

ANTICIPATING INTEGRALS FOR A CLASS  
OF MARTINGALES

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# ANTICIPATING INTEGRALS FOR A CLASS OF MARTINGALES

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**Abstract.** An anticipating stochastic integral is proposed for “normal martingales”. It agrees with the Skorohod integral in the Brownian case. A variational derivative of Malliavin type is also defined. An *integration by parts* formula is given which has some subtle and important differences from the formula in the Brownian case. Existence and uniqueness of solutions of linear SDEs with anticipating exogenous driving terms is also established.

**Keywords.** Anticipating stochastic integral, variational derivative, stochastic integration by parts, normal martingales, homogeneous chaos, multiple stochastic integral, anticipating stochastic differential equation, Azéma’s martingale, structure equation.

*Mathematics subject classification (1991):* 60H05, 60H07, 60H10.

## §1. Introduction.

In this paper we propose an anticipating integral for general martingales possessing two key properties: that  $\langle M, M \rangle_t = t$ , and that  $M$  possesses the *chaos representation property* (CRP). Martingales having the first property have been called “normal” by Dellacherie, Maisonneuve and Meyer (1992, p.199) and are discussed in section 2. Our integral is based on chaos expansion and as such it is analogous to the Skorohod integral as developed by Nualart and Pardoux (1988). When the integrand is predictable it reduces to the usual martingale integral as presented, for example, in Protter (1990); in the case when  $M$  is a Brownian motion it is exactly the Skorohod integral.

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There are many similarities between our martingale anticipating integral and the Skorohod integral, but there are also some important (and non trivial) differences. Many of these differences come from one key fact: in the Brownian case  $[B, B]_t = \langle B, B \rangle_t = t$ , while in our case only  $\langle M, M \rangle_t = t$ , and  $[M, M]_t$  is random (see Protter (1990) for all unexplained notation). For example there are two ways to describe a variational derivative (also known as the “Malliavin derivative” in the Brownian case), and they are equivalent in the Brownian case but not in ours. This leads to some subtle differences, and null sets exist for measures arising from kernels on the product space  $\Omega \times \mathbb{R}_+$ .

Our paper is organized as follows. In sections 3 and 4 we recall what chaos is, and in section 4 we give the basic definitions of a variational derivative and an anticipating integral, and we establish some elementary properties which are analogues of the Brownian (Skorohod) case. Here there already arises a difference: in the Brownian case one can use the equivalence of the two definitions of the variational derivative to give elegant proofs (using the “integration by parts” formula) right from the beginning (see the elegant presentation of Nualart (1995), for example), whereas such tools are not available for our integral.

In section 5 we continue our study of properties of the Skorohod integral and we include a formula inspired by preliminary work of Russo and Vallois (1994, Theorem 5.3). Here the subtlety of the differences with the Brownian case begin to become readily apparent, and correct definitions with respect to reference measures need to be made.

In section 6 we give some preliminary results on stochastic differential equations. Our results are a far cry from the beautiful results already established for the Skorohod integral (see, e.g., Buckdahn (1994)), but on the other hand they are very general in that they hold for any normal martingale  $M$ . Our method relies exclusively on the Chaos Representation Property of a normal martingale, therefore neither path regularity nor  $L^p$  estimations of the anticipating integrals are required.

The reader will note that we do not include results establishing path regularity of the integral, and hence we are as yet unable to establish a change of variables formula. We hope to address these issues in a future work.

## §2. Normal Martingales.

The following definition is from Dellacherie, Maisonneuve and Meyer (1992, p.199 ff).

**Definition 2.1.** A martingale  $M$  is called normal if  $\langle M, M \rangle_t = t$ .

In this paper we will be interested in normal martingales that have an extra property: the Chaos Representation Property (CRP) (see Section 3). Note that  $[M, M]_t - \langle M, M \rangle_t = [M, M]_t - t$  is a martingale, and because CRP implies the predictable representation property (PRP), we have that

$$(2.1) \quad [M, M]_t = t + \int_0^t \varphi_s dM_s$$

for some predictable  $\varphi$ . In other words, equation (2.1) is *necessary* if a martingale  $M$  is normal with CRP. Emery (1989) first presented and studied equation (2.1) and has called it the *structure equation*. If one begins with the structure equation (2.1), then one can show that it has unique solutions under quite general conditions on  $\varphi$  (see Meyer (1989) or Kurtz-Protter (1991, p.1044–1045)). The simplest structure equations are those of Brownian motion and the compensated Poisson process, which are respectively:

$$[M, M]_t = t, \quad \text{and} \quad [M, M]_t = t + cM_t, \quad (c > 0).$$

An important special class was studied by Emery, which we write in differential form:

$$(2.2) \quad d[M, M]_t = dt + (\alpha + \beta M_{t-}) dM_t; \quad M_0 = x.$$

Especially if  $\alpha = 0$  and  $-2 \leq \beta \leq 0$ , Emery proved the important result that a solution  $M$  of (2.2) is normal and has CRP. This gives a whole family of normal martingales with CRP for which our results apply. Note that  $\beta = 0$  is Brownian motion,  $\beta = -1$  is Azéma's martingale (see, e.g., Protter (1990, p.180–185) for a treatment of Azéma's martingale). When  $\beta = -2$ ,  $M$  is known as the "parabolic martingale" with  $|M_t| = \sqrt{t}$ , a.s., first studied by Protter-Sharpe (1979) and M. Barlow (1981). Vallois (1995) has recently demonstrated an interesting connection between the Brownian range process and parabolic martingales, thus showing that if  $M_t = \sqrt{2}B_{\theta(\sqrt{t})}$ , where  $B$  is standard Brownian

motion and  $\theta$  is the inverse of its range process; then  $M$  is a normal martingale with CRP too. It is worth pointing out that any solution  $M$  of (2.2) has no continuous martingale part: that is,  $[M, M]_t^c = 0$ , for all  $t$  (see Emery (1989)). Finally, Russo and Vallois (1994) have extended Emery's results a bit by considering the equation:

$$(2.3) \quad d[M, M]_t = dt + \beta(t)M_t - dM_t.$$

They show that if  $-2 \leq \beta(t) \leq 0$ , for all  $t$ , then  $M$  is again normal with CRP.

The preceding discussion indicates, hopefully, that there is a significant family of normal martingales having CRP already known to exist, and prospects of more.

