A NEW LOOK AT WARRANT PRICING AND RELATED OPTIMAL STOPPING PROBLEMS

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1 General Ideas

In several papers on sequential Bayes testing and change-point detection (see for instance [1], Chapter II of [7], or [19]) the following argument is used: The Bayes risk $R(T)$ is represented for all stopping times T with $R(T) < \infty$ as

(R) $$R(T) = Eg(L_T)\,,$$

where L_t denotes a certain stochastic process connected to the likelihood process evaluated at time t and where g is a positive function with a unique minimum, let's say at a^*. Then we have

$$R(T) = Eg(L_T) \geq g(a^*)\ .$$

If L_t is a time-continuous process and passes a^* with probability one, the optimal stopping time will be

$$T^* = \inf\{t > 0 | L_t = a^*\}\ .$$

If L_t is discrete in time, one will usually not hit a^* exactly, therefore one has to stop ahead of a^*. This is also the case for the 'parking problem' described in the book of Chow, Robbins and Siegmund ([3], page 45 and 60). There $g(x) = |x|$ and $L_n = X_1 + ... + X_n$ where the X_i are geometrically distributed. Therefore M. Woodroofe has called situations as described above 'generalized parking problems' (see [19]). Of course for time-continuous processes L_t the solution is trivial, when one has the representation (R). Nevertheless to find a representation of type (R) is sometimes not obvious (see e. g. [1]).

One can combine the above technique with the one recently used by Shepp and Shiryayev ([14]). This yields an easy method to handle also some tricky optimal stopping problems.

Since the examples are formulated more naturally as maximization problems, we switch for convenience from minimization to maximization. To explain our technique more thoroughly let $X = \left(X_t; 0 \leq t < \infty\right)$ denote a continuous stochastic process for which we want to maximize EX_T over all finite stopping times T with respect to some filtration $\mathcal{F} = (\mathcal{F}_t; 0 \leq t < \infty)$. We will discuss a general approach to transform such a problem to a generalized parking problem. The basic idea is to find another continuous stochastic process Y adapted to $\mathcal{F}$, a function g with a maximum uniquely located at some point y^* and a positive martingale M with $M_0 = 1$ such that

$$X_t = g(Y_t)M_t$$

for $0 \leq t < \infty$. By the properties of g we have

$$X_t = g(Y_t)M_t \leq g(y^*)M_t\ .$$

Since M is a martingale we obtain for any finite stopping time T

$$EX_T \leq g(y^*) \ .$$

In order to prove the optimality of

$$T^* = \inf\{t > 0 | Y_t = y^*\}$$

one needs only to show that

$$P(T^* < \infty) = 1 \quad \text{and} \quad EM_{T^*} = M_0 = 1 \ .$$

This can be seen as follows:

$$\begin{aligned} EX_{T^*} &= Eg(Y_{T^*})M_{T^*}1_{\{T^*<\infty\}} \\ &= g(y^*)EM_{T^*} \\ &= g(y^*). \end{aligned}$$

To reformulate this argument let Q denote the probability measure on $\mathcal{F}_\infty = \sigma(\mathcal{F}_t; t \geq 0)$ with

$$\frac{dQ}{dP}\Big|_{\mathcal{F}_t} = M_t \qquad \text{for } 0 \leq t < \infty \ .$$

(In the problems we are considering such a probability measure always exists since we assume $\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t)$.)
Then we have for all stopping times T with $P(T < \infty) = 1$

$$EX_T = E_Q g(Y_T) \ .$$

This means that we have tranformed the initial stopping problem into a generalized parking problem with respect to a new probability measure Q. To prove $EM_{T^*} = 1$, is equivalent to show

$$Q(T^* < \infty) = 1 \ .$$

We note that T^* still maximizes the quantity $EX_T 1_{\{T<\infty\}}$, if $P(T^* < \infty) < 1$ but $Q(T^* < \infty) = 1$ holds.

A crucial point in some of our arguments is to establish the martingale property for continuous local martingales. Sufficient conditions for that are given in the book of Protter ([10], p.35, 66). Our technique works especially well for problems with exponentially discounting. Many problems of option pricing have this feature.

In the sections 2.1, 2.2 we discuss some classical results on American call and put options. In section 2.3 we take a more general viewpoint which leads also to results on two-sided problems. As a consequenc e we calculate the values of American straddles and strangles with infinite horizon in section 2.5. In section 2.6 we take another look on the classical problem of stopping $E\frac{W_T}{T+1}$. Finally section 2.6 discusses Russian options.

Our paper has close links to Moerbeke's excellent survey paper ([18]), which lead us to the be lieve that there is a general principle underlying our examples. So far we were unable to formulate it in a mathematical statement.

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2 Examples

2.1 The Warrant Pricing Problem

Let $\mathbf{R}$ denote the real and $\mathbf{R}^+$ the positive real numbers. Let W denote standard Brownian motion with measure P. Let $\sigma \in \mathbf{R}^+$ and $\mu \in \mathbf{R}$. Let X denote the geometric Brownian motion given by

$$X_t = \exp(\sigma W_t + (\mu - \sigma^2/2)t)$$

Problem 1 *Find a stopping time T of X that maximizes*

$$E\left\{e^{-rT}\left(X_T - K\right)^+ 1_{\{T<\infty\}}\right\} ,$$

where K and r are constants with $K > 0$ and $r > \max\{0, \mu\}$.

This problem has been considered first by H. P. McKean [8] and P. A. Samuelson [11] and later by Moerbeke [17]; see also [6]. Note that for $r \leq \mu$ the problem is trivial. In that case it is advantagous to wait as long as possible.

Let

$$\alpha = -\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right) + \sqrt{\frac{2r}{\sigma^2} + \left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2}$$

and

$$C^* = \max_{K \leq x < \infty} \left\{(x-K)^+ x^{-\alpha}\right\} .$$

Let x^* denote the unique point in (K,∞) where the function $(x-K)^+x^{-\alpha}$ attains its maximum C^*. A straightforward argument yields

$$x^* = \frac{\alpha}{\alpha-1}K \quad \text{and} \quad C^* = \frac{1}{\alpha-1}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha} K^{1-\alpha} \,.$$

Note that the condition $r > \max\{0,\mu\}$ implies $\alpha > 1$. We will solve the above problem under the additional assumption $x^* > 1$.

