EMPIRICAL BAYES ESTIMATION OF RELIABILITY CHARACTERISTICS FOR AN EXPONENTIAL FAMILY*

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Abstract

Empirical Bayes (EB) estimators for mean life time and reliability function are constructed when failure time has p.d.f.

$$f(t/\theta) = \frac{\alpha}{\Gamma(\nu)} \frac{t^{\alpha\nu-1}}{\theta^{\nu}} exp\left(-\frac{t^{\alpha}}{\theta}\right), \; t > 0, \; \theta \in \Theta,$$

which includes gamma and Weibul density functions as particular cases with $\alpha = 1$ and $\nu = 1$, respectively. It is assumed that $\Theta = (a_1, a_2)$, $a_1 > 0$, and a prior distribution of θ is completely unknown and unspecified. It is shown that under squared loss prior risks of EB estimators of mean lifetime and reliability are $O\left(N^{-1}(\ln N)^{1+\alpha}\right)$ which is better than polynomial rates of convergence obtained before.

KEY WORDS: mean lifetime, reliability function, nonparametric empirical Bayes estimation

1 Introduction

Estimation of two reliability characteristics, the mean life time and the survival function, of some equipment given failure times $T_1, T_2, ..., T_N$ is one of the main problems of reliability theory. Here $T_1, T_2, ..., T_N$ are i.i.d. random variables and have conditional probability

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density function (p.d.f.) $f(t/\theta)$ given $\theta > 0$. In most of practical applications $f(t/\theta)$ is gamma or Weibull distribution density. However, sometimes none of these distributions fit well enough. As such we consider $f(t/\theta)$ of the form

$$f(t/\theta) = \frac{\alpha}{\Gamma(\nu)} \frac{t^{\alpha\nu-1}}{\theta^{\nu}} exp\left(-\frac{t^{\alpha}}{\theta}\right) I(t > 0), \quad \theta \in \Theta, \tag{1.1}$$

where $I(A)$ is the indicator function of the set A, and the parameter space Θ is a subset of $(0, \infty)$, the positive real line. It is easy to see that gamma-distribution and Weibull distribution are particular cases of (1.1) when $\alpha = 1$ and $\nu = 1$, respectively. In this situation the mean life time $m(\theta)$ and the survival function $U(\theta, t)$ have the forms

$$m(\theta) = \int_0^{\infty} t f(t/\theta) dt = \theta^{\frac{1}{\alpha}} \ (\Gamma(\nu))^{-1} \Gamma(\nu + \alpha^{-1}) \tag{1.2}$$

$$U(\theta, t) = \int_t^{\infty} f(x/\theta) dx = (\Gamma(\nu))^{-1} \Gamma(\nu, \theta^{-1} t^{\alpha}), \tag{1.3}$$

where $\Gamma(a, x)$ is incomplete gammma-function (see 8.350, Gradshtein & Ryzhik(1980)).

A number of authors have dealt with estimation of (1.2) and (1.3) in non-Bayesian as well as in Bayesian contexts (see, for instance, Ananda(1992), Belyaev(1992), Calabria(1992), Tiwari & Zalkikar(1993)). However, both non-Bayesian and Bayesian have examined the problem from extreme point of view in the sense that for the former, θ is a fixed unknown constant, whereas for the latter, θ is a random variable with **known** distribution.

The purpose of the present paper is estimation of the mean life time $m(\theta)$ and the survival function $U(t, \theta)$ from empirical Bayes point of view where θ is a random variable with unknown distribution. Thus we have observations $T_1, T_2, ..., T_N$ on failure times of N units where $T_i, i = 1, ..., N$, are distributed according to density function $f(t/\theta_i)$ of the form (1.1). Here θ_i are unobservable, i.i.d. random variables having an unknown p.d.f. $g(\theta)$ with support in Θ.

Suppose the observation $T_{N+1} \equiv y$ on failure time of $(N+1)$-th unit is made. Let $\theta_{N+1} \equiv z$ be the value of random parameter θ generating the value y and the reliability characteristics (1.2) and (1.3) of $(N+1)$-th unit. Thus the goal is to estimate mean life time $m(z)$ and reliability function $U(z, t)$ of $(N+1)$-th unit. Here we shall be concerned with a slightly more general problem, namely we construct empirical Bayes (EB) estimators of z^b for $b > 0$ and of $U(z, t)$. Since variance $v(\theta)$ of lifetime has the form

$$v(\theta) = \int_0^{\infty} t^2 f(t/\theta) dt - m^2(\theta) = \theta^{\frac{2}{\alpha}} \left[\frac{\Gamma\left(\nu + \frac{2}{\alpha}\right)\Gamma(\nu) - \Gamma^2(\nu + \alpha^{-1})}{\Gamma^2(\nu)} \right] \tag{1.4}$$

it will enable us to get EB estimators of both the mean and variance of the lifetime distribution.

The problems similar to those covered in this paper were considered for special distributions like exponential, gamma and Weibull distributions by Chiou(1993), Lahiri & Park(1991), Li(1984), Nakao & Liu(1990) among others. However no attempt has been made to construct empirical Bayes estimators of reliability function for the general family of conditional p.d.f.'s (1.1), may be because in this case $U(\theta, t)$ has a complicated form (1.3). No assumptions on parametric form of $g(\theta)$ are made in this paper. Estimators constructed here have $O\left(N^{-1}(lnN)^{1+\alpha}\right)$ rate of convergence to asymptotic optimality which is better than polynomial rates obtained earlier for exponential and gamma conditional distributions, e.g. see Singh(1976, 1979), Singh & Wei(1992), Walter & Hamedani (1991). Research conducted here is in part based on technique proposed in Penskaya(1992, 1993) but is significantly noval from all the works mentioned above.

Let us define

$$p(y) = \int_0^\infty f(y/\theta)g(\theta)d\theta, \tag{1.5}$$

$$\Phi(y,b) = \int_0^\infty \theta^b f(y/\theta)g(\theta)d\theta, \tag{1.6}$$

$$\Psi(y,t) = \int_0^\infty U(\theta,t) f(y/\theta)g(\theta)d\theta, \tag{1.7}$$

and denote the mathematical expectations with respect to densities $g(\theta)$, $f(t/\theta)$, $p(x)$ and $\prod_{i=1}^N p(x_i)$, respectively, by $\mathbf{E}_g$, $\mathbf{E}_f$, $\mathbf{E}_p$ and $\mathbf{E}_{p^N}$.