### §3. Multiple Integrals and Preliminaries.

Since the recent book of Dellacherie, Maisonneuve, and Meyer (1992) gives a lovely treatment of multiple integrals for normal martingales, we do not give one here, but content ourselves with a definition. We let  $\Sigma_n$  be an "increasing simplex" of  $\mathbb{R}_+^n$ :

$$(3.1) \quad \Sigma_n = \{(t_1, \dots, t_n) \in \mathbb{R}_+^n : 0 < t_1 < \dots < t_n\},$$

and we extend a function  $f$  defined on  $\Sigma_n$  by making  $f$  *symmetric* on  $\mathbb{R}_+^n$ . We can then define

$$(3.2) \quad I_n(f) = n! \int_{\Sigma_n} f(t_1, \dots, t_n) dM_{t_1} \cdots dM_{t_n}.$$

This has the advantage of working with traditional adapted integrands for our martingale integrals. Note that the domain  $\Sigma_n$  and its symmetrizations do not cover  $\mathbb{R}_+^n$ : we are ignoring the diagonals. As Meyer (1976) has pointed out, this ignores terms such as

$$\int_{\{t_1=t_2\}} f(t_1, t_2) dM_{t_1} dM_{t_2} = \int_0^\infty f(t, t) d[M, M]_t,$$

which need not be trivial in our case. We avoid this problem by making the following convention:

**Convention 3.1.** *Functions  $f$  defined on the simplex  $\Sigma_n$  are automatically extended to  $\mathbb{R}_+^n$  to be symmetric and to be zero on the diagonals.*

More precisely, let us define a class of functions for  $n \in \mathbb{N}$  and  $T = [0, 1]$  by

$$L_s^2(T^n) = \{f \in L^2(T^n) \mid f \text{ is symmetric in all variables}\}.$$

Note that elements in  $L_s^2(T^n)$  are in fact equivalence classes, and we shall always choose a representative that vanishes on the “diagonal” set:  $\Delta_n = \{(t_1, \dots, t_n) \mid \exists t_i = t_j, i \neq j\}$ . Such a selection of the representative will not affect our discussion when we are treating the iterated integral and its norm in  $L^2(T^n)$ , since in the former case the set  $\Delta_n$  is never involved, and in the latter  $\Delta_n$  is a null set under the Lebesgue measure. However, it will become crucial when some measure involving  $d[M, M]_t$  is considered, as in general  $d[M, M]_t$  will charge the Lebesgue-null set.

The *multiple integral* with respect to  $M$  defined in (3.2) will be considered defined for every  $f \in L_s^2(T^n)$  (or  $L^2(\Sigma_n)$ ); and it is known (see, e.g., Meyer (1993)) that for each  $f \in L_s^2(T^n)$

$$(3.3) \quad \|I_n(f)\|_{L^2(\Omega)}^2 = (n!)^2 \|f\|_{L^2(\Sigma_n)}^2 = n! \|f\|_{L^2(T^n)}^2.$$

**Definition 3.2.** Let  $\mathcal{G} = \sigma\{M_t; t \geq 0\}$ , the  $\sigma$ -field generated by a (normal) martingale  $M$ . Let  $\mathcal{H}_n$  be the  $n$ -th homogeneous chaos:  $\mathcal{H}_n = I_n(f)$ , where  $f$  ranges over all  $f \in L^2(\Sigma_n)$ . If  $L^2(\mathcal{G}, dP) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$  (the direct sum), then we say  $M$  possesses the *Chaos Representation Property (CRP)*.

**Remark 3.3.** The hypothesis  $\langle M, M \rangle_t = t$  has been used to define multiple integrals in a nice way, which in turn are needed to define CRP. One could require only  $d\langle M, M \rangle_t \ll dt$  to define the multiple integral, but if  $\langle M, M \rangle_t$  were random one would lose the property that different chaoses are orthogonal. Meyer (1976, p. 325) also points out that one could assume that  $\langle M, M \rangle_t = c(t)$  where  $c(\cdot)$  is non-random, but we do not pursue such generality here, partially because to date there are no known interesting examples of martingales  $M$  such that  $\langle M, M \rangle_t = c(t) \neq t$  and which have CRP.

In what follows we shall always assume that a normal martingale  $M$  with CRP is given on the probability space  $(\Omega, \mathcal{F}, P)$ , and that  $\mathcal{F}$  is generated by  $M$ . Thus, for any

random variable  $F \in L^2(\mathcal{F}, dP)$  ( $= L^2(\Omega)$ ) we have by CRP that there exists a sequence of functions  $f_n \in L^2_s(T^n)$ ,  $n = 1, 2, \dots$ , such that  $F = \sum_{n=0}^{\infty} I_n(f_n)$ . Hence

$$(3.4) \quad \infty > \|F\|_{L^2(\Omega)}^2 = \sum_{n=0}^{\infty} \|I_n(f_n)\|_{L^2(\Omega)}^2 = \sum_{n=0}^{\infty} n! \|f_n\|_{L^2(T^n)}^2.$$

We shall also use the following notation throughout the paper. If  $f \in L^2(T^n)$  and  $g \in L^2(T^m)$ , we denote  $f \otimes g \in L^2(T^{n+m})$  by

$$f \otimes g(t_1, \dots, t_n; s_1, \dots, s_m) = f(t_1, \dots, t_n)g(s_1, \dots, s_m);$$

and if  $I \subseteq T = [0, 1]$ , and  $k \in \mathbb{N}$ , we denote by  $1_I^{\otimes k}$  a function in  $L^2(T^k)$  such that

$$1_I^{\otimes k}(t_1, \dots, t_k) = \prod_{i=1}^k 1_I(t_i).$$

#### §4. A Derivative Operator and an Anticipating Integral.

Consider the following subset  $\mathcal{D} \subset L^2(\Omega)$ :

$$\mathcal{D} := \left\{ F = \sum_{n=0}^{\infty} I_n(f_n) \mid \sum_{n=0}^{\infty} n n! \|f_n\|_n^2 < \infty \right\},$$

where  $\|\cdot\|_n := \|\cdot\|_{L^2(T^n)}$ . To be consistent with the usual notation in the literature (e.g., Nualart and Pardoux (1988)), we shall write  $\mathcal{D} = \mathbb{D}_{1,2}$ . It is easily seen that  $\mathbb{D}_{1,2}$  is dense in  $L^2(\Omega)$ , since every element in the finite Hilbert sum of chaoses belongs to  $\mathbb{D}_{1,2}$ . The derivative operator is analogous to what is often called the *Malliavin derivative* in the Brownian case, and it is defined as a linear operator  $D : \mathbb{D}_{1,2} \subset L^2(\Omega) \rightarrow L^2(T \times \Omega)$ , by