Theorem 1 *Let $r > \max\{0,\mu\}$ and $K > 1 - 1/\alpha$, then*

$$\sup_T E\left\{e^{-rT}\left(X_T - K\right)^+ 1_{\{T<\infty\}}\right\} = E\left\{e^{-rT^*}\left(X_{T^*} - K\right)^+ 1_{\{T^*<\infty\}}\right\} = C^* \,,$$

with

$$T^* = \inf\{t > 0 | X_t = x^*\} \ .$$

Under the additional assumption

$$\mu - \frac{\sigma^2}{2} \geq 0$$

it holds that $P(T^* < \infty) = 1$. In this case T^* also maximizes

$$E\left\{e^{-rT}\left(X_T - K\right)^+\right\}$$

among all stopping times T.

Proof: Let M_t denote the process

$$e^{-rt}X_t^{\alpha}.$$

It holds

$$\frac{(\alpha\sigma)^2}{2} + \alpha\left(\mu - \frac{\sigma^2}{2}\right) = r.$$

Therefore we have

$$M_t = \exp\left\{\alpha\sigma W_t - \frac{(\alpha\sigma)^2}{2}t\right\}$$

and M is a positive martingale with $M_0 = 1$. By the choice of C^* for all $0 \leq t < \infty$ it holds

$$e^{-rt}(X_t - K)^+ = (X_t - K)^+ X_t^{-\alpha} M_t \leq C^* M_t.$$

On the set $\{T^* < \infty\}$ we have

$$\left(X_{T^*} - K\right)^+ X_{T^*}^{-\alpha} = C^*.$$

Therefore it is sufficient to show $EM_{T^*}1_{\{T^*<\infty\}} = 1$. Let Q denote the probability measure on the σ-algebra $\sigma(W_s; 0 \le s < \infty)$ with

$$\frac{dQ}{dP}\Big|_{\mathcal{F}_t^W} = M_t$$

for $0 \le t < \infty$, where $\mathcal{F}_t^W = \sigma(W_s; 0 \le s \le t)$. Under the probability measure Q the process W is a Brownian motion with drift $\alpha\sigma$. Therefore under Q the process $\log X$ is a Brownian motion with drift (per unit time)

$$\sigma^2\alpha + \mu - \frac{\sigma^2}{2} = \sigma^2\sqrt{\frac{2r}{\sigma^2} + \left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)} > 0.$$

Since $\log x^* > 0$ we get $Q(T^* < \infty) = 1$. If the additional condition $\mu - \frac{\sigma^2}{2} \ge 0$ holds, then the drift of $\log X$ is nonnegative. □

2.2 Perpetual American Put Options

Let W denote standard Brownian motion. Let $\sigma \in \mathbf{R}^+$ and $\mu \in \mathbf{R}$. Let X denote the exponential Brownian motion given by

$$X_t = \exp(\sigma W_t + (\mu - \sigma^2/2)\,t) \; .$$

The following problem is treated in [5] for the case $r = \mu$. See also [11].

Problem 2 *Find a stopping time T of X that maximizes*

$$E\left\{e^{-rT}(K - X_T)^+ 1_{\{T<\infty\}}\right\} \; ,$$

where K and r are constants with $K > 0$ and $r > 0$.

Put

$$\alpha = \left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right) + \sqrt{\frac{2r}{\sigma^2} + \left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2} \; .$$

and

$$C^* = \max_{0 \le x \le K}\left\{(K - x)^+ x^\alpha\right\} \; .$$

Let x^* denote the unique point in $(0, K)$ where the function $(K - x)^+ x^\alpha$ attains its maximum C^*. A straightforward computation yields

$$x^* = \frac{\alpha K}{\alpha + 1}$$

and

$$C^* = \frac{1}{\alpha+1}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha} K^{1+\alpha} .$$

Note that $\alpha > 0$. We will solve the above problem under the additional assumption $x^* < 1$.

Theorem 2 *Let $r > 0$ and $K < 1 + 1/\alpha$, then*

$$\sup_T E\left\{e^{-rT}\left(K - X_T\right)^+ 1_{\{T<\infty\}}\right\} = E\left\{e^{-rT^*}\left(K - X_{T^*}\right)^+ 1_{\{T^*<\infty\}}\right\} = C^* ,$$

with

$$T^* = \inf\left\{t > 0 | X_t = x^*\right\} .$$

Under the additional assumption

$$\mu - \frac{\sigma^2}{2} \leq 0$$

it holds that $P(T^* < \infty) = 1$. In this case T^* also maximizes

$$E\left\{e^{-rT}\left(K - X_T\right)^+\right\}$$

among all stopping times T.

Proof: Let M_t denote the process

$$e^{-rt} X_t^{-\alpha} .$$

Then

$$\frac{(\alpha\sigma)^2}{2} + \alpha\left(\mu - \frac{\sigma^2}{2}\right) = r .$$

Therefore it holds

$$M_t = \exp\left\{-\alpha\sigma W_t - \frac{(\alpha\sigma)^2}{2} t\right\}$$

and so M is a positive martingale with $M_0 = 1$. By the choice of C^* it holds for all $0 \leq t < \infty$ that

$$e^{-rt}(K - X_t)^+ = (K - X_t)^+ X_t^{\alpha} M_t \leq C^* M_t .$$

Therefore we have for all stopping times T

$$Ee^{-rT}(K - X_T)^+ 1_{\{T<\infty\}} \leq C^* .$$

For the stopping time T^* holds

$$Ee^{-rT^*}(K - X_{T^*})^+ 1_{\{T^*<\infty\}} = C^* E M_{T^*} 1_{\{T^*<\infty\}} .$$

Let Q denote the probability measure on $\sigma(W_s; 0 \leq s < \infty)$ defined by

$$\frac{dQ}{dP}\Big|_{\mathcal{F}_t^W} = M_t$$

for $0 \leq t < \infty$, where $\mathcal{F}_t^W = \sigma(W_s; 0 \leq s \leq t)$. Under the probability measure Q the process W is a Brownian motion with drift $-\alpha\sigma$. Therefore $\log X$ is a Brownian motion with drift

$$-\alpha\sigma^2 + \mu - \frac{\sigma^2}{2} = - - \sigma^2\sqrt{\frac{2r}{\sigma^2} + \left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2} \ .$$

This yields $Q(T^* < \infty) = 1$ since $\log x^* < 0$. Therefore it holds

$$EM_{T^*}1_{\{T^*<\infty\}} = 1 \ .$$

□

2.3 Exponentially Discounted Functions of Brownian Motion with Drift – one-sided boundaries

The arguments in **2.1** and **2.2** are very similar. In fact the examples of these sections can be seen as particular cases of the following general theorem.

Let W denote standard Brownian motion. Let $\sigma \in \mathbf{R}^+$ and $\mu \in \mathbf{R}$. Let X denote the Brownian motion given by

$$X_t = \sigma W_t + \mu t \ .$$

Let h denote a measurable real-valued function. Let r be a strictly positive constant. The following problem is treated for $\mu = 0$ in Moerbeke [17] and [18].