If we knew the prior density $g(\theta)$, then Bayes estimators $\beta(y,b)$ of θ^b and $\Upsilon(y,t)$ of survival function that minimize squared error losses (see Zacks(1971)) can be calculated as

$$\beta(y,b) = \frac{\Phi(y,b)}{p(y)}, \tag{1.8}$$

$$\Upsilon(y,t) = \frac{\Psi(y,t)}{p(y)}. \tag{1.9}$$

However, as prior density is unknown, these minimum expected loss estimators are not available to us for use. So we construct empirical Bayes estimators $\beta_N(y,b) = \beta_N(y,b;\ T_1,T_2,...,T_N)$ and $\Upsilon_N(y,t) = \Upsilon_N(y,t;\ T_1,T_2,...,T_N)$ as the estimators of (1.8) and (1.9), respectively, from observations $T_1,T_2,...,T_N$. With these notations our EB estimators $m_N(y)$ and $v_n(y)$ of mean lifetime (1.2) and variance of lifetime (1.4), respectively, are given by

$$m_N(y) = (\Gamma(\nu))^{-1}\,\Gamma(\nu+\alpha^{-1})\beta_N(y,1/\alpha); \tag{1.10}$$

$$v_N(y) = \left[\frac{\Gamma\left(\nu+\frac{2}{\alpha}\right)\Gamma(\nu) - \Gamma^2\left(\nu+\alpha^{-1}\right)}{\Gamma^2(\nu)}\right]\beta_N(y,2/\alpha). \tag{1.11}$$

An EB estimator $\varrho_N(y)$ of a parametric function $r(\theta)$ may be characterized by the posterior risk

$$R(y;\varrho_N) = (p(y))^{-1}\mathbf{E}_{p^N}\int_0^\infty (\varrho_N(y) - r(\theta))^2 f(y/\theta)g(\theta)d\theta,$$

or by prior risk

$$\mathbf{E}_p R(y;\varrho_N) = \mathbf{E}_{p^N}\int_0^\infty\int_0^\infty (\varrho_N(y) - r(\theta))^2 f(y/\theta)g(\theta)d\theta dy.$$

It is easy to notice that both risks can be broken into two components. The first components $R(y;\varrho) = (p(y))^{-1}\int_0^\infty(\varrho(y) - r(\theta))^2 f(y/\theta)g(\theta)d\theta$ and $R(\varrho) = \mathbf{E}_p R(y;\varrho)$ are, respectively, the posterior and the prior risks of the Bayes estimator $\varrho(y)$ and they are independent of $\varrho_N(y)$. So that we shall characterize EB estimators by second components

$$\Delta(y;\varrho_N) = \mathbf{E}_{p^N}(\varrho_N(y) - \varrho(y))^2 \tag{1.12}$$

$$\Delta(\varrho_N) = \mathbf{E}_p\Delta_N(y;\varrho_N). \tag{1.13}$$

that are, respectively, posterior and prior risks of EB estimator $\varrho_N(y)$.

In general, for $R(y;\varrho_N)$ to be finite or for $\Delta(y;\varrho_N)$ to converge to zero for each y, some constraints on θ are necessary. We assume that the parameter space is an interval

$$\Theta = [a_1, a_2];\quad 0 < a_1 < a_2 < \infty, \tag{1.14}$$

with known a_1 and a_2.

We say that EB estimator ϱ_N is *asymptotically optimal* with rates $O(\tau_N)$ of convergence to optimality if $\Delta(\varrho_N) = O(\tau_N)$ as $N \to \infty$. We also say that EB estimator ϱ_N is *pointwise asymptotically optimal* with rates $O(\tau_N)$ of convergence to optimality if there exists a positive function $C(y)$ such that $\Delta(y;\varrho_N) \le C(y)\tau_N$ for any value of y and N.

In what follows we demonstrate that both EB estimators of θ^b and of $U(\theta,t)$ are pointwise optimal with rates of convergence $O(N^{-1}\ln N)$. It seems that this rate of convergence can not be substantially improved. Actually, in the case of EB estimation of θ^b the lower bounds for $\Delta(y;\beta_N)$ over the class of all possible estimators of $\beta(y,b)$ from observations $T_1, T_2, ..., T_N$ was shown to be $O(N^{-1}\ln N(\ln\ln N)^{-1})$ (see Penskaya(1995)). However the difference between upper and lower bound is probably due to the fact that the lower bound is not exact. We also establish that prior risks have the following rates of convergence

$$\Delta(\beta_N) = O(N^{-1}(\ln N)^{1+\alpha}),\quad \Delta(\Upsilon_N) = O(N^{-1}(\ln N)^{1+\alpha})$$

which is better than polynomial convergence rates obtained before (see Singh (1976, 1979, 1992)).

2 Construction of empirical Bayes estimators

Let us break EB estimation of (1.8) and (1.9) into three steps. The first step is estimation of the numerators $\Phi(y,b)$ and $\Psi(y,t)$ that are linear functionals of the prior density $g(\theta)$. The second stage is estimation of denominator $p(y)$ which is the marginal density of T_1 and the third is estimation of the ratios (1.8) and (1.9).

Following to the approach of Penskaya(1993), to estimate $\Phi(y,b)$ and $\Psi(y,t)$ we first wish to find functions $\varphi_\varepsilon(y,x)$ and $\psi_\varepsilon(x,y,t)$ such that for every ε and y

$$\mathbf{E}_p\left[\varphi_\varepsilon(y,x)\right]^{2j} < \infty, \quad \mathbf{E}_p\left[\psi_\varepsilon(x,y,t)\right]^{2j} < \infty, \; j = 1,2; \tag{2.1}$$

$$\lim_{\varepsilon\to 0}\int_0^\infty \varphi_\varepsilon(y,x) f(x/\theta)dx = \theta^b f(y/\theta); \tag{2.2}$$

$$\lim_{\varepsilon\to 0}\int_0^\infty \psi_\varepsilon(x,y,t) f(x/\theta)dx = U(\theta,t) f(y/\theta). \tag{2.3}$$

Having found such functions, our proposed estimators of $\Phi(y,b)$ and $\Psi(y,t)$ are then given by

$$\Phi_N(y,b) = N^{-1}\sum_{j=1}^N \varphi_\varepsilon(y,T_j), \quad \varepsilon = \varepsilon(N); \tag{2.4}$$

$$\Psi_N(y,t) = N^{-1}\sum_{j=1}^N \psi_\varepsilon(y,T_j,t) \quad \varepsilon = \varepsilon(N). \tag{2.5}$$