$$(4.1) \quad D_t F := \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)), \quad t \in [0, 1],$$

whenever  $F$  has the chaotic expansion  $F = \sum_{n=0}^{\infty} I_n(f_n)$ . It is easy to see that

$$(4.2) \quad \begin{aligned} \|D_t F\|_{L^2(T \times \Omega)}^2 &= \int_0^1 \left\| \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)) \right\|_{L^2(\Omega)}^2 dt \\ &= \int_0^1 \sum_{n=1}^{\infty} n^2 (n-1)! \|f(\cdot, t)\|_{L^2(T^{n-1})}^2 dt \\ &= \sum_{n=1}^{\infty} n n! \int_0^1 \|f(\cdot, t)\|_{L^2(T^{n-1})}^2 dt = \sum_{n=1}^{\infty} n n! \|f\|_n^2 < \infty, \end{aligned}$$

for all  $F \in \mathbb{D}_{1,2}$ . Note that since our definition is analogous to the one normally used for the Brownian chaotic expansion, we can derive several properties using exactly the same methods as those used in the Brownian case. Surprisingly, however, the definition is not compatible with a Sobolev space structure, as of course the case for Brownian motion. We shall explain this via an example at the end of this section. Here we give two properties of the operator  $D$  for ready reference.

**Lemma 4.1.** (1) Suppose that  $F \in L^2(\Omega)$ . Then  $D_t F = 0$ , for all  $t \in [0, 1]$  if and only if  $F$  is a constant (non-random).

(2) Suppose that  $F \in L^2(\Omega)$  and is  $\mathcal{F}_t$ -measurable. Then  $D_s F = 0$  for all  $s > t$ .

*Proof.* This is identical to the Brownian case, so we omit it.  $\square$

We now turn to the definition of an anticipating integral, which is analogous to that of the Skorohod integral in the Brownian case. Note that since  $D$  is a densely defined operator, we can define its *adjoint operator*, denoted by  $\delta$ , in the usual way. Namely, Let

$$\mathcal{D}^* := \mathcal{D}(\delta) := \left\{ G \in L^2(T \times \Omega) : \exists C > 0, \left| E \int_0^1 G(t, \cdot) D_t F dt \right| \leq C \|F\|, \forall F \in \mathbb{D}_{1,2} \right\},$$

and the adjoint operator  $\delta : L^2(T \times \Omega) \mapsto L^2(\Omega)$  is defined by the equation:

$$(4.3) \quad E(\delta(G)F) = E \int_0^1 G(t, \cdot) D_t F dt, \quad \forall F \in \mathbb{D}_{1,2}, G \in \mathcal{D}^*.$$

Since every element in  $L^2(T \times \Omega)$  also has a chaos expansion, we can write  $G(t, \cdot) = \sum_{n=0}^{\infty} I_n(g_n(\cdot, t))$ , where  $g_n$ 's are deterministic functions, jointly measurable in all variables, and symmetric in the first  $n$  variables. Following the same arguments as those in Nualart (1995), one can show that the set  $\mathcal{D}^*$  is dense in  $L^2(T \times \Omega)$  and that

$$(4.4) \quad \mathcal{D}^* = \left\{ G(t, \cdot) = \sum_{n=0}^{\infty} I_n(g_n(\cdot, t)) : \sum_{n=0}^{\infty} (n+1)! \|\bar{g}_n^s\|_{L^2(T^{n+1})} < \infty \right\},$$

where  $\bar{g}^s$  denotes the symmetrization of  $g$  in all variables. In other words, the set  $\mathcal{D}^*$  is of the same form as the usual notion  $\text{Dom}(\delta)$  in the literature of the Skorohod integral (cf., e.g., Nualart (1995, Proposition III.3)), and therefore we will not distinguish the two in the sequel. Consequently, as the adjoint operator of  $D$ ,  $\delta$  is a densely defined, closed



operator, which in turn shows that the operator  $D$  is closable (in fact, it is not too hard to show that  $D$  is closed by definition). We have the following definition.

**Definition 4.2.** *The adjoint operator of  $D$ , denoted by  $\delta : L^2(T \times \Omega) \mapsto L^2(\Omega)$ , is called the anticipating integral of the element in  $\mathcal{D}^* = \text{Dom}(\delta)$ . Furthermore, for any  $u \in \text{Dom}(\delta)$ , we denote*

$$\delta(u) = \int_0^1 u_s \delta M_s.$$

Also, we can define the set  $\mathbb{L}^{1,2}$  as usual by

$$(4.5) \quad \mathbb{L}^{1,2} := \left\{ u_t = \sum_{n=0}^{\infty} I_n(f_n(\cdot, t)) \mid \sum_{n=0}^{\infty} nn! \|f_n\|_{L^2(T^{n+1})}^2 < \infty \right\}.$$

Noting that  $\sum nn! \|f_n\|_{L^2(T^{n+1})}^2 < \infty$  is equivalent to  $\sum (n+1)! \|f_n\|_{L^2(T^{n+1})}^2 < \infty$ , and the obvious inequality

$$(4.6) \quad \|\bar{f}_n^s\|_{L^2(T^{n+1})} \leq \|f_n\|_{L^2(T^{n+1})},$$

we have  $\mathbb{L}^{1,2} \subset \text{Dom}(\delta)$ , as in the Brownian case. The following results for our anticipating integral can also be proved using exactly the same arguments as those used, for example, in Nualart (1995). We give only the statements.

**Proposition 4.3.** (1) *Suppose that  $u \in \text{Dom}(\delta)$ , and that  $u$  has chaotic expansion  $u_t = \sum_{n=0}^{\infty} I_n(g_n(\cdot, t))$ , where  $g(\cdot, t) \in L_s^2(T^n)$ , for a.a.  $t \in T$ . Then*

$$\delta(u) = \sum_{n=0}^{\infty} I_{n+1}(\bar{g}_n^s).$$

(2) *Suppose that  $u, v \in \mathbb{L}^{1,2}$ . Then*

$$E(\delta(u)\delta(v)) = E \int_0^1 u_s v_s ds + E \int_0^1 \int_0^1 D_s u_t D_t v_s ds dt.$$

*In particular, if  $u = v \in \mathbb{L}^{1,2}$ , then*

$$E(\delta(u)^2) = E \int_0^1 u_s^2 ds + E \int_0^1 \int_0^1 D_s u_t D_t u_s ds dt.$$

(3) Suppose that  $u$  is a process in the space  $\mathbb{L}^{1,2}$  such that for almost all  $t \in [0, 1]$ ,  $D_t u \in \text{Dom}(\delta)$ , and there is a version of the process  $\{\int_T(D_t u_s)\delta M_s; t \geq 0\}$  which is in  $L^2(T \times \Omega)$ . Then  $\delta(u) \in \mathbb{D}_{1,2}$ , and it holds that

$$D_t(\delta(u)) = \int_0^1 (D_t u_s)\delta M_s + u_t.$$

□

A natural question now is whether the definition of our anticipating integral is a generalization of the usual Itô-type stochastic integral. Note that the proof in the Brownian case of this result (see, e.g., Nualart (1995)) uses the fact that any adapted,  $L^2$  process can be approximated by elementary adapted processes; and that the Skorohod and Itô integrals coincide on elementary adapted processes. The latter result relies on an integration by parts formula which in our case is more complicated and not easily applicable, as we shall see in the next section. Therefore the usual Brownian technique does not seem to apply. Our technique uses only the CRP property (and not integration by parts), so it also gives an alternative (new) proof in the Brownian case.