Problem 3 *Find a stopping time T with respect to $\mathcal{F}^X$ that maximizes*

$$E\left\{e^{-rT}h(X_T)1_{\{T<\infty\}}\right\} \ .$$

Let

$$\alpha_1 = -\frac{\mu}{\sigma^2} + \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}}$$

and

$$\alpha_2 = -\frac{\mu}{\sigma^2} - \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}}$$

denote the solutions of the quadratic equation

$$\frac{(\alpha\sigma)^2}{2} + \alpha\mu = r \ .$$

Of course $\alpha_2 < 0 < \alpha_1$. Therefore the processes $M_t^{(1)}$ and $M_t^{(2)}$ given by

$$M_t^{(1)} = e^{-rt} e^{\alpha_1 X_t} \quad \text{and} \quad M_t^{(2)} = e^{-rt} e^{\alpha_2 X_t}$$

are positive martingales.

Theorem 3 *If*

$$0 < C_1 = \sup_{x \in \mathbf{R}} \left(e^{-\alpha_1 x} h(x) \right) < \infty$$

and there exists a point $x_1 > 0$ with

$$C_1 = e^{-\alpha_1 x_1} h(x_1) \ ,$$

then

$$\sup_T E\left\{ e^{-rT} h(X_T) 1_{\{T<\infty\}} \right\} = C_1$$

and the supremum is attained for T^* with

$$T^* = \inf\{t > 0 | \ X_t = x_1\} \ .$$

Proof: Let $Q^{(1)}$ denote the probability measure on $\sigma(W_s; 0 \leq s < \infty)$ with

$$\left.\frac{dQ^{(1)}}{dP}\right|_{\mathcal{F}_t^W} = M_t^{(1)}$$

for all $0 \leq t < \infty$. We have for all stopping times T

$$\begin{aligned} E\left\{ e^{-rT} h(X_T) 1_{\{T<\infty\}} \right\} &= E\left\{ M_T^{(1)} e^{-\alpha_1 X_T} h(X_T) 1_{\{T<\infty\}} \right\} \\ &= E_{Q^{(1)}}\left\{ e^{-\alpha_1 X_T} h(X_T) 1_{\{T<\infty\}} \right\} \ . \end{aligned}$$

By the definition of C_1 we obtain for all stopping times T

$$E\left\{ e^{-rT} h(X_T) 1_{\{T<\infty\}} \right\} \leq C_1 \ .$$

On the set $\{T^* < \infty\}$ it holds

$$e^{-\alpha_1 X_{T^*}} h(X_{T^*}) = C_1$$

and so we have

$$E\left\{e^{-rT^*}h(X_{T^*})1_{\{T^*<\infty\}}\right\} = C_1 Q^{(1)}(T^* < \infty) \ .$$

To complete the proof it is therefore sufficient to show $Q^{(1)}(T^* < \infty) = 1$. Under $Q^{(1)}$ the process W is a Brownian motion with drift $\alpha_1\sigma$. This yields the desired result. □

Remark *If $P(T^* < \infty) = 1$ holds, then T^* also maximizes*

$$Ee^{-rT}h(X_T)$$

among all stopping times T.

Example (See [18], p. 553-554) *For $\mu = 0$ and $\sigma = 1$ we have $\alpha_1 = \sqrt{2r}$. For a sufficiently smooth function h the point x_1 satisfies the equation*

$$\left(h(x)e^{-\sqrt{2r}x}\right)' = 0 \ .$$

Since

$$\left(h(x)e^{-\sqrt{2r}x}\right)' = \left(h'(x) - \sqrt{2r}h(x)\right)e^{-\sqrt{2r}x}$$

holds, x_1 also solves

$$\frac{d}{dx}\log h(x) = \sqrt{2r} \ .$$

For $h(x) = x$ it is easy to check the above conditions. We obtain

$$T^* = \inf\left\{t > 0 \middle| W_t = \frac{1}{\sqrt{2r}}\right\}$$

and

$$\sup_T E\left(e^{-rT}W_T\right) = \frac{1}{\sqrt{2r}}e^{-1} \ .$$

With similar arguments as above one can also prove the following theorem.

Theorem 4 *If*

$$0 < C_2 = \sup_{x \in \mathbf{R}} \left(e^{-\alpha_2 x} h(x) \right) < \infty$$

and there exists a point $x_2 < 0$ *with*

$$C_2 = e^{-\alpha_2 x_2} h(x_2) \ ,$$

then

$$\sup_T E \left\{ e^{-rT} h(X_T) 1_{\{T<\infty\}} \right\} = C_2$$

and the supremum is attained for T^* *with*

$$T^* = \inf\{t > 0 | \ X_t = x_2\} \ .$$

Note that the conditions of Theorem 3 and Theorem 4 are mutually exclusive. Suppose for example that the conditions of Theorem 3 are satisfied, that is

$$0 < C_1 = \sup_{x \in \mathbf{R}} \left(e^{-\alpha_1 x} h(x) \right) < \infty$$

and there exists a point $x_1 > 0$ with

$$C_1 = e^{-\alpha_1 x_1} h(x_1) \ .$$

Then for all $x < 0$ with $h(x) > 0$ it holds

$$e^{-\alpha_1 x_1} h(x_1) \geq e^{-\alpha_1 x} h(x) > e^{-\alpha_2 x} h(x) \ .$$

Since $x_1 > 0$ we have $e^{-\alpha_2 x_1} h(x_1) > e^{-\alpha_1 x_1} h(x_1)$ and so we obtain

$$e^{-\alpha_2 x_1} h(x_1) > e^{-\alpha_2 x} h(x)$$

for all $x < 0$ with $h(x) > 0$. This inequality clearly also holds for $x < 0$ with $h(x) \leq 0$. Therefore $\sup_{x \in \mathbf{R}} (e^{-\alpha_2 x} h(x))$ cannot be attained at some point $x_2 < 0$.