Furthermore if we can find functions $\varphi(x,y)$ and $\psi(x,y,t)$ independent of ε such that they satisfy (2.1), then formulas (2.4) and (2.5) will give us unbiased estimators of $\Phi(y,b)$ and $\Psi(y,t)$ with $O(N^{-1})$ variance. So let us first search for functions $\varphi(x,y)$ and $\psi(x,y,t)$. In this case, equations (2.2) and (2.3) can be written as

$$\int_0^\infty exp\left(-\theta^{-1}x^\alpha\right) x^{\alpha\nu-1}\varphi(x,y)dx = \theta^b exp\left(-\theta^{-1}y^\alpha\right) y^{\alpha\nu-1}; \tag{2.6}$$

$$\int_0^\infty exp\left(-\theta^{-1}x^\alpha\right) x^{\alpha\nu-1}\psi(x,y,t)dx = [\Gamma(\nu)]^{-1}\Gamma\left(\nu, -\frac{t^\alpha}{\theta}\right) exp\left(-\theta^{-1}y^\alpha\right) y^{\alpha\nu-1}. \tag{2.7}$$

Let us solve equations (2.6) and (2.7) in turn.

To find $\varphi(x,y)$ we change variable $\xi = x^\alpha$ in the integral in formula (2.6) and denote

$$\tilde{\varphi}(y,x) = \alpha^{-1}x^{\nu-1}y^{\frac{1}{\alpha}-\nu}\varphi\left(x^{\frac{1}{\alpha}}, y^{\frac{1}{\alpha}}\right).$$

Then equation (2.6) takes the form

$$\int_0^\infty exp\left(-\theta^{-1}(x-y)\right)\tilde{\varphi}(y,x)dx = \theta^b \tag{2.8}$$

and thus we only need to find a solution $\tilde{\varphi}(y,x)$ of the last equation such that $\mathbf{E}_p(\tilde{\varphi}(y,x))^{2j} < \infty,\ j=1,2$. It is easy to see that a solution of (2.8) has the form

$$\tilde{\varphi}(y,x) = [\Gamma(b)]^{-1}(x-y)^{b-1}I(x \le y)$$

provided $b > 3/4$. If $b \le 3/4$ let us use the formula (see Erdélyi et al., 4.14.30)

$$\int_0^\infty exp(-yx)x^{\varrho/2}J_\varrho(2\sqrt{ax})dx = a^{\varrho/2}y^{-\varrho-1}exp(-ay^{-1}),\ \ \varrho > -1. \qquad (2.9)$$

Here $J_\varrho(x)$ is Bessel function of the first kind (see Gradshtein et.al., 8.402). Putting in (2.9) $a = \varepsilon^{-1},\ y = \theta^{-1},\ \varrho = b-1$, and using simple equality

$$\int_0^\infty exp(-x\theta^{-1})x^{b-1}dx = \Gamma(b)\theta^b \qquad (2.10)$$

we obtain

$$\int_0^\infty \left[\frac{x^{b-1}}{\Gamma(b)} - (x\varepsilon)^{\frac{b-1}{2}} J_{b-1}\left(2\sqrt{\frac{x}{\varepsilon}}\right)\right] exp\left(-x/\theta\right)dx = \theta^b\left[1 - exp(-\theta/\varepsilon)\right]. \qquad (2.11)$$

Since $\tilde{\varphi}_\varepsilon(y,x)$ should "approximately satisfy" equation (2.8) and $exp(-\theta\varepsilon^{-1})$ turns to zero as $\varepsilon \to 0$, the previous equality gives us

$$\tilde{\varphi}_\varepsilon(y,x) = \left[\frac{(x-y)^{b-1}}{\Gamma(b-1)} - ((x-y)\varepsilon)^{\frac{b-1}{2}} J_{b-1}\left(2\sqrt{\frac{x-y}{\varepsilon}}\right)\right] I(x \ge y).$$

Returning to $\varphi(x,y)$ and $\varphi_\varepsilon(y,x)$, we get finally

$$\varphi_\varepsilon(y,x) \equiv \varphi(x,y) = \alpha y^{\alpha\nu-1}x^{\alpha-\alpha\nu}\,\frac{(x^\alpha - y^\alpha)^{b-1}}{\Gamma(b)}\,I(x \ge y) \qquad (2.12)$$

for the case $b > 3/4$, and

$$\varphi_\varepsilon(y,x) = \alpha\frac{y^{\alpha\nu-1}}{x^{\alpha\nu-\alpha}}\left[\frac{(x^\alpha - y^\alpha)^{b-1}}{\Gamma(b)} - (\varepsilon(x^\alpha - y^\alpha))^{\frac{b-1}{2}}J_{b-1}\left(2\sqrt{\frac{x^\alpha - y^\alpha}{\varepsilon}}\right)\right] I(x \ge y). \qquad (2.13)$$

for the case $0 \le b \le 3/4$. Let us choose $\varepsilon = 2a_1(\ln N)^{-1}$.

For solution of equation (2.7) recall that $\Gamma(\nu, \theta^{-1}t^\alpha) = \alpha\theta^{-\nu}\int_{t^\alpha}^\infty exp(-u/\theta)u^{\nu-1}du$. So, if we find a function $\psi(x,y,u,t)$ satisfying the equation

$$\int_0^\infty exp\left(-\theta^{-1}x^\alpha\right)x^{\alpha\nu-1}\psi(x,y,u,t)dx = \alpha\frac{\theta^{-\nu}y^{\alpha\nu-1}u^{\nu-1}}{\Gamma(\nu)}exp\left(-\theta^{-1}(y^\alpha + u)\right) \qquad (2.14)$$

then we may choose

$$\psi(x,y,t) = \int_{t^\alpha}^{\infty} \psi(x,y,u,t)du \tag{2.15}$$

and after that check whether function (2.15) satifies (2.1). By transformation $\xi = x^\alpha$ we express (2.14) in a form

$$\int_0^\infty exp\left(-\theta^{-1}(x - y^\alpha - u)\right) x^{\nu-1} y^{1-\alpha\nu} u^{1-\nu} \psi(x^{\frac{1}{\alpha}}, y, u, t) dx = \alpha(\Gamma(\nu))^{-1}\theta^{-\nu}.$$

Now, denoting

$$\tilde{\psi}(x,y,u,t) = x^{\nu-1} y^{1-\alpha\nu} u^{1-\nu} \psi(x^{\frac{1}{\alpha}}, y, u, t), \tag{2.16}$$

we arrive at simple equation that gives us

$$\tilde{\psi}(x,y,u,t) = \alpha[\Gamma(\nu)]^{-2}(x - y^\alpha - u)^{\nu-1} I(x - y^\alpha - u \geq 0) \tag{2.17}$$

Taking into account (2.15) and (2.16) we finally obtain

$$\psi(x,y,t) = \frac{\alpha}{\Gamma^2(\nu)} x^{\alpha-\alpha\nu} y^{\alpha\nu-1} (x^\alpha - y^\alpha)^{2\nu-1} B(\frac{t^\alpha}{x^\alpha - y^\alpha}; \nu, \nu)\, I(t^\alpha \leq x^\alpha - y^\alpha). \tag{2.18}$$

Here

$$B(A;\nu,\mu) = \int_A^1 u^{\nu-1}(1-u)^{\mu-1} du \tag{2.19}$$

is incomplete beta-function (see Gradshtein & Ryzhik(1980)).