**Proposition 4.4.** *Suppose that  $u \in L^2(T \times \Omega)$  and is predictable. Then  $u \in \text{Dom}(\delta)$  and*

$$\delta(u) = \int_0^1 u_t dM_t.$$

where the right side above is in the semimartingale (or “Itô”) sense.

*Proof.* Let us first assume that  $u_t = I_n(f_n(\cdot, t))$ , where  $f_n$  is some  $L^2$ -function, symmetric in the first  $n$  variables. Since  $u$  is predictable, we have

$$u_t = n! \int_0^t \left\{ \int_{t_1 < \dots < t_n} f_n(t_1, \dots, t_n, t) dM_{t_1} \cdots dM_{t_{n-1}} \right\} dM_{t_n} = I_n(f_n(\cdot, t) 1_{[0,t]}^{\otimes n}).$$

Now by definition,

$$\begin{aligned} \delta(u) &= I_{n+1}(\overline{f_n(\cdot, *) 1_{[0,*]}^{\otimes n}(\cdot)^s}) \\ &= (n+1)! \int_0^1 \left\{ \int_{t_1 < \dots < t_n < t} \overline{f_n(t_1, \dots, t_n, t) 1_{[0,t]}^{\otimes n}(t_1, \dots, t_n)^s} dM_{t_1} \cdots dM_{t_n} \right\} dM_t. \end{aligned}$$

where “ $\cdot$ ” stands for the first  $n$  variables, and “ $*$ ” stands for the last variable;  $\overline{f_n(\cdot, *) 1_{[0,*]}^{\otimes n}(\cdot)^s}$  is the symmetrization of  $f_n(\cdot, *) 1_{[0,*]}^{\otimes n}(\cdot)$  in all variables. Note that since

$f_n$  is symmetric in the first  $n$  variables, a little computation shows that on the set  $\Sigma_{n+1} = \{t_1 < t_2 < \cdots < t_n < t\}$ , we have

$$\begin{aligned} \overline{f_n(t_1, \dots, t_n, t) 1_{[0,t]}^{\otimes n}(t_1, \dots, t_n)}^s &= \frac{1}{(n+1)!} \sum_{\sigma_n} f_n(\sigma_n(1), \dots, \sigma_n(n), t) \\ &= \frac{1}{(n+1)} f_n(t_1, \dots, t_n, t), \end{aligned}$$

where  $\sigma_n$  runs over all the permutations of  $\{1, \dots, n\}$ . Thus

$$\delta(u) = n! \int_0^1 \left\{ \int_{t_1 < \dots < t_n < t} f_n(t_1, \dots, t_n, t) dM_{t_1} \cdots dM_{t_n} \right\} dM_t = \int_0^1 u_t dM_t.$$

Now let us suppose that  $u_t = \sum_{n=0}^{\infty} I_n(f_n(\cdot, t)) \in L^2(T \times \Omega)$ , that is,

$$(4.7) \quad E \int_0^1 |u_t|^2 dt = \sum_{n=0}^{\infty} n! \int_0^1 \|f_n(\cdot, t)\|_n^2 dt = \sum_{n=0}^{\infty} n! \|f_n\|_{n+1}^2 < \infty.$$

Since  $u_t$  is predictable, it can be easily checked that every  $I_n(f_n(\cdot, t))$  must have the form  $I_n(f_n(\cdot, t)) = I_n(f_n(\cdot, t) 1_{[0,t]}^{\otimes n})$ . Therefore, using the same argument as before, we have

$$I_{n+1}(\overline{f_n^s}) = \delta(I_n(f_n(\cdot, \cdot))) = \int_0^1 I_n(f_n(\cdot, t)) dM_t,$$

and

$$\begin{aligned} (n+1)! \|\overline{f_n^s}\|_{n+1}^2 &= E |I_{n+1}(\overline{f_n^s})|^2 = E \int_0^t |I_n(f_n(\cdot, t))|^2 dt = n! \int_0^t \|f_n(\cdot, t)\|_n^2 dt \\ &= n! \|f_n\|_{n+1}^2. \end{aligned}$$

Thus (4.7) gives that

$$\sum_{n=0}^{\infty} (n+1)! \|\overline{f_n^s}\|_{n+1}^2 = \sum_{n=0}^{\infty} n! \|f_n\|_{n+1}^2 < \infty,$$

hence  $u \in \text{Dom}(\delta)$ , and by Proposition 4.3-(1),

$$\delta(u) = \sum_{n=0}^{\infty} I_{n+1}(\overline{f_n^s}) = \int_0^1 \left\{ \sum_{n=0}^{\infty} I_n(f_n(\cdot, t)) \right\} dM_t = \int_0^1 u_t dM_t,$$

proving the proposition. □

*An Ocone-Haussmann-Clark Type Formula.*

Let  $M$  be a normal martingale with CRP. For any  $F \in L^2(\Omega)$  we can write

$$(4.8) \quad F = \sum_{n=0}^{\infty} I_n(f_n) = E(F) + \sum_{n=1}^{\infty} I_n(f_n) = E(F) + \int_0^t u_t dM_t,$$

where  $u$  is a predictable process given by

$$(4.9) \quad u_t = \sum_{n=1}^{\infty} n! \int_{t_1 < t_2 < \dots < t_{n-1} < t} f_n(t_1, \dots, t_{n-1}, t) dM_{t_1} \dots dM_{t_{n-1}}.$$

In the Brownian case, the predictable process  $u$  can be further described using the ‘‘Malliavin derivative’’ operator, known as the Ocone-Haussmann-Clark formula (see, e.g., Ocone (1984)). Our definition of the ‘‘Malliavin derivative’’ operator  $\{D_t\}_{t \geq 0}$  also allows us to derive an analogue of that formula, as we see in the following theorem.