2.4 Exponentially Discounted Functions of Brownian Motion with Drift – two-sided boundaries

The method of section **2.3** can be extended to treat problems with two-sided boundaries. We will now consider the problem to maximize

$$E \left\{ e^{-rT} h(X_T) 1_{\{T<\infty\}} \right\}$$

for a function h with the properties:

$$\sup_{x\le 0}\left(e^{-\alpha_1 x}h(x)\right) > \sup_{x\ge 0}\left(e^{-\alpha_1 x}h(x)\right) \quad , \quad \sup_{x\ge 0}\left(e^{-\alpha_2 x}h(x)\right) > \sup_{x\le 0}\left(e^{-\alpha_2 x}h(x)\right) \quad ,$$

$$0 < \sup_{x\ge 0}\left(e^{-\alpha_1 x}h(x)\right) < \infty$$

and

$$0 < \sup_{x\le 0}\left(e^{-\alpha_2 x}h(x)\right) < \infty \, .$$

In this case we can neither apply Theorem 3 nor Theorem 4. Examples for such functions are given by $h(x) = x^2$ or $h(x) = \max\{(L - e^x)^+, (e^x - K)^+\}$ (if $\alpha_1 > 1$). The basic idea is to replace the martingales $M^{(1)}$ and $M^{(2)}$ by the martingale M with

$$M_t = pM_t^{(1)} + (1-p)M_t^{(2)} \, ,$$

where $p \in (0,1)$ is suitable chosen. We have for all stopping times T

$$E\{e^{-rT}h(X_T)\} = E\left\{M_T \frac{h(X_T)}{pe^{\alpha_1 X_T} + (1-p)e^{\alpha_2 X_T}}\right\} \, .$$

To find a proper p we will now study the function

$$\frac{h(x)}{pe^{\alpha_1 x} + (1-p)e^{\alpha_2 x}}$$

more closely. Since the sets $\{x \ge 0 | h(x) > 0\}$ and $\{x \le 0 | h(x) > 0\}$ are both nonempty, we have

$$\sup_{x\ge 0} \frac{h(x)}{pe^{\alpha_1 x} + (1-p)e^{\alpha_2 x}} = \sup_{x\ge 0; h(x)>0} \frac{h(x)}{pe^{\alpha_1 x} + (1-p)e^{\alpha_2 x}} = \left(\inf_{x\ge 0; h(x)>0} \frac{pe^{\alpha_1 x} + (1-p)e^{\alpha_2 x}}{h(x)}\right)^{-1}$$

and

$$\sup_{x\le 0} \frac{h(x)}{pe^{\alpha_1 x} + (1-p)e^{\alpha_2 x}} = \sup_{x\le 0; h(x)>0} \frac{h(x)}{pe^{\alpha_1 x} + (1-p)e^{\alpha_2 x}} = \left(\inf_{x\le 0; h(x)>0} \frac{pe^{\alpha_1 x} + (1-p)e^{\alpha_2 x}}{h(x)}\right)^{-1} \, .$$

Note that for all $p \in (0,1)$ it holds

$$0 < \sup_{x\ge 0; h(x)>0} \frac{h(x)}{pe^{\alpha_1 x} + (1-p)e^{\alpha_2 x}} \le \frac{1}{p} \sup_{x\ge 0}\left(e^{-\alpha_1 x}h(x)\right) < \infty$$

and

$$0 < \sup_{x\le 0; h(x)>0} \frac{h(x)}{pe^{\alpha_1 x} + (1-p)e^{\alpha_2 x}} \le \frac{1}{1-p} \sup_{x\le 0}\left(e^{-\alpha_2 x}h(x)\right) < \infty \, .$$

For fixed x with $h(x) > 0$ the function

$$\frac{pe^{\alpha_1 x} + (1-p)e^{\alpha_2 x}}{h(x)}$$

is linear in p. Therefore the functions $m_1(p)$ and $m_2(p)$ given by

$$m_1(p) = \inf_{x \geq 0; h(x) > 0} \frac{pe^{\alpha_1 x} + (1-p)e^{\alpha_2 x}}{h(x)}$$

and

$$m_2(p) = \inf_{x \leq 0; h(x) > 0} \frac{pe^{\alpha_1 x} + (1-p)e^{\alpha_2 x}}{h(x)}$$

are concave functions on $(0,1)$ with values in $(0,\infty)$. The function m_1 is nondecreasing and the function m_2 is nonincreasing. It holds that

$$0 < \lim_{p \to 1} m_1(p) = \frac{1}{\sup_{x \geq 0}(e^{-\alpha_1 x} h(x))} < \infty$$

and

$$0 < \lim_{p \to 0} m_2(p) = \frac{1}{\sup_{x \leq 0}(e^{-\alpha_2 x} h(x))} < \infty \, .$$

We have further

$$\lim_{p \to 0} m_1(p) = \inf_{x \geq 0; h(x) > 0} \frac{e^{\alpha_2 x}}{h(x)} = \frac{1}{\sup_{x \geq 0}(e^{-\alpha_2 x} h(x))}$$

and

$$\lim_{p \to 1} m_2(p) = \inf_{x \leq 0; h(x) > 0} \frac{e^{\alpha_1 x}}{h(x)} = \frac{1}{\sup_{x \leq 0}(e^{-\alpha_1 x} h(x))} \, ,$$

with the convention that $\frac{1}{+\infty} = 0$. Since

$$\sup_{x \geq 0} \left(e^{-\alpha_2 x} h(x)\right) > \sup_{x \leq 0} \left(e^{-\alpha_2 x} h(x)\right)$$

we obtain

$$\lim_{p \to 0} (m_1(p) - m_2(p)) < 0 \, .$$

In a similar way we can show that

$$\lim_{p \to 1} (m_1(p) - m_2(p)) > 0 \, .$$

Therefore the function $m_1(p) - m_2(p)$ is a continuous function for $p \in (0,1)$ with at least one zero. This means there exists at least one $p^* \in (0,1)$ such that

$$\sup_{x \geq 0} \frac{h(x)}{p^* e^{\alpha_1 x} + (1-p^*)e^{\alpha_2 x}} = \sup_{x \leq 0} \frac{h(x)}{p^* e^{\alpha_1 x} + (1-p^*)e^{\alpha_2 x}} \, .$$

In general p^* is not necessarily unique.

We will now show that p^* is unique, if there exists a point $\tilde{x} > 0$ with

$$e^{-\alpha_1 \tilde{x}} h(\tilde{x}) = \sup_{x \geq 0} \left(e^{-\alpha_1 x} h(x) \right) .$$

Suppose there exist p^* and p^{**} with $0 < p^* < p^{**} < 1$ such that

$$m_1(p^*) - m_2(p^*) = 0 = m_1(p^{**}) - m_2(p^{**}) .$$

This implies

$$0 \geq m_1(p^*) - m_1(p^{**}) = m_2(p^*) - m_2(p^{**}) \geq 0$$

and so

$$m_1(p^*) - m_1(p^{**}) = m_2(p^*) - m_2(p^{**}) = 0 .$$

Since m_1 is concave and nondecreasing this yields $m_1(p^{**}) = m_1(p)$ for all $p \in (p^{**}, 1)$. Therefore we have

$$m_1(p^{**}) = \lim_{p \to 1} m_1(p) = \frac{1}{\sup_{x \geq 0}(e^{-\alpha_1 x} h(x))} .$$

