = XZ$, and so

$$E\{\, X[F(X) - E(F(X) \mid \mathcal{F}(T-))\,]\} = 0. \tag{4}$$

Consider inequality (2) with $\mathcal{F}(T-)$ playing the role of $\mathcal{G}$. Equation (4) implies that the *unconditional* expectations of each side of (2) agree, and so we must have equality *a.s.* in (2). Lemma 2 now tells us that $F(X)$ is $\mathcal{F}(T-)$-measurable. Therefore $X = \Delta A_T$ is also $\mathcal{F}(T-)$-measurable, since F was chosen to be strictly increasing. This establishes (P2). □

It is pleasing that conditions (P1) and (P2) are established in a similar way: by constructing a martingale that allows us to extract information from the condition $E[A, M] = 0$. Theorem 5 is one half of the following theorem, which states the equivalence of the concepts of naturality and predictability for processes of integrable variation. It is an immediate consequence of Theorem 5 and Theorem III.23 of [12].

Theorem 6. Let A be an adapted càdlàg process of integrable variation with $A_0 = 0$. Then A is natural if and only if it is predictable.

5. The Fundamental Theorem of Local Martingales

Our proof of this basic result follows Jia-an Yan's [11], as presented in [12]. Several technicalities may be bypassed when Theorem 5 above is on hand.

Theorem 7 (The Fundamental Theorem of Local Martingales). Let M be a local martingale. Then M is the sum of a local martingale N having bounded jumps and a local martingale L of finite variation on compacts.

Proof: Suppose w.l.o.g. that $M_0 = 0$. By stopping, we can further suppose w.l.o.g. that M is a uniformly integrable martingale. Let β be a given positive constant. We set

$$C_t = \sum_{0<s\leq t} |\Delta M_s| I_{(|\Delta M_s|>\beta)}.$$

We show that C is locally integrable. Define

$$R_n = \inf\{t \ : \ C_t \geq n \text{ or } |M_t| \geq n\}.$$

Then $|\Delta M(R_n)| \leq |M(R_n)| + |M(R_n-)| \leq |M(R_n)| + n$. So $C(R_n) \leq C(R_n-) + |\Delta M(R_n)| \leq |M(R_n)| + 2n$, which is in L^1. Since $R_n \uparrow \infty$, C is locally integrable. By stopping, we can now further assume w.l.o.g. that C is integrable: $E(C_\infty) < \infty$.

Now define B by

$$B_t = \sum_{0<s\leq t} \Delta M_s I_{(|\Delta M_s|>\beta)}.$$

In fact, C_∞ is the total variation of B, and, since it is integrable, it is elementary that B is a quasimartingale. Therefore B has the decomposition

$$B = L + A,$$

where L is a local martingale and A is a natural process. Since $|B_t|$ and $|A_t|$ are both bounded by the total variation of B, namely C_∞ (Theorems III.7 and III.8 of [12]), which is integrable, it follows that the local martingales $L = B - A$ and

$$N = M - L = (M - B) - A \tag{5}$$

are in fact uniformly integrable. We have identified the required decomposition $M = N + L$. L has integrable (and therefore finite) variation.

It remains only to show that the jumps of N are bounded. Let T be any stopping time. If $\Delta A_T = 0$ *a.s.* then $|\Delta N_T| \leq \beta$. This is because of (5) and the fact that the jumps of $M - B$ are bounded by β in magnitude. Suppose on the other hand that $P(\Lambda) > 0$ where $\Lambda = (\Delta A_T > 0)$. Let $R = T_\Lambda$, which is to say that $R = T$ on Λ and $R = \infty$ elsewhere. R is a stopping time as $\Lambda \in \mathcal{F}(T)$. Decompose R into its accessible part R_1 and its totally inaccessible part R_2, with $R = R_1 \wedge R_2$. Since A is natural, it cannot jump at a totally inaccessible time by Theorem 5, whence $R_2 \equiv \infty$. Let T_k be a sequence of predictable stopping times enveloping R_1. It suffices to show that $\Delta N(T_k)$ is bounded by 2β. We show that in fact ΔN_S is bounded by 2β for any predictable stopping time S. Let S be predictable, and let (S_n) be a sequence of stopping times announcing S. Then we have

$$E\left(N_S \mid \bigvee_{n=1}^{\infty} \mathcal{F}(S_n)\right) = \lim_{n\to\infty} E(N_S \mid \mathcal{F}(S_n)) = \lim_{n\to\infty} N(S_n) = N(S-),$$

so that $E(\Delta N_S \mid \mathcal{F}(S-)) = 0$. It follows that

$$\begin{aligned}
\Delta N_S &= \Delta N_S - E(\Delta N_S \mid \mathcal{F}(S-)) \\
&= \Delta(M - B + A)_S - E(\Delta(M - B + A)_S \mid \mathcal{F}(S-)) \\
&= \Delta(M - B)_S - E(\Delta(M - B)_S \mid \mathcal{F}(S-)) + \Delta A_S - E(\Delta A_S \mid \mathcal{F}(S-)) \\
&= \Delta(M - B)_S - E(\Delta(M - B)_S \mid \mathcal{F}(S-)).
\end{aligned}$$

The final equality is due to Theorem 5 (P2) and naturality of A. Thus ΔN_S is bounded by 2β, since $|\Delta(M - B)|$ is bounded by β by construction. This implies that all the jumps of N are bounded by 2β, and completes the proof. □

6. Remarks on Accessible Stopping Times

In [12], p. 122, the reader is referred to Doob [4], pp. 483–487, for a proof of what is here Theorem 5. Unfortunately, there is an error in the last step of that proof. The error is in the claim that, for T an *accessible* stopping time, the process $t \to I_{[T,\infty)}(t)$ is predictable. To draw this conclusion, T must itself be a predictable stopping time. The error appears on pp. 430–1 and 487 of [4].

Here is a classic example. Let T be a random variable taking values 1 and 2 with equal probability. Let $X(t) = I_{[T,\infty)}(t)$, $t \geq 0$, and equip the process X with its natural filtration. Then T is the jump time of X, and is thus a stopping time. It is accessible, since T is either 1 or 2, and constants are predictable stopping times. To see that T is not predictable, let S be a stopping time with $S < T$ *a.s.* Then $P(S < 1) \geq 1/2$, and so for some $a < 1$ we have $P(S < a) > 0$. But the event $(S < a)$ is in $\mathcal{F}_a$, which is the trivial σ-field for $a < 1$. Therefore $P(S < a) = 1$. It follows that there cannot exist and announcing sequence S_n for T, and so we conclude that T is not predictable. That the process X is not predictable may be see by showing that it is not natural, by Theorem 6. To this end, let M be the martingale $M(t) = (2T - 3)I_{[1,\infty)}(t)$. Then $[X, M] = -1/2 \neq 0$, and so X is not natural.

We remark that accessible times that are not predictable rarely arise naturally and are of little interest in themselves. In fact, they are *times of discontinuity* of the filtration ([9], VII.T45). They present a technical problem that must be surmounted in proving results such as Theorem 5.

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