**Theorem 4.5.** *Let  $M$  be a normal martingale with CRP, and let  $F \in L^2(\Omega)$ . If  $F \in \mathbb{D}_{1,2}$  then*

$$(4.10) \quad F = E(F) + \int_0^1 {}^p(D_t F) dM_t,$$

where  ${}^p H_t$  denotes the predictable projection of a process  $H$ .

*Proof.* Comparing (4.10) with (4.8), we see that we need only show  $u_t = {}^p(D_t F)$ . Note that  $u$  is already predictable, so if  $u = {}^o(D_t F)$ , the optional projection of  $D_t F$ , then we would have  ${}^p(D_t F) = {}^p({}^o(D_t F)) = {}^p(u_t) = u_t$ ,  $\forall t \in [0, 1]$ , and we are done. Therefore it suffices to show that  $u$  is the optional projection of  $D_t F$ , or equivalently, (see Dellacherie (1972))

$$(4.11) \quad u_\tau = E\{D_\tau F | \mathcal{F}_\tau\}, \quad \text{for any stopping time } \tau \in [0, 1],$$

where  $D_\tau F$  is defined to be  $D_t F|_{t=\tau}$ . To this end, we first note that by definition we have

$$(4.12) \quad D_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)) = \sum_{n=1}^{\infty} n! \int_0^1 h_n(t_{n-1}, t) dM_{t_{n-1}},$$

where  $h_n(s, t) = \int_{t_1 < \dots < t_{n-2} < s} f_n(t_1, \dots, t_{n-1}, s, t) dM_{t_1} \dots dM_{t_{n-2}}$ . Clearly, for each fixed  $t \in [0, 1]$ ,  $h_n(\cdot, t)$  is predictable, so if we define  $M_n(r, t) := \int_0^r h_n(s, t) dM_s$ ,  $r \in [0, 1]$ , then

$M_n(\cdot, t)$  is a martingale with parameter  $t \in [0, 1]$ . Thus for any fixed  $t \in [0, 1]$  and any stopping time  $\tau \in [0, 1]$ , one has  $E\{M_n(1, t) - M_n(\tau, t)|\mathcal{F}_\tau\} = 0$ . By a standard Monotone Class argument, one can show that for any stopping time  $\tau \in [0, 1]$ , it holds that

$$(4.13) \quad E\{M_n(1, \tau) - M_n(\tau, \tau)|\mathcal{F}_\tau\} = 0, \quad \text{a.s.}$$

Further, recall that  $f_n(\cdot, t, t) = 0$ , we have by (4.9) that  $\sum_{n=1}^{\infty} n!M_n(t, t) = u_t, \forall t, \text{ a.s.}$  . Consequently, we see from (4.12) and (4.13) that, for any stopping time  $\tau \in [0, 1]$ ,

$$E\{D_\tau F|\mathcal{F}_\tau\} = \sum_{n=1}^{\infty} n!E\{M_n(1, \tau)|\mathcal{F}_\tau\} = \sum_{n=1}^{\infty} n!E\{M_n(\tau, \tau)|\mathcal{F}_\tau\} = u_\tau, \text{ a.s. ,}$$

here the last equality is due to the predictability of  $u$ . This proves (4.11), whence the theorem.  $\square$

We remark that in the Brownian case this result is customarily stated more simply by writing  $u_t = E\{D_t F|\mathcal{F}_t\}$ , for fixed  $t$ . It is implicit that in such an expression, one really means that  $u_t$  is the optional projection of  $D_t F$ .

### *Indefinite Integrals.*

We now study the possibility of defining indefinite anticipating integrals. Note that even in the Brownian case, it is not true that for any  $u \in \text{Dom}(\delta)$ , the process  $\{u_s 1_{[0, t]}(s) : s \in [0, 1]\} \in \text{Dom}(\delta)$  (see Nualart and Pardoux (1988)). We have the following definition.

**Definition 4.6.** Suppose that the process  $u \in \text{Dom}(\delta)$  is such that for any  $t \in [0, 1]$ , it holds that  $u \cdot 1_{[0, t]}(\cdot) \in \text{Dom}(\delta)$ . Then the *indefinite anticipating stochastic integral* of  $u$  on  $[0, t]$  is defined by

$$\int_0^t u_s \delta M_s := \delta(u \cdot 1_{[0, t]}) = \int_0^1 u_s 1_{[0, t]}(s) dM_s.$$

The following lemma gives a description of the class of processes in  $\text{Dom}(\delta)$  for which the indefinite integral exists.

**Lemma 4.6.** Suppose that  $u \in \text{Dom}(\delta)$ , and  $u_s = \sum_n I_n(f_n(\cdot, s))$ . Then for any  $t \in [0, 1]$ , the process  $u \cdot 1_{[0, t]}(\cdot) \in \text{Dom}(\delta)$  if and only if

$$(4.14) \quad \sum_{n=0}^{\infty} (n+1)! \| \overline{f_n(\cdot, *) 1_{[0, t]}(*)} \|_{n+1}^2 < \infty.$$

Consequently, a sufficient condition for the process  $u \cdot 1_{[0,t]}(\cdot)$  to belong to  $\text{Dom}(\delta)$  is that  $u \in \mathbb{L}^{1,2}$ .

*Proof.* The proof for (4.14) is direct, by using the definition. As for the second assertion, it suffices to recall the definition of  $\mathbb{L}^{1,2}$  and the fact that

$$\|\overline{f_n(\cdot, *)1_{[0,t]}(*)}\|_{n+1}^2 \leq \|f_n(\cdot, *)1_{[0,t]}(*)\|_{n+1}^2 \leq \|f_n\|_{n+1}^2.$$

□

To end this section we give an example which shows that one cannot define the derivative operator in the usual way in our case to obtain a Sobolev space structure for the space  $\mathbb{D}_{1,2}$ . In fact the example somehow shows that the two definitions (Sobolev space vs. chaos expansion) are compatible if and only if the process  $[M, M]$  is deterministic.