This is a contradiction to

$$\begin{aligned} m_1(p^{**}) &\leq \frac{p^{**} e^{\alpha_1 \tilde{x}} + (1 - p^{**}) e^{\alpha_2 \tilde{x}}}{h(\tilde{x})} < \frac{e^{\alpha_1 \tilde{x}}}{h(\tilde{x})} \\ &= \frac{1}{\sup_{x \geq 0}(e^{-\alpha_1 x} h(x))} . \end{aligned}$$

Theorem 5 *Let p^* be chosen such that*

$$\sup_{x \geq 0} \frac{h(x)}{p^* e^{\alpha_1 x} + (1 - p^*) e^{\alpha_2 x}} = \sup_{x \leq 0} \frac{h(x)}{p^* e^{\alpha_1 x} + (1 - p^*) e^{\alpha_2 x}} = C^* .$$

If there exist points $x_1 > 0$ and $x_2 < 0$ such that

$$\frac{h(x_1)}{p^* e^{\alpha_1 x_1} + (1 - p^*) e^{\alpha_2 x_1}} = \frac{h(x_2)}{p^* e^{\alpha_1 x_2} + (1 - p^*) e^{\alpha_2 x_2}} = C^* ,$$

then

$$\sup_T \left\{ E e^{-rT} h(X_T) 1_{\{T < \infty\}} \right\} = C^*$$

and the supremum is attained for

$$T^* = \inf\{t > 0 | X_t = x_1 \text{ or } X_t = x_2\} .$$

Proof: For all stopping times T it holds

$$E\left\{e^{-rT}h(X_T)1_{\{T<\infty\}}\right\} = E\left\{M_T\frac{h(X_T)}{p^*e^{\alpha_1 X_T}+(1-p^*)e^{\alpha_2 X_T}}1_{\{T<\infty\}}\right\} \leq C^* E M_T 1_{\{T<\infty\}} ,$$

where

$$M_t = p^* M_t^{(1)} + (1-p^*)M_t^{(2)} .$$

Therefore it only remains to show that $E\{M_{T^*}1_{\{T^*<\infty\}}\} = 1$. Let Q denote the probability measure on $\sigma(W_s; 0 \leq s < \infty)$ with

$$\frac{dQ}{dP}\Big|_{\mathcal{F}_t^W} = M_t$$

for all $0 \leq t < \infty$, where $\mathcal{F}_t^W = \sigma(W_s; 0 \leq s \leq t)$. Let B denote a standard Brownian motion and Y a random variable that is independent of B with

$$P(Y = \alpha_1\sigma) = p^* = 1 - P(Y = \alpha_2\sigma) .$$

Under the probability measure Q the process W has the same distribution as $(B_t + Yt; 0 \leq t < \infty)$. Therefore it holds

$$Q(T^* < \infty) = 1 .$$

□

Remark *In general it is not possible to determine p^*, x_1 and x_2 explicitly. One particular situation in which it is straightforward to determine p^* is, when h is symmetric around zero (i.e. $h(x) = h(-x)$ for all x) and the Brownian motion X has drift zero. Then we have $\alpha_2 = -\alpha_1 = -\sqrt{2r/\sigma^2}$, $p^* = \frac{1}{2}$ and $x_2 = -x_1$. Even in this particular case there seems to be no explizit expression for x_1. If in addition h is sufficiently smooth the point x_1 is a solution of the differential equation*

$$h'(x)\left\{e^{\alpha x} + e^{-\alpha x}\right\} = \alpha h(x)\left\{e^{\alpha x} - e^{-\alpha x}\right\}$$

with $\alpha = \sqrt{2r/\sigma^2}$.

2.5 Perpetual straddle and strangle options

Theorem **5** above can be used to determine the value and optimal exercise strategy of a perpetual straddle or strangle option. A strangle (straddle) is a combination of a put with exercise prize L (K) and a call with exercise prize K on the same security, where $L \leq K$. If we model the price of the underlying process by a geometric Brownian motion

$$\exp\left\{\sigma W_t + (\mu - \frac{\sigma^2}{2})t\right\} ,$$

with W a standard Brownian motion, then we have to solve the following problem. Let

$$X_t = \sigma W_t + \tilde{\mu} t ,$$

with $\tilde{\mu} = \mu - \frac{\sigma^2}{2}$ and

$$h(x) = \begin{cases} L - e^x & x \leq \log L \\ 0 & \log L \leq x \leq \log K \\ e^x - K & x \geq \log K \end{cases} .$$

The task is to find a stopping time T that maximizes

$$E\left\{e^{-rT} h(X_T)\right\} ,$$

where $r > 0$. Note that different than in the sections **2.1** and **2.2** the process X is now the logarithm of the prize of the underlying security. Let

$$\alpha_1 = -\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right) + \sqrt{\frac{2r}{\sigma^2} + \left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2}$$

and

$$\alpha_2 = -\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right) - \sqrt{\frac{2r}{\sigma^2} + \left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2} .$$

Then $\alpha_2 < 0 < \alpha_1$. Since the value of a straddle or strangle is larger than the value of the corresponding call, we assume as in section **2.1** that the inflation factor r satisfies $r > \mu$. This implies $\alpha_1 > 1$ and so the constants α_1 and α_2 and the function h fulfill the conditions of Theorem **5**. Moreover we assume that

$$\log L \leq 0 = X_0 \leq \log K .$$

Under these conditions we obtain from Theorem **5** the following result.

Corollary 1 *Let x_1, x_2 and p^* be the unique solutions with $x_1 > \log K$, $x_2 < \log L$ and $0 < p < 1$ of the following system of equations:*

$$\begin{aligned}
\frac{e^{x_1}-K}{p^*e^{\alpha_1 x_1}+(1-p^*)e^{\alpha_2 x_1}} &= \frac{L-e^{x_2}}{p^*e^{\alpha_1 x_2}+(1-p^*)e^{\alpha_2 x_2}} \\
\frac{e^{x_1}}{e^{x_1}-K} &= \frac{p^*\alpha_1 e^{\alpha_1 x_1}+(1-p^*)\alpha_2 e^{\alpha_2 x_1}}{p^*e^{\alpha_1 x_1}+(1-p^*)e^{\alpha_2 x_1}} \\
\frac{-e^{x_2}}{L-e^{x_2}} &= \frac{p^*\alpha_1 e^{\alpha_1 x_2}+(1-p^*)\alpha_2 e^{\alpha_2 x_2}}{p^*e^{\alpha_1 x_2}+(1-p^*)e^{\alpha_2 x_2}}
\end{aligned}$$