The second problem is estimation of $p(y)$. Since $p(y)$ is a usual marginal density we construct kernel estimator of the form (see Singh(1979), Nadaraya(1989))

$$p_N(y) = \alpha \frac{y^{\alpha\nu-1}}{Nh} \sum_{j=1}^{N} K\left(\frac{y^\alpha - T_j^\alpha}{h}\right) (T_j)^{\alpha-\alpha\nu}. \tag{2.20}$$

Here we select kernels $K(x)$ and h as follows (see (1.14))

$$K(x) = (-x)^{-1/2} J_1\left(2\sqrt{(-x)}\right) I(x < 0), \quad h = 2a_1 (\ln N)^{-1}. \tag{2.21}$$

Now it remaines to estimate fractions $\Phi(y,b)/p(y)$ and $\Psi(y,t)/p(y)$ (see formulae (1.5) - (1.7)). Since $\theta \in [a_1; a_2]$ we denote

$$H(z) = \begin{cases} 0, & \text{if } z < 0, \\ z, & \text{if } 0 \leq z \leq a_2 \\ a_2, & \text{if } z > a_2 \end{cases} \tag{2.22}$$

Our proposed empirical Bayes estimators in the absence of minimum expected loss Bayes estimators $\beta(y,b)$ and $\Upsilon(y,t)$ are, respectively,

$$\beta_N(y,b) = H\left(\Phi_N(y,b)[p_N(y)]^{-1}\right), \tag{2.23}$$

and

$$\Upsilon_N(y,t) = H\left(\Psi_N(y,t)[p_N(y)]^{-1}\right). \tag{2.24}$$

Here $\Phi_N(y,b)$, $\Psi_N(y,t)$ and $p_N(y)$ are constructed according to formulas (2.4), (2.5) and (2.20), respectively, with $\varphi_\varepsilon(y,x)$, $\psi(x,y,t)$ and $K(x)$ given by (2.12), (2.13), (2.18) and (2.21). We should remind the reader that EB estimators of mean lifetime and variance of lifetime have, respectively, the forms (1.10) and (1.11).

3 Convergence rates of EB estimators

In this section we show that our proposed EB estimators $\beta_N(y,b)$ and $\Upsilon_N(y,t)$ are both asymptotically optimal and pointwise asymptotically optimal and investigate the rates of convergence to optimality. For that purpose we examine posterior and prior risks (1.12), (1.13). In particular we show that there exist positive functions $C_1(y)$ and $C_2(y)$ such that

$$\Delta(y;\beta_N) \le C_1(y)N^{-1}\ln N; \quad \Delta(y;\Upsilon_N) \le C_2(y)N^{-1}\ln N. \tag{3.1}$$

We also demonstrate that for prior risks the following relations are valid

$$\Delta(\beta_N) = O\left(N^{-1}(\ln N)^{1+\alpha}\right); \quad \Delta(\Upsilon_N) = O\left(N^{-1}(\ln N)^{1+\alpha}\right) \tag{3.2}$$

as $N \to \infty$. It means that for $N \to \infty$ posterior risks of EB estimators of mean lifetime and of survival function are $O\left(N^{-1}\ln N\right)$ and prior risks are $O\left(N^{-1}(\ln N)^{1+\alpha}\right)$.

Let us evaluate posterior risks (1.12) of estimators (2.23) and (2.24). To do that we denote

$$d^2_{N1}(y,b) = \mathbf{E}_{p^N}(\Phi_N(y,b) - \Phi(y,b))^2,$$
$$d^2_{N2}(y,t) = \mathbf{E}_{p^N}(\Psi_N(y,t) - \Psi(y,t))^2,$$
$$d^2_{N3}(y) = \mathbf{E}_{p^N}(p_N(y) - p(y))^2.$$

So now our objective is to find upper bounds for $d^2_{N1}(y,b)$, $d^2_{N2}(y,t)$ and $d^2_{N3}(y)$ and establish a relationship between $\Delta(y;\beta_N)$, $\Delta(\beta_N)$, $\Delta(y;\Upsilon_N)$, $\Delta(\Upsilon_N)$ and $d^2_{N1}(y,b)$, $d^2_{N2}(y,t)$ $d^2_{N3}(y)$.

Let us first calculate $d^2_{N1}(y,b)$. We have two different cases here. If $b > 3/4$ then we produce estimator (2.4) with $\varphi_\varepsilon(y,x) \equiv \varphi(x,y)$ given by formula (2.12). In this situation $\Phi_N(y,b)$ is unbiased estimator of $\Phi(y,b)$, and therefore $d^2_{N1}(y,b) = N^{-1}\sigma^2(y,b)$, where

$$\sigma^2(y,b) = \int \varphi^2(y,x)p(x)dx =$$

$$= \frac{\alpha y^{\alpha\nu-1}}{\Gamma^2(b)} \int_{a_1}^{a_2} f(y/\theta)g(\theta)d\theta \int_y^\infty x^{\alpha(1-\nu)}(x^\alpha - y^\alpha)^{2b-2} exp(-x^\alpha/\theta)\alpha x^{\alpha-1}dx.$$

Introducing the new variable $z = x^\alpha - y^\alpha$ and using inequality (since $\alpha\nu > 1$)

$$(z+y^\alpha)^{1-\nu} \le (z+y^\alpha)^{1/\alpha-\nu}(z+y^\alpha)^{1-1/\alpha} \le y^{1-\alpha\nu}(z+y^\alpha)^{1-1/\alpha} \tag{3.3}$$

we obtain

$$\sigma^2(y,b) \le \int_{a_1}^{a_2} f(y/\theta)g(\theta)\left[\frac{\alpha}{\Gamma^2(b)}\int_0^\infty (z+y^\alpha)^{1-1/\alpha} z^{2b-2} exp(-z/\theta)dz\right] d\theta.$$

The expression in square brackets is bounded by $C(y^{\alpha-1} + \chi_\alpha)$ where χ_α is the indicator $\chi_\alpha = I(\alpha \ge 1)$ and the constant C is independent of y and N. Therefore we get

$$d_{N1}^2(y,b) \le CN^{-1}p(y)(y^{\alpha-1} + \chi_\alpha), \quad b > 3/4. \tag{3.4}$$

Note that here and in what follows we use notation C for different constants independent of y and N.