*Example.* Consider a symmetric function  $f(s, t) = 1_{(a,b]}(s)1_{(a,b]}(t)$ . The second chaos  $I_2(f)$  can be computed as

$$(4.15) \quad \begin{aligned} I_2(f) &= 2! \int_{0 < s < t \leq 1} f(s, t) dM_s dM_t = 2 \int_a^b \int_a^{t-} dM_s dM_t = 2 \int_a^b (M_{t-} - M_a) dM_t \\ &= (M_b - M_a)^2 - \{[M, M]_b - [M, M]_a\}. \end{aligned}$$

Here, the last equality is due to Itô's formula. Now consider the function  $F(x_1, x_2) = (x_2 - x_1)^2$ , and define a smooth functional  $\xi = F(M_a, M_b)$ . Let us define the derivative  $D_t \xi$  in a way analogous to one of the equivalent definitions in the Brownian case:

$$(4.16) \quad \begin{aligned} D_t \xi &= D_t(M_b - M_a)^2 = \frac{\partial F}{\partial x_1}(M_a, M_b)1_{[0,a]}(t) + \frac{\partial F}{\partial x_2}(M_a, M_b)1_{[0,b]}(t) \\ &= 2(M_b - M_a)1_{(a,b]}(t). \end{aligned}$$

However, by our definition

$$D_t I_2(f) = 2I_1(f(\cdot, t)) = 2 \int_0^1 1_{(a,b]}(s) dM_s \cdot 1_{(a,b]}(t) = 2(M_b - M_a)1_{(a,b]}(t).$$

We can substitute this into (4.16) and compare it with (4.15) to see that the two definitions coincide if and only if  $D_t\{[M, M]_b - [M, M]_a\} = 0$ , for all  $t \in T$ . By Lemma 4.1-(1), this means that  $[M, M]_b - [M, M]_a$  must be constant. If we look at the structure equation

(2.1), this amounts to saying that  $\varphi \equiv 0$ , therefore the two definitions are in contradiction and cannot hold simultaneously unless  $M = B$ , the Brownian motion.

## §5. Integration by Parts Formulae.

In this section we study an *integration by parts formula* given in Theorem 5.4. This formula differs from the one in the Brownian case in an important way, since it explicitly involves  $[M, M]_t$ , whose role is hidden in the Brownian case (in that case  $[M, M]_t = t$ ). Another difference is more subtle, in that one has to be careful what *versions* of the multiple integrals one takes. A key step in this direction was taken by Russo and Vallois (1994), which is presented here in Theorem 5.2; one difference in our treatment of Theorem 5.2 is that we take care to make precise which versions of the multiple integrals we are using. It is worth pointing out here that because in our case  $\mathbb{D}_{1,2}$  does not have a Sobolev space structure, and thus as we mentioned in the previous section all the proofs of these properties will depend solely on the chaos expansion. Therefore sometimes they are more complicated than those of the Brownian case.

Let us first introduce some notation. Define for each  $n \in \mathbb{N}$ , a vector space

$$(5.1) \quad S_n := \text{span} \left\{ \overline{\otimes^n 1_{I^i}(\cdot)}^s \mid I^i = (a_i, b_i]; a_1 < b_1 \leq a_2 < b_2 \leq \dots \leq a_n < b_n \right\}.$$

It is clear that  $S_n$  is dense in both  $L^2(\Sigma_n)$  and  $L^2_s(T^n)$ . Define for  $f, g \in L^2(T^n)$ ,

$$(5.2) \quad \langle f, g \rangle_n = \int_{T^n} f(t_1, \dots, t_n) g(t_1, \dots, t_n) dt_1 \cdots dt_n.$$

We define a measure  $\mu$  on the space  $T \times \Omega$  in terms of the underlying martingale  $M$  (known as the Doléans-Dade measure): for  $t \in [0, 1]$  and  $B \in \mathcal{F}$ ,

$$(5.3) \quad \mu([0, t] \times B) := E\{1_B[M, M]_t\}.$$

Then it is clear that  $L^2(d\mu) = L^2(dt \times dP)$  if and only if  $[M, M]_t = t$ . In other words, in the Brownian case  $L^2(d\mu)$  is superfluous. Let us denote for any  $F, G \in L^2(d\mu)$  that

$$\langle F, G \rangle_{L^2(d\mu)} = E \int_0^1 F_t G_t d[M, M]_t,$$

and for  $n, m \in \mathbb{N}$  and  $f \in S_n, g \in S_m$  that

$$(5.4) \quad (f, g)_{n,m} = \langle I_{n-1}(f(\cdot, \cdot)), I_{m-1}(g(\cdot, \cdot)) \rangle_{L^2(d\mu)}.$$

When  $n = m$ , we denote  $(f, g)_{n,n} = (f, g)_n$ . We first give a lemma whose proof can be found in Russo and Vallois (1994).

**Lemma 5.1.** *Suppose that  $f \in S_n$  and  $g \in S_1$ . Then it holds that*

$$I_n(f)I_1(g) = I_{n+1}(f \otimes g) + n \int_0^\infty I_{n-1}(f(\cdot, t)g(t)d[M, M]_t.$$

Next, we prove the following *isometry* property:

**Lemma 5.2.** *If  $f \in S_n$  and  $g \in S_m, n, m \in \mathbb{N}$ , then it holds that*

$$(f, g)_{n,m} = \begin{cases} 0, & n \neq m; \\ (n-1)! \langle f, g \rangle_n, & n = m. \end{cases}$$

*Proof.* We split the proof into two different cases.

*Case 1. ( $n = m$ ):* It suffices to consider the case when  $f = g$ , as the general case can be derived by polarization. In other words, we shall prove that

$$(5.5) \quad (f, f)_n = (n-1)! \|f\|_n^2, \quad n = 1, 2, \dots.$$

Further, it is clear that we need only consider those functions

$$f(t_1, \dots, t_k, t) = \overline{\left( \prod_{i=1}^k 1_{I^i}(t_i) \right)} 1_{I^{k+1}}(t), \quad I^i = (a_i, b_i], i = 1, \dots, k+1,$$

where  $a_1 < b_1 \leq a_1 < b_2 \leq \dots \leq a_k < b_k \leq a_{k+1} < b_{k+1}$ . We proceed by induction. For  $n = 1, I_0(f(t)) = f(t)$ , so there is nothing to prove. Suppose that (5.5) is true for  $n = k$ , and consider the case  $n = k + 1$ . Note that  $n - 1 = k$  and

$$(5.6) \quad \begin{aligned} I_k(f(\cdot, t)) &= k! \int_{0 < t_1 < \dots < t_{k-1} < s < 1} f(t_1, \dots, t_{k-1}, s, t) dM_{t_1} \dots dM_{t_{k-1}} dM_s \\ &= k \left\{ \int_0^{t-} I_{k-1}(f(\cdot, s, t) 1_{[0,s]}^{\otimes k-1}) dM_s + \int_{t+}^\infty I_{k-1}(f(\cdot, s, t) 1_{[0,s]}^{\otimes k-1}) dM_s \right\} \\ &= V_1(t) + V_2(t). \end{aligned}$$