Let C^* denote the common value of

$$\frac{e^{x_1}-K}{p^*e^{\alpha_1 x_1}+(1-p^*)e^{\alpha_2 x_1}} \quad \text{and} \quad \frac{L-e^{x_2}}{p^*e^{\alpha_1 x_1}+(1-p^*)e^{\alpha_2 x_1}} .$$

Then

$$\sup_T Ee^{-rT}\max\left\{L-e^{\sigma W_T+\left(\mu-\frac{\sigma^2}{2}\right)T}, 0, e^{\sigma W_T+\left(\mu-\frac{\sigma^2}{2}\right)T}-K\right\} = C^*$$

and the supremum is attained for T^* with

$$T^* = \inf\left\{t>0 | e^{\sigma W_t+\left(\mu-\frac{\sigma^2}{2}\right)t} = e^{x_1} \quad \text{or} \quad e^{\sigma W_t+\left(\mu-\frac{\sigma^2}{2}\right)t} = e^{x_2}\right\} .$$

Proof: Since

$$\sup_{x\geq 0} e^{-\alpha_1 x}h(x) = e^{-\alpha_1\tilde{x}}h(\tilde{x})$$

for $\tilde{x} = \log\frac{\alpha_1}{\alpha_1-1} + \log K$, there exist a unique p^* with

$$\sup_{x\geq 0}\frac{h(x)}{p^*e^{\alpha_1 x}+(1-p^*)e^{\alpha_2 x}} = C^* = \sup_{x\leq 0}\frac{h(x)}{p^*e^{\alpha_1 x}+(1-p^*)e^{\alpha_2 x}} .$$

For any $p\in(0,1)$ we have

$$\lim_{x\to\infty}\frac{h(x)}{pe^{\alpha_1 x}+(1-p)e^{\alpha_2 x}} = 0 .$$

As $h(x)\geq 0$ for all x and $h(\log K)=0$ for any fixed $p\in(0,1)$ the function

$$\frac{h(x)}{pe^{\alpha_1 x}+(1-p)e^{\alpha_2 x}}$$

assumes its maximum over $(\log K, \infty)$ at some point x in $(\log K,\infty)$. Each such point is a solution of

$$\left(\frac{h(x)}{pe^{\alpha_1 x}+(1-p)e^{\alpha_2 x}}\right)' = 0 .$$

On $(\log K, \infty)$ this equation is equivalent to

$$\frac{e^x}{e^x - K} = \frac{p\alpha_1 e^{\alpha_1 x} + (1-p)\alpha_2 e^{\alpha_2 x}}{p e^{\alpha_1 x} + (1-p) e^{\alpha_2 x}} .$$

The function $\frac{e^x}{e^x - K}$ is strictly decreasing on $(\log K, \infty)$ and the function

$$\frac{p\alpha_1 e^{\alpha_1 x} + (1-p)\alpha_2 e^{\alpha_2 x}}{p e^{\alpha_1 x} + (1-p) e^{\alpha_2 x}}$$

is strictly increasing. Therefore there is at most one solution of

$$\left(\frac{h(x)}{p e^{\alpha_1 x} + (1-p) e^{\alpha_2 x}}\right)' = 0$$

in $(\log K, \infty)$. With similar arguments one can show that for any fixed $p \in (0,1)$ the function

$$\frac{h(x)}{p e^{\alpha_1 x} + (1-p) e^{\alpha_2 x}}$$

assumes its maximum over the interval $(-\infty, \log L)$ at the point, which is the unique solution of

$$\frac{-e^x}{L - e^x} = \frac{p\alpha_1 e^{\alpha_1 x} + (1-p)\alpha_2 e^{\alpha_2 x}}{p e^{\alpha_1 x} + (1-p) e^{\alpha_2 x}}$$

in $(-\infty, \log L)$. □

2.6 Parabolic Boundaries

Let W denote standard Brownian motion and h be a measurable function such that

$$\sup_{x \in \mathbf{R}} \frac{h(x)}{H(x)} < \infty ,$$

where

$$H(x) = \int_0^\infty e^{ux - \frac{u^2}{2}} u^{2\beta - 1} du ,$$

with $\beta \in \mathbf{R}^+$. We further assume that the supremum of $h(x)/H(x)$ over $\mathbf{R}$ is attained at a unique point x^* and that this supremum is strictly positive. Let C^* denote

$$C^* = \frac{h(x^*)}{H(x^*)} .$$

Let $x_0 < x^*$ and

$$X_t = W_t + x_0$$

for $0 \le t < \infty$. We will now consider

Problem 4 *Find a stopping time T of X that maximizes*

$$E\left\{(T+1)^{-\beta}h\left(\frac{X_T}{\sqrt{T+1}}\right)\right\} .$$

This problem is treated by Moerbeke ([18]) (under different assumptions on h).

Theorem 6 *Under the above assumptions it holds*

$$\sup_T E\left\{(T+1)^{-\beta}h\left(\frac{X_T}{\sqrt{T+1}}\right)\right\} = E\left\{(T^*+1)^{-\beta}h\left(\frac{X_{T^*}}{\sqrt{T^*+1}}\right)\right\} = H(x_0)C^* ,$$

where

$$T^* = \inf\left\{t > 0 \Big| \frac{X_t}{\sqrt{t+1}} = x^*\right\} .$$

Proof: Let M_t denote the process

$$(t+1)^{-\beta}H\left(\frac{X_t}{\sqrt{t+1}}\right) \Big/ H(x_0) .$$

It holds

$$(t+1)^{\beta}\int_0^{\infty} e^{uX_t-\frac{u^2}{2}t}e^{-\frac{u^2}{2}}u^{2\beta-1}du \;=\; H\left(\frac{X_t}{\sqrt{t+1}}\right) .$$

Moreover we have

$$e^{uX_t-\frac{u^2}{2}t} = e^{ux_0}e^{uW_t-\frac{u^2}{2}t}$$

and

$$Ee^{uX_t-\frac{u^2}{2}t} = e^{ux_0} .$$

Therefore $(M_t; 0 \le t < \infty)$ is a positive martingale with $EM_0 = 1$ and by the definition of C^* we have

$$(t+1)^{-\beta}h\left(\frac{X_t}{\sqrt{t+1}}\right) = H(x_0)\frac{h\left(\frac{X_t}{\sqrt{t+1}}\right)}{H\left(\frac{X_t}{\sqrt{t+1}}\right)}M_t \le H(x_0)C^*M_t .$$