Now let us consider the case of $0 < b \le 3/4$. In this situation estimator $\Phi_N(y,b)$ has the form (2.13). Taking into account formulas (2.6), (2.10), (2.11) and (2.12) we present bias of the estimator as follows

$$b_\varepsilon(y) = \int_{a_1}^{a_2}\left[\int_0^\infty \alpha\frac{y^{\alpha\nu-1}}{x^{\alpha\nu-\alpha}}(\varepsilon(x^\alpha - y^\alpha))^{\frac{b-1}{2}} J_{b-1}\left(2\sqrt{\frac{x^\alpha - y^\alpha}{\varepsilon}}\right) f(x/\theta)dx\right] g(\theta)d\theta.$$

By changing variables $z = x^\alpha - y^\alpha$ and applying the equality (2.11) we get

$$b_\varepsilon(y) = \int_{a_1}^{a_2} \theta^b f(y/\theta) exp(-\theta/\varepsilon)g(\theta)d\theta \le a_2^b \, exp(-a_1/\varepsilon)\, p(y). \tag{3.5}$$

Variance of estimator (2.13) is $N^{-1}\sigma_\varepsilon^2(y,b)$ where

$$\sigma_\varepsilon^2(y,b) = \int_{a_1}^{a_2}\int_y^\infty \frac{\alpha^2 y^{2\alpha\nu-2}}{x^{2\alpha\nu-2\alpha}}\, Q_b^2(x^\alpha - y^\alpha, \varepsilon) f(x/\theta)g(\theta)dxd\theta \tag{3.6}$$

and function $Q_b(z,\varepsilon)$ has the form $Q_b(z,\varepsilon) = z^b\varepsilon^{-1}q_b(z\varepsilon^{-1})$ with

$$q_b(x) = (x\Gamma(b))^{-1} - x^{-\frac{b+1}{2}} J_{b-1}\left(2\sqrt{x}\right). \tag{3.7}$$

Using series representation of Bessel function of the first kind (use formula 8.402 of Gradshtein & Ryzhik, 1980, with $\varrho = b-1, z = 2\sqrt{x}$) we obtain the expansion for $q_b(x)$

$$q_b(x) = x^b \sum_{k=0}^\infty \frac{(-1)^k x^k}{(k+1)!\,\Gamma(b+k+1)} \tag{3.8}$$

Now changing variables in (3.6) and using inequality (3.3) we obtain

$$\sigma_\varepsilon^2(y,b) = \varepsilon^{2b-1} \int_{a_1}^{a_2} f(y/\theta)g(\theta) \left[\int_0^\infty (z\varepsilon + y^\alpha)^{1-\frac{1}{\alpha}} e^{-\frac{z\varepsilon}{\theta}} z^{2b} q_b^2(z)\, dz \right] d\theta.$$

Let us break the integral in square brackets into two parts so that

$$\sigma_\varepsilon^2(y,b) = \varepsilon^{2b-1} \int_{a_1}^{a_2} f(y/\theta)g(\theta) \left[I_1(\varepsilon,\theta) + I_2(\varepsilon,\theta) \right] d\theta \tag{3.9}$$

with

$$I_1(\varepsilon,\theta) = \int_0^1 (z\varepsilon + y^\alpha)^{1-\frac{1}{\alpha}} e^{-\frac{z\varepsilon}{\theta}} z^{2b} q_b^2(z) dz, \quad I_2(\varepsilon,\theta) = \int_1^\infty (z\varepsilon + y^\alpha)^{1-\frac{1}{\alpha}} e^{-\frac{z\varepsilon}{\theta}} z^{2b} q_b^2(z) dz.$$

From the absolute convergence of series (3.8) it follows that $q_b(x)$ is bounded therefore

$$I_1(\varepsilon,\theta) \le C \int_0^1 (z\varepsilon + y^\alpha)^{1-\frac{1}{\alpha}} dz \le C(y^{\alpha-1} + \chi_\alpha). \tag{3.10}$$

For the sake of construction of upper bounds for $I_2(\varepsilon,\theta)$ we use formula 8.471.1 of Gradshtein and Ryzhik(1980): $zJ_{b-1}(z) + zJ_{b+1}(z) = 2bJ_b(z)$, which yields

$$q_b(x) = (x\Gamma(b))^{-1} - 2bx^{-\frac{b+2}{2}} J_b(2\sqrt{x}) + x^{-\frac{b+1}{2}} J_{b+1}(2\sqrt{x}). \tag{3.11}$$

Now it is easy to see that $I_2(\varepsilon,\theta)$ is majorized by the sum of integrals

$$I_2(\varepsilon,\theta) \le C \left(I_{2,1}(\varepsilon,\theta) + I_{2,2}(\varepsilon,\theta) \right). \tag{3.12}$$

where the first integral is

$$I_{2,1}(\varepsilon,\theta) = [\Gamma(b)]^{-2} \int_1^\infty (z\varepsilon + y^\alpha)^{1-\frac{1}{\alpha}} e^{-\frac{z\varepsilon}{\theta}} z^{2b-2} dz \le C\varepsilon^{1-2b} \tag{3.13}$$

and the second integral has the form

$$I_{2,2}(\varepsilon,\theta) = \int_1^\infty (z\varepsilon + y^\alpha)^{1-\frac{1}{\alpha}} e^{-\frac{z\varepsilon}{\theta}} z^{2b} \left\{ z^{-(b+2)} J_b^2(2\sqrt{z}) + z^{-(b+1)} J_{b+1}^2(2\sqrt{z}) \right\} dz.$$

By changing variables and using the fact that $\theta \le a_1$, last expression may be majorized by

$$I_{2,2}(\varepsilon,\theta) = C \int_1^\infty (z^2\varepsilon + y^\alpha)^{1-\frac{1}{\alpha}} e^{-\frac{z^2\varepsilon}{a_1}} \left\{ z^{2b-3} J_b^2(2z) + z^{2b-1} J_{b+1}^2(2z) \right\} dz.$$