Here we used the fact that  $f(\cdot, t, t) = 0$ ,  $\forall t \in T$  and  $f(\cdot, s, t) = 0$ , for  $s > 1$ . Since for  $(t_1, \dots, t_{k-1}, s) \in C_k$ , and  $s < t$ , we have

$$f(t_1, \dots, t_{k-1}, s, t) = \prod_{i=1}^{k-1} 1_{(a_i, b_i]}(t_i) 1_{(a_k, b_k]}(s) 1_{(a_{k+1}, b_{k+1})}(t),$$

it holds that

$$(5.7) \quad I_{k-1}(f(\cdot, s, t) 1_{[0, s]}^{\otimes k-1}) = I_{k-1}(\widehat{f}(\cdot, s) 1_{[0, s]}^{\otimes k-1}(\cdot)) 1_{I^{k+1}}(t) = g_k(s) 1_{I^{k+1}}(t),$$

where  $\widehat{f}(t_1, \dots, t_{k-1}, s) := \prod_{i=1}^{k-1} 1_{(a_i, b_i]}(t_i) 1_{(a_k, b_k]}(s)$ ;  $g_k(s) := I_{k-1}(\widehat{f}(\cdot, s) 1_{[0, s]}^{\otimes k-1}(\cdot))$ ; and  $I^{k+1} = (a_{k+1}, b_{k+1}]$ . It is easily seen that  $g_k(\cdot)$  is predictable, and so is  $1_{I^{k+1}}(\cdot) \int_0^{\cdot} g_k^2(s) ds$ . Noting that  $E[M, M]_t = t$ , we have

$$(5.8) \quad \begin{aligned} \|V_1(\cdot)\|_{L^2(d\mu)}^2 &= \int_0^\infty EV_1(t)^2 dt = k^2 \int_0^\infty E \left\{ \int_0^{t-} 1_{I^{k+1}}(t) g_k(s) dM_s \right\}^2 dt \\ &= k^2 E \int_0^\infty \left\{ \int_0^t 1_{I^{k+1}}(t) g_k^2(s) ds \right\} d[M, M]_t. \end{aligned}$$

In order to determine a similar evaluation for  $\|V_2\|_{L^2(d\mu)}^2$ , let us define a stopping time  $\tau_t := \inf\{s \geq 0 : [M, M]_s > t\} \wedge 1$ . Then we have

$$(5.9) \quad \begin{aligned} \|V_2(\cdot)\|_{L^2(d\mu)}^2 &= k^2 E \int_0^\infty \left\{ \int_{(t, \infty)} 1_{I^{k+1}}(t) g_k(s) dM_s \right\}^2 d[M, M]_t \\ &= k^2 E \int_0^\infty 1_{I^{k+1}}(\tau_t) \left\{ \int_{\tau_t+}^\infty g_k(s) dM_s \right\}^2 dt \\ &= k^2 \int_0^\infty E \left\{ \int_{\tau_t+}^\infty 1_{I^{k+1}}(\tau_t) g_k(s) dM_s \right\}^2 dt. \end{aligned}$$

Define for fixed  $t$  a process  $G(t, s, \omega) = 1_{I^{k+1}}(\tau_t(\omega)) g_k(s, \omega)$ , then  $G(t, \cdot, \cdot)$  is predictable for  $s > \tau_t$ . Thus

$$E \left\{ \int_{\tau_t+}^\infty 1_{I^{k+1}}(\tau_t) g_k(s) dM_s \right\}^2 = E \int_{\tau_t+}^\infty 1_{I^{k+1}}(\tau_t) g_k^2(s) ds.$$

Therefore the right side of (5.9) becomes

$$k^2 \int_0^\infty E \int_{\tau_t+}^\infty 1_{I^{k+1}}(\tau_t) g_k^2(s) ds dt = k^2 E \int_0^\infty \left\{ \int_{t+}^\infty 1_{I^{k+1}}(t) g_k^2(s) ds \right\} d[M, M]_t.$$

Combining this with (5.8) and (5.9), and applying Fubini's Theorem to  $d[sd[M, M]]_t$  (path by path), we have (recall (5.7) for the definition of  $g_k$ ) that

$$(5.10) \quad \begin{aligned} \|V_1\|_{L^2(d\mu)}^2 + \|V_2\|_{L^2(d\mu)}^2 &= k^2 E \int_0^\infty \left\{ \int_0^\infty 1_{I^{k+1}}(t) g_k^2(s) ds \right\} d[M, M]_t \\ &= k^2 E \int_0^\infty \int_0^\infty I_{k-1}^2(f(\cdot, s, t) 1_{[0,s]}^{\otimes k-1}(\cdot)) d[M, M]_t ds. \end{aligned}$$

Moreover, by (5.8) and (5.9) we also see that

$$\begin{aligned} \|V_1\|_{L^2(d\mu)}^2 &= k^2 E \int_0^\infty 1_{I^{k+1}}(t) \left\{ \int_0^t g_k^2(s) ds \right\} dt \\ &\leq k^2 (k-1)! \int_0^\infty \int_0^t \|f(\cdot, s, t) 1_{[0,s]}^{\otimes k-1}(\cdot)\|_{k-1}^2 ds dt \\ &\leq k k! \|f\|_{k+1}^2 < \infty; \end{aligned}$$

and, again by Fubini's theorem, that

$$(5.11) \quad \begin{aligned} \|V_2\|_{L^2(d\mu)}^2 &= k^2 E \int_0^\infty \left\{ \int_{t+}^\infty 1_{I^{k+1}}(t) g_k^2(s) ds \right\} d[M, M]_t \\ &= k^2 E \int_0^\infty \left\{ \int_{[0,s]}^\infty 1_{I^{k+1}}(t) g_k^2(s) d[M, M]_t \right\} ds \\ &= k^2 \int_0^\infty E \left\{ \int_0^\infty I_{k-1}^2(f(\cdot, s, t) 1_{[0,s]}^{\otimes k-1}(\cdot)) 1_{[0,s]}(t) d[M, M]_t \right\} ds, \end{aligned}$$

by the definition of  $g_k$ . Therefore, if we define for each fixed  $s$  a function  $h_s(\cdot, t) = f(\cdot, s, t) 1_{[0,s]}^{\otimes k-1}(\cdot) 1_{[0,s]}(t)$ , then  $h_s(\cdot, t)$  is symmetric in the first  $k-1$  variables. Using the induction hypothesis, we have

$$(5.12) \quad (h_s, h_s)_k = E \int_0^\infty I_{k-1}^2(f(\cdot, s, t) 1_{[0,s]}^{\otimes k-1}(\cdot)) 1_{[0,s]}(t) d[M, M]_t = (k-1)! \|h_s\|_k^2.$$