This implies

$$E\left\{(T+1)^{-\beta}h\left(\frac{X_T}{\sqrt{T+1}}\right)\right\} \le H(x_0)C^*EM_T$$

for all $\mathcal{F}^X$–stopping times T. As M is a positive martingale with $M_0 = 1$ we obtain from this that for all $\mathcal{F}^X$–stopping times T

$$E\left\{(T+1)^{-\beta}h\left(\frac{X_T}{\sqrt{T+1}}\right)\right\} \le H(x_0)C^*$$

holds. On the set $\{T^* < \infty\}$ one has

$$\frac{h\left(\frac{X_{T^*}}{\sqrt{T^*+1}}\right)}{H\left(\frac{X_{T^*}}{\sqrt{T^*+1}}\right)} = C^*$$

and therefore

$$(T^*+1)^{-\beta} h\left(\frac{X_{T^*}}{\sqrt{T^*+1}}\right) = C^* H(x_0) M_{T^*} .$$

In order to complete the proof it is therefore sufficient to show

$$P(T^* < \infty) = 1 \quad \text{and} \quad EM_{T^*} = 1 .$$

The law of the iterated logarithm yields immediately $P(T^* < \infty) = 1$. Let ρ denote the probability measure on $\mathbf{R}^+$ with Lebesgue–density

$$\frac{1}{H(x_0)} e^{ux_0 - \frac{u^2}{2}} u^{2\beta-1} .$$

Let Q denote the probability measure on $\sigma(W_s; 0 \le s < \infty)$ with

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t^W} = \int_0^\infty e^{uX_t - \frac{u^2}{2}t} \rho(du) = \frac{1}{H(x_0)} \int_0^\infty e^{uX_t - \frac{u^2}{2}t} e^{-\frac{u^2}{2}} u^{2\beta-1} du = M_t ,$$

for $0 \le t < \infty$, where $\mathcal{F}_t^W = \sigma(W_s; 0 \le s \le t)$. Let B denote a standard Brownian motion and Y a random variable with distribution ρ that is independent of B. Under Q the process $(X_t; 0 \le t < \infty)$ has the same distribution as $(x_0 + B_t + Y_t; 0 \le t < \infty)$. Therefore

$$Q(T^* < \infty) = 1$$

and so the assertion follows. □

Example (A classical stopping problem) *We now consider the special case* $h(x) = x$, $x_0 = 0$, *and* $\beta = \frac{1}{2}$. *That means we want to maximize*

$$E\frac{W_T}{T+1} .$$

This problem is treated in [13] and [15] and was initiated by [2] and [4]. An easy calculation shows that

$$\int_0^\infty e^{ux - \frac{u^2}{2}} du \quad = \quad \sqrt{2\pi} e^{\frac{x^2}{2}} \Phi(x) ,$$

where

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \ .$$

Differentation yields the following transzendental equation for the threshold x^* (see [13])

$$\begin{aligned} 0 &= H(x) - xH'(x) \\ &= (1 - x^2) \int_0^\infty e^{ux - \frac{u^2}{2}} du - x \ . \end{aligned}$$

Remark *Let T_a denote the stopping time*

$$T_a = \inf\{t > 0 | x_0 + W_t \geq a\sqrt{t+1}\} \ .$$

Since M_t is a martingale the optional stopping theorem yields

$$E\left(T_a + 1\right)^{-\beta} = \frac{H(x_0)}{H(a)}$$

for $a > x_0$ and $\beta > 0$. Note that $\sup_{-\infty < z \leq a} H(z) = H(a) < \infty$. This is a special case of results of Novikov [9] and Shepp [12].

2.7 Perpetual Russian Options

Let W denote standard Brownian motion. Let $\sigma \in \mathbf{R}^+$ and $\mu \in \mathbf{R}$. Let X denote the geometric Brownian motion given by

$$X_t = \exp(\sigma W_t + (\mu - \sigma^2/2)t)$$

and let S_t denote the running maximum of X

$$S_t = \max_{0 \leq s \leq t} X_s \ .$$

We will now consider the following problem

Problem 5 *Find a stopping time T that maximizes*

$$E\left(e^{-rT} S_T\right)$$

under all stopping times of the process X. r is a positive constant satisfying $r > \mu$.

Shepp and Shiryaev proved the following result. We formulate it in their notation.

Theorem 7 *Let*

$$\gamma_1 = -\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right) - -\sqrt{\frac{2r}{\sigma^2} + \left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2} ,$$

$$\gamma_2 = -\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right) + \sqrt{\frac{2r}{\sigma^2} + \left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2}$$

and

$$\alpha = \left(\frac{1 - 1/\gamma_1}{1 - 1/\gamma_2}\right)^{1/(\gamma_2 - \gamma_1)} .$$

Then

$$\sup_T E\left(e^{-rT} S_T\right) = E\left(e^{-rT^*} S_{T^*}\right) ,$$

where

$$T^* = \inf\left\{t > 0 | \frac{S_t}{X_t} = \alpha\right\} .$$

We will now derive the result of Shepp and Shiryayev in [14] with our technique.

Proof: The continuous semimartingale $\frac{X_t}{S_t}$ satisfies the stochastic differential equation

$$d\frac{X_t}{S_t} = \frac{1}{S_t} dX_t - \frac{X_t}{S_t^2} dS_t$$

since S has increasing paths. For sufficiently smooth functions h we therefore obtain

$$\begin{aligned} e^{-rt} S_t h\left(\frac{X_t}{S_t}\right) &= \int_0^t e^{-ru} S_u \left\{-rh\left(\frac{X_u}{S_u}\right) + \mu \frac{X_u}{S_u} h'\left(\frac{X_u}{S_u}\right) + \frac{\sigma^2}{2}\left(\frac{X_u}{S_u}\right)^2 h''\left(\frac{X_u}{S_u}\right)\right\} du \\ &\quad + \int_0^t e^{-ru} \left\{-\frac{X_u}{S_u} h'\left(\frac{X_u}{S_u}\right) + h\left(\frac{X_u}{S_u}\right)\right\} dS_u \\ &\quad + \sigma \int_0^t e^{-ru} X_u h'\left(\frac{X_u}{S_u}\right) dW_u . \end{aligned}$$

The process S is flat off the set $\{t | S_t = X_t\}$. Therefore

$$\int_0^t e^{-ru} \left\{-\frac{X_u}{S_u} h'\left(\frac{X_u}{S_u}\right) + h\left(\frac{X_u}{S_u}\right)\right\} dS_u = \int_0^t e^{-ru} \left\{-h'(1) + h(1)\right\} dS_u .$$

Hence for sufficiently smooth functions h the process $e^{-rt} S_t h\left(\frac{X_t}{S_t}\right)$ is a local martingale if h satisfies

$$\begin{aligned} 0 &= -rh(x) + \mu x h'(x) + \frac{\sigma^2}{2} x^2 h''(x) \quad \text{for all } x \in (-\infty, 1] \\ 0 &= h'(1) - h(1) . \end{aligned}$$