Applying formula 6.574.2 of Gradshtein and Ryzhik(1980)

$$\int_0^\infty J_\nu^2(at) t^{-\lambda} dt < \infty, \quad 2\nu + 1 > \lambda > 0,$$

and inequality $(z^2\varepsilon + y^\alpha)^{1-\frac{1}{\alpha}} e^{-\frac{z^2\varepsilon}{a_1}} z^\varrho \le C\varepsilon^{-\varrho/2}$, $z, \varrho > 0$, we obtain

$$I_{2,2}(\varepsilon, \theta) \le C(\chi_\alpha + y^{\alpha-1})(\varepsilon^{-b/3} + 1). \tag{3.14}$$

Combining (3.9), (3.10) and (3.12) – (3.14), we finally arrive at the following upper bound for $\sigma_\varepsilon^2(y, b)$

$$\sigma_\varepsilon^2(y, b) \le Cp(y)\, \varepsilon^{\frac{3b}{2}-1}(\chi_\alpha + y^{\alpha-1}). \tag{3.15}$$

Choosing $\varepsilon = 2a_1 (\ln N)^{-1}$ and taking into account finiteness of $p(y)$ we obtain from (3.5) and (3.15) that for $0 < b \le 3/4$

$$d_{N1}^2(y, b) \le CN^{-1}(\ln N)^{1-\frac{3b}{2}} p(y)\, (\chi_\alpha + y^{\alpha-1}). \tag{3.16}$$

Let us evaluate the error $d_{N2}^2(y, t)$. Since the estimator (2.5) is unbiased estimator of $\Psi(y, t)$, then $d_{N2}^2(y, t) = N^{-1}\sigma^2(y, t)$ where

$$\sigma^2(y, t) = \int_{a_1}^{a_2} \int_0^\infty \psi^2(x, y, t) f(x/\theta) g(\theta) dx d\theta. \tag{3.17}$$

To produce upper bounds for $\sigma^2(y, t)$ we use the apparent fact that (see (2.19)) $B(A, \nu, \nu) \le \nu^{-1}(1 - A)^\nu$ for any $A \in (0, 1)$, which implies

$$\psi(x, y, t) \le \frac{\alpha}{\nu\Gamma^2(\nu)} x^{\alpha-\alpha\nu} y^{\alpha\nu-1}(x^\alpha - y^\alpha)^{\nu-1}(x^\alpha - y^\alpha - t^\alpha)^\nu. \tag{3.18}$$

The substitution of (3.18) into (3.17) and change of variables give

$$\sigma^2(y, t) \le \frac{\alpha}{\Gamma^4(\nu)} \frac{y^{\alpha\nu-1}}{\nu^2} \int_{a_1}^{a_2} exp\left(-\theta^{-1}t^\alpha\right) f(y/\theta) Q_1(t, y) g(\theta) d\theta \tag{3.19}$$

where

$$Q_1(t, y) = \int_0^\infty (z + y^\alpha + t^\alpha)^{1/\alpha - \nu + 1 - 1/\alpha}(z + t^\alpha)^{2\nu-2} z^{2\nu} exp\left(-z/\theta\right) dz.$$

By use of inequality (3.3) we derive

$$Q_1(t, y) \le y^{1-\alpha\nu} \int_0^\infty (z + y^\alpha + t^\alpha)^{1-1/\alpha}(z + t^\alpha)^{2\nu-2} z^{2\nu} exp\left(-z/\theta\right) dz.$$

Denote

$$V(t) \equiv t^v,\ v = min\{0, \alpha(\nu - 1)\chi_\alpha\}. \tag{3.20}$$

Substituting $Q_1(t, y)$ into (3.19) and using the fact that for every positive ϱ the supremum $\sup_x [x^\varrho exp(-x^\varrho/\theta)]$ is finite, we arrive at

$$d_{N2}^2(y, t) \le CN^{-1} p(y)(\chi_\alpha + y^{\alpha-1}) V(t). \tag{3.21}$$

Now it remaines to get upper bounds for $d^2_{N3}(y)$. For this purpose we remind that $d^2_{N3}(y) = b^2_h(y) + D^2(h,y)$ where

$$b_h(y) = \alpha h^{-1} y^{\alpha\nu-1} \int_{-\infty}^{\infty} u^{\alpha-\alpha\nu} K\left(\frac{y^\alpha - u^\alpha}{h}\right) p(u)du - p(y), \tag{3.22}$$

$$D^2(h,y) = \alpha^2 N^{-1} h^{-2} y^{2\alpha\nu-2} \int_{-\infty}^{\infty} u^{2\alpha-2\alpha\nu} K^2\left(\frac{y^\alpha - u^\alpha}{h}\right) p(u)du \tag{3.23}$$

are, respectively, bias and variance of estimator (2.20). Substituting the expression for $p(y)$ into (3.22), (3.23), changing variables $u^\alpha = z$ and taking into account (3.3), we get

$$b_h(y) = \int_{a_1}^{a_2} f(y/\theta) \left[\int_0^\infty K(-z) exp(-hz/\theta)dz - 1\right] g(\theta)d\theta,$$

$$D^2(h,y) = (Nh)^{-1} \int_{a_1}^{a_2} f(y/\theta) \left[\int_0^\infty K^2(-z)(y^\alpha + hz)^{1-1/\alpha} exp(-zh/\theta)dz\right] g(\theta)d\theta.$$

Recall that $K(z) = 0$ for $z > 0$ (see (2.21)). Using the following relations (see Gradshtein & Ryzhik (1980))

$$\int_{-\infty}^{\infty} K^2(z)dz = 1/2, \qquad \int_0^\infty K(-z)exp(-zh/\theta)dz = 1 - exp(-\theta/h),$$

similarly to the case of $d^2_{N1}(y,b)$ we obtain that

$$b_h(y) \le C(\chi_\alpha + y^{\alpha-1}) exp(-a_1/h) p(y); \qquad D^2(h,y) \le C(Nh)^{-1}(\chi_\alpha + y^{\alpha-1})p(y). \tag{3.24}$$

From (3.24) it immediately follows that as soon as $h \sim 2a_1(\ln N)^{-1}$ we get

$$d^2_{N3}(y) \le CN^{-1} \ln N(\chi_\alpha + y^{\alpha-1})p(y). \tag{3.25}$$

At this point as we constructed upper bounds for $d^2_{N1}(y,b)$, $d^2_{N2}(y,t)$ and $d^2_{N3}(y)$, our goal is to obtain upper bounds for $\Delta(y;\beta_N)$ and $\Delta(y;\Upsilon_N)$. For this purpose we show that function $H(z)$ given by formula (2.22) satisfy the following assertion: *if* $\mathbf{E}(\xi_N - \xi)^2 < \delta_1$ *and* $\mathbf{E}(\eta_N - \eta)^2 < \delta_2$, $\eta \ne 0$, *then*

$$\mathbf{E}\left[H\left(\xi_N/\eta_N\right) - H\left(\xi/\eta\right)\right]^2 \le \delta_1 \eta^{-2}(16a_2^2 + 2) + \delta_2 \eta^{-2}(24a_2^2 + 8a_2 + 2). \tag{3.26}$$

To prove the statement we partition the domain Ω^2 into two parts $U = \{\omega : |\xi_N - \xi| \le \eta/2, |\eta_N - \eta| \le \eta/2\}$ and $\bar{U} = \Omega \setminus U$. Then the following relationship is apparent

$$\mathbf{E}\left[H\left(\xi_N/\eta_N\right) - H\left(\xi/\eta\right)\right]^2 \le \int_U \left(\xi_N/\eta_N - \xi/\eta\right)^2 dP + 4a_2^2 P(\bar{U}).$$

Application of Taylor's expansion and Chebyshev inequality result in (3.26).

Substitution of $\Phi_N(y,b), \Phi(y,b), p_N(y)$ and $p(y)$ for ξ_N, ξ, η_N and η in formula (3.26), respectively, and taking into account (3.4), (3.16), (3.21) and (3.25) yeild

$$\Delta(y;\beta_N) \leq CN^{-1}\ln N(\chi_\alpha + y^{\alpha-1})(p(y))^{-1}. \tag{3.27}$$

By use of the similar procedure we obtain upper bound for posterior risk of $\Upsilon_N(y,t)$

$$\Delta(y;\Upsilon_N) \leq CN^{-1}\ln N\ V(t)(\chi_\alpha + y^{\alpha-1})(p(y))^{-1}. \tag{3.28}$$

Here function $V(t)$ is determined by formula (3.20).

Now finally we are able to find the asymptotic expression for prior risks for estimators $\beta_N(y,b)$ and $\Upsilon_N(y,t)$. Let us first produce upper bound for $\Delta(\beta_N)$. To do that, we break the interval $(0;\infty)$ into two parts $(0;A)$ and $(A;\infty)$. Thus $\Delta(\beta_N)$ turns out to be

$$\Delta(\beta_N) = \int_0^A \Delta(y;\beta_N)p(y)dy + \int_A^\infty \Delta(y;\beta_N)p(y)dy \equiv \Delta_{N,1}(\beta_N) + \Delta_{N,2}(\beta_N).$$

By simple transformations we get $\Delta_{N,1}(\beta_N) \leq CN^{-1}\ln N(A\chi_\alpha + A^\alpha)$. After that, as $p(y) \leq Cy^{\alpha\nu-1}exp\{-y^\alpha/a_1\}$ and $\Delta(y;\beta_N) \leq 4a_2^2$, we obtain that

$$\Delta_{N,2}(\beta_N) \leq CA^{\alpha(\nu-1)}exp\{-A^\alpha/a_1\}.$$

Now, choosing $A = [a_1(ln(N) + (\nu-3)lnln(N) - (\nu-2)lnlnln(N))]^{1/\alpha}$, and combining last two formulas, we arrive at

$$\Delta(\beta_N) \leq CN^{-1}(\ln N)^{\alpha+1}. \tag{3.29}$$

Procceding similarly in the case of $\Upsilon_N(y,t)$ we derive the required upper bound

$$\Delta(\Upsilon_N) \leq CN^{-1}(\ln N)^{\alpha+1}V(t). \tag{3.30}$$

4 Simulation and discussion

In Sections 2 and 3 we obtained upper bounds for posterior and prior risks of EB estimators $\beta_N(y,b)$ and $\Upsilon_N(y,t)$ (see (3.27) – (3.30)). These bounds depend on constants C which in its turn depends on unknown prior density $g(\theta)$. Nevertheless the constant C can be majorized over all possible choices of $g(\theta)$; to do that we simply need to substitute C by its numerical value in every inequality. However evaluation of these upper limits is not of much sense since the bounds we get through this procedure will be grossly overestimated. Besides that, upper bounds (3.27) and (3.28) for posterior risks depend not only on the value of C but also on the value of unknown marginal density $p(y)$.

Thus it seems interesting to study the behaviour of prior and posterior risks by Monte-Carlo simulations. Let us perform simulations in the case of EB estimation of mean lifetime, i.e. EB estimation of θ^b with $b = 1$. For this purpose we choose $a_1 = 1$, $a_2 = 2$, $\alpha = 1$, $\nu = 2$ and $g(\theta) = (\theta \ln 2)^{-1}, \theta \in [1, 2]$. In this situation marginal density $p(y)$ and functional $\Phi(y) \equiv \Phi(y, 1)$ have, respectively, the forms

$$p(y) = (0.5 + y^{-1})e^{-0.5y} - (1 + y^{-1})e^{-y}, \quad \Phi(y) = e^{-0.5y} - e^{-y} \tag{4.1}$$

and $\beta(y) = \Phi(y)/p(y)$.

For the sake of evaluation of prior risk of EB estimator of θ we generate $m = 200$ independent samples $(T_{j1}, T_{j2}, ..., T_{jN}, T_{jN+1} \equiv y_j)$, $j = 1, ..., m$, of the size $(N + 1)$ according to the density $p(y)$ and construct EB estimators $\beta_N^{(j)}(y_j)$, $j = 1, ..., m$, for each sample. Then we use

$$\hat{\Delta}_N = m^{-1} \sum_{j=1}^{m} \left(\beta_N^{(j)}(y_j) - \beta(y_j) \right)^2 \tag{4.2}$$

as asymptotic approximation for $\Delta(\beta_N)$. According to formula (3.29), $\hat{\Delta}_N$ is of the order $O\left(N^{-1}(\ln N)^2\right)$.

Table 1 presents values of $\hat{\Delta}_N$ and of weighted prior risk $\hat{\zeta}_N = N(\ln N)^{-2}\hat{\Delta}_N$ for $N = 100, 200, ..., 2000$. It is easy to notice that in our example $\hat{\zeta}_N$ is less than 1 for every N.