A solution of these equations is given by (see [14])

$$h(x) = \frac{1}{\gamma_2 \alpha^{\gamma_1} - \gamma_1 \alpha^{\gamma_2}} \left(\gamma_2 \left(\alpha x\right)^{\gamma_1} - -\gamma_1 \left(\alpha x\right)^{\gamma_2}\right) ,$$

where γ_1, γ_2 and α are as above. Note that we have normalized the solution in a different way than in [14]. This particular solution moreover satisfies $h(1) = 1$ and it is easy to see that

$$\inf_{0<x\leq 1} h(x) = h\left(\alpha^{-1}\right) > 0$$

holds. Since $M_t = e^{-rt} S_t h\left(\frac{X_t}{S_t}\right)$ is a positive local martingale we obtain

$$E\left(e^{-rT} S_T\right) = E\left(M_T \frac{1}{h\left(\frac{X_T}{S_T}\right)}\right) \leq \frac{1}{h(\alpha^{-1})} E M_T \leq \frac{1}{h(\alpha^{-1})} .$$

On the event $\{T^* < \infty\}$ it holds

$$h\left(\frac{X_{T^*}}{S_{T^*}}\right) = h\left(\alpha^{-1}\right) .$$

A straightforward argument (see [14], equation (2.15)) shows that $P(T^* < \infty) = 1$. Therefore it is only left to show that $EM_{T^*} = 1$. A sufficient condition for this equality is

$$E \sup_{0\leq t\leq T^*} e^{-rt} S_t h\left(\frac{X_t}{S_t}\right) < \infty .$$

For $0 \leq t \leq T^*$ we have

$$0 < \frac{1}{\alpha} \leq \frac{X_t}{S_t} \leq 1 .$$

The continuous function h is bounded on the compact interval $[1/\alpha, 1]$. Hence it is sufficient to show that

$$E \sup_{0\leq t\leq T^*} e^{-rt} S_t < \infty .$$

For all $t \geq 0$ it holds

$$e^{-rt} \sup_{0\leq u\leq t} X_u \leq \sup_{0\leq u\leq t} e^{-ru} X_u$$

and hence

$$\sup_{0\leq t<\infty} e^{-rt} S_t \leq \sup_{0\leq t<\infty} \sup_{0\leq u\leq t} e^{-ru} X_u = \sup_{0\leq u<\infty} e^{-ru} X_u .$$

This yields (see [14])

$$E \sup_{0\leq t\leq T^*} e^{-rt} S_t \leq E \sup_{0\leq u<\infty} e^{-ru} X_u < \infty .$$

The estimation of the r.h.s. is a straightforward calculation (see [14]). □

The arguments in the proof of Theorem 7 can also be used to discuss a perpetual put with path-dependent strike S_t. This means we can consider the following problem.

Problem 6 *Find a stopping time T^* that maximizes*

$$E\left(e^{-rT}(S_T - X_T)1_{\{T<\infty\}}\right)$$

under all stopping times T with respect to the process X. Here r is a positive constant satisfying $r > \mu$.

We have the representation

$$e^{-rt}(S_t - X_t) = M_t \frac{1 - \frac{X_t}{S_t}}{h(X_t/S_t)} .$$

The function $(1-x)/h(x)$ assumes its maximum over $(0,1)$ uniquely at some point x^*. This yields the optimality of the stopping time T^* with

$$T^* = \inf\{t > 0 | X_t \leq x^* S_t\} .$$

Remark on the structure of M: *The local martingale M appearing in the proof above satisfies*

$$M_t = \sigma \int_0^t e^{-ru} X_u h'\left(\frac{X_u}{S_u}\right) dW_u = \sigma \int_0^t M_u \frac{X_u}{S_u} \frac{h'\left(\frac{X_u}{S_u}\right)}{h\left(\frac{X_u}{S_u}\right)} dW_u .$$

The function Λ with

$$\Lambda(x) = x\frac{h'(x)}{h(x)} = \gamma_1\gamma_2 \frac{\left(\alpha x\right)^{\gamma_1} - -\left(\alpha x\right)^{\gamma_2}}{\gamma_2\left(\alpha x\right)^{\gamma_1} - -\gamma_1\left(\alpha x\right)^{\gamma_2}}$$

is bounded on **R**. *Therefore the stochastic integral Y with*

$$Y_t = \sigma \int_0^t \Lambda\left(\frac{X_u}{S_u}\right) dW_u$$

is well-defined. The process M satisfies the Doléans equation

$$dM_t = M_t dY_t$$

with $M_0 = 1$. Hence we have (see [10], p. 77)

$$M_t = \exp\left\{\sigma \int_0^t \Lambda\left(\frac{X_u}{S_u}\right) dW_u - \frac{\sigma^2}{2} \int_0^t \Lambda\left(\frac{X_u}{S_u}\right)^2 du\right\} .$$

Since $\Lambda\left(\frac{X_u}{S_u}\right)$ is a bounded process it is immediately clear that M is a martingale. Let Q denote the probability measure on $\sigma(W_s; 0 \leq s < \infty)$ with

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t^W} = M_t ,$$

$\mathcal{F}_t^W = \sigma(W_s; 0 \leq s \leq t)$. Under the probability measure Q the process

$$W_t - \sigma \int_0^t \Lambda\left(\frac{X_u}{S_u}\right) du$$

is a standard Brownian motion. Note that $\Lambda(x) < 0$ holds for $x > 1/\alpha$ and $\Lambda(1/\alpha) = 0$.

Remark on Laplacetransforms: *Let T_a denote the stopping time*

$$T_a = \inf\{t > 0 | X_t \leq aS_t\} .$$

The function f with

$$f(x) = \gamma_2 x^{\gamma_1} - \gamma_1 x^{\gamma_2}$$

satisfies

$$\begin{aligned} 0 &= -rf(x) + \mu x f'(x) + \frac{\sigma^2}{2} x^2 f''(x) \quad \text{for all } x \in (-\infty, 1] \\ 0 &= f'(1) . \end{aligned}$$

This implies that the process $e^{-rt} f(X_t/S_t)$ is a nonnegative local martingale. Moreover for any $0 < a < 1$ it holds on $0 \leq t \leq T_a$ that

$$e^{-rt} f\left(\frac{X_t}{S_t}\right) \leq f(a) .$$

The optional sampling theorem therefore yields for $0 < a < 1$ and $r > 0$ that

$$Ee^{-rT_a} = \frac{\gamma_2 - \gamma_1}{\gamma_2 a^{\gamma_1} - \gamma_1 a^{\gamma_2}} .$$

This is special case of the results in [16].

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