For investigation of posterior risk $\Delta(y; \beta_N)$ we generate $m = 200$ independent samples $(T_{j1}, T_{j2}, ..., T_{jN})$, $j = 1, ..., m$. Then we construct estimators $\beta_N^{(j)}(y)$ based on the j-th sample and estimate posterior risk at a point y by

$$\hat{\Delta}_N(y) = m^{-1} \sum_{j=1}^{m} \left(\beta_N^{(j)}(y) - \beta(y) \right)^2. \tag{4.3}$$

We also calculate weighted posterior risk $\hat{\zeta}_N(y) = N(\ln N)^{-1}\hat{\Delta}_N(y)$. Table 2 containes the values of $\hat{\Delta}_N(y)$ and $\hat{\zeta}_N(y)$ for different values of y when $N = 100$. The values of $p(y)$ are placed in the last column.

Dependences between $\hat{\zeta}_N(y)$ and y and $\hat{\zeta}_N(y)$ and $p(y)$ are displayed on Figure 1 and Figure 2, respectively. Figure 2 shows that $\hat{\zeta}_N(y)$ increases as $p(y)$ decreases. In another words, the less probable value we obtain, the bigger error we get. It seems that this effect can be weaken by the choice of h depending on y, i.e. $h = h(y)$. However, it is a subject of future investigation.

Table 1.
Prior risk of EB estimator

N	$\hat{\Delta}_N$	$\hat{\zeta}_N$
100	.10291	.48524
200	.08614	.61367
300	.05516	.50869
400	.04153	.46279
500	.06202	.80289
600	.04151	.60865
700	.04027	.65681
800	.03955	.70815
900	.03460	.67306
1000	.03032	.63531
1100	.03068	.68811
1200	.02326	.55530
1300	.02630	.66507
1400	.02840	.75763
1500	.02031	.56948
1600	.02322	.68249
1700	.02488	.76454
1800	.02511	.80439
1900	.02098	.69952
2000	.01740	.60241

Table 2.
Posterior risk of EB estimator,N=100

y	$\hat{\Delta}_N(y)$	$\hat{\zeta}_N(y)$	$p(y)$
.01	.1700	3.6905	.0037
.50	.0735	1.5969	.1274
1.00	.0752	1.6331	.1740
1.50	.0732	1.5898	.1792
2.00	.0788	1.7107	.1649
2.50	.0803	1.7447	.1429
3.00	.0924	2.0064	.1196
3.50	.1000	2.1710	.0977
4.00	.1137	2.4690	.0786
4.50	.1140	2.4750	.0625
5.00	.1253	2.7210	.0494
5.50	.1460	3.1698	.0388
6.00	.1556	3.3779	.0303
6.50	.1797	3.9012	.0236
7.00	.2018	4.3830	.0184
7.50	.2507	5.4448	.0143
8.00	.2575	5.5911	.0111

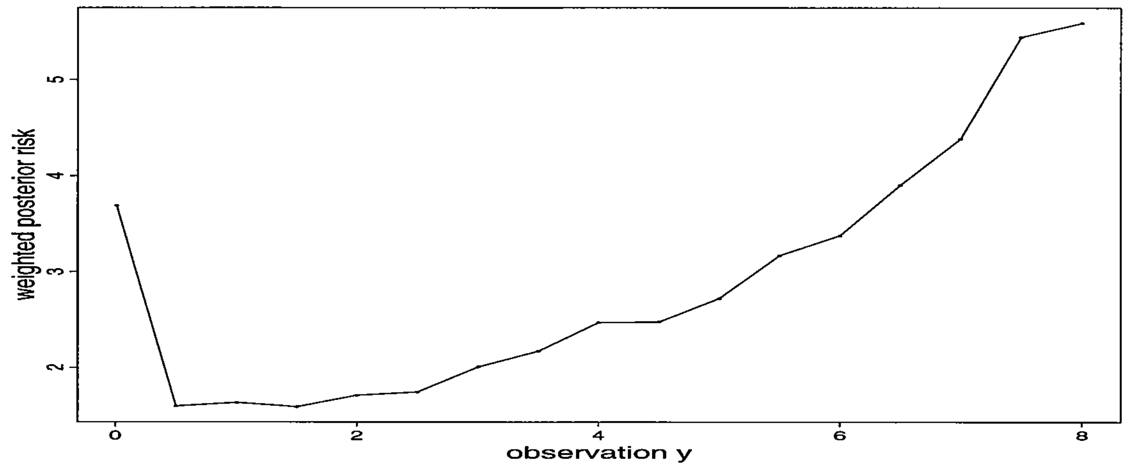

Figure 1: Dependence between the weighted posterior risk $\hat{\zeta}_N(y)$ and y.

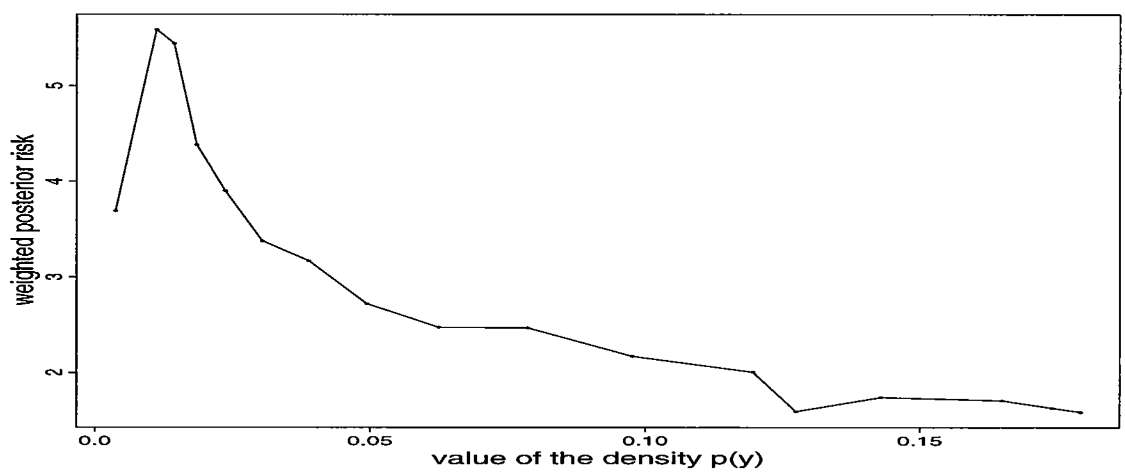

Figure 2: Dependence between the weighted posterior risk $\hat{\zeta}_N(y)$ and $p(y)$i.

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