Combining (5.12) with (5.11), we obtain that

$$(5.13) \quad \|V_2\|_{L^2(d\mu)}^2 = k^2 (k-1)! \int_0^\infty \|h_s\|_k^2 ds \leq k^2 (k-1)! \|f\|_{k+1}^2 < \infty.$$

In other words, we have shown that  $V_1, V_2 \in L^2(d\mu)$ , whence  $V_1 V_2 \in L^1(d\mu)$ . We now show that

$$(5.14) \quad E \int_0^\infty V_1(t) V_2(t) d[M, M]_t = 0.$$

In fact, using Lebesgue's change of time lemma (cf. e.g., Dellacherie (1972, p.91)),

$$\infty > E \int_0^\infty |V_1(t)V_2(t)|d[M, M]_t = E \int_0^\infty |V_1(\tau_t)V_2(\tau_t)|dt = \int_0^\infty E|V_1(\tau_t)V_2(\tau_t)|dt,$$

we see that for a.e.  $t \in [0, 1]$ ,  $E|V_1(\tau_t)V_2(\tau_t)| < \infty$ . Hence

$$\begin{aligned} E\{V_1(\tau_t)V_2(\tau_t)\} &= E\{V_1(\tau_t)E\{V_2(\tau_t)|\mathcal{F}_{\tau_t}\}\} \\ &= E\left\{V_1(\tau_t)E\left\{\int_{\tau_t+}^\infty 1_{I^{k+1}}(\tau_t)g_k(s)dM_s\middle|\mathcal{F}_{\tau_t}\right\}\right\}. \end{aligned}$$

Since  $I^{k+1}(\tau_t)g_k(s)$  is predictable on  $s > \tau_t$  for a.a.  $t$ , we have  $EV_1(\tau_t)V_2(\tau_t) = 0$  for a.a.  $t$ , whence

$$E \int_0^\infty V_1(t)V_2(t)d[M, M]_t = \int_0^\infty E\{V_1(\tau_t)V_2(\tau_t)\}dt = 0,$$

proving (5.14). Now by (5.6), (5.10) and (5.14) we get

$$\begin{aligned} (5.15) \quad \|I_k(f(\cdot, \cdot))\|_{L^2(d\mu)}^2 &= \|V_1(\cdot) + V_2(\cdot)\|_{L^2(d\mu)}^2 = \|V_1\|_{L^2(d\mu)}^2 + \|V_2\|_{L^2(d\mu)}^2 \\ &= k^2 \int_0^\infty E\left\{\int_0^\infty I_{k-1}^2(f(\cdot, s, t)1_{[0,s]}^{\otimes k-1}(\cdot))d[M, M]_t\right\}ds := k^2 J. \end{aligned}$$

Using the induction hypothesis again, we see that

$$E\left\{\int_0^\infty I_{k-1}^2(f(\cdot, s, t)1_{[0,s]}^{\otimes k-1}(\cdot))d[M, M]_t\right\} = (k-1)! \|f(\cdot, s, \cdot)1_{[0,s]}^{\otimes k-1}(\cdot)\|_k^2,$$

Thus

$$\begin{aligned} J &= \int_0^\infty (k-1)! \|f(\cdot, s, \cdot)1_{[0,s]}^{\otimes k-1}(\cdot)\|_k^2 ds \\ &= (k-1)! \int_0^\infty \left\{ \int_0^\infty \left[ \int_0^s \cdots \int_0^s |f(t_1, \dots, t_{k-1}, s, t)|^2 dt_1 \cdots dt_{k-1} \right] dt \right\} ds \\ &= (k-1)!(k-1)! \int_0^\infty \left\{ \int_{0 < t_1 < \cdots < t_{k-1} < s < \infty} |f(t_1, \dots, t_{k-1}, s, t)|^2 dt_1 \cdots dt_{k-1} ds \right\} dt \\ &= ((k-1)!)^2 \cdot \frac{1}{k!} \int_0^\infty \left\{ \int_{T^k} |f(t_1, \dots, t_{k-1}, s, t)|^2 dt_1 \cdots dt_{k-1} ds \right\} dt \\ &= \frac{((k-1)!)^2}{k!} \|f\|_{k+1}^2. \end{aligned}$$

Therefore

$$\|I_k(f(\cdot, \cdot))\|_{L^2(d\mu)}^2 = k^2 J = k^2 \frac{((k-1)!)^2}{k!} \|f\|_{k+1}^2 = k! \|f\|_{k+1}^2,$$

and (5.5) is proved.

*Case 2. ( $n \neq m$ ):* The proof is very similar, we give only a sketch. We shall again use induction, but this time on  $k = n \wedge m$ . Assume that  $k = 1$ . Let  $n = 1$  and  $m \geq 2$ . Then notice that  $I_0(f(\cdot, t)) = f(t)$ , we have

$$(f, g)_{1,m} = E \int_0^\infty I_0(f(\cdot, t)) I_{m-1}(g(\cdot, t)) d[M, M]_t = E \int_0^\infty I_{m-1}(g(\cdot, t)) f(t) d[M, M]_t.$$

Since  $g \in S_m$  and  $f \in S_1$ , we have by Lemma 5.1 that

$$m \int_0^\infty I_{m-1}(g(\cdot, t)) f(t) d[M, M]_t = I_m(g) I_1(f) - I_{m+1}(g \otimes h).$$

Therefore  $(f, g)_{1,m} = \frac{1}{m} \{E I_m(g) I_1(f) - E I_{m+1}(g \otimes f)\} = 0$ . Next assume that the conclusion is true for  $n \wedge m = k$ , consider the case when  $n = k$  and  $m \geq k + 1$ .

$$\begin{cases} V_1(t) = (n-1) \int_0^{t-} 1_{I^{n+1}}(t) \widehat{f}_n(s) dM_s; & V_2(t) = (n-1) \int_{t+}^\infty 1_{I^{n+1}}(t) \widehat{f}_n(s) dM_s; \\ U_1(t) = (m-1) \int_0^{t-} 1_{I^{m+1}}(t) \widehat{g}_m(s) dM_s; & U_2(t) = (m-1) \int_{t+}^\infty 1_{I^{m+1}}(t) \widehat{g}_m(s) dM_s, \end{cases}$$

where

$$\widehat{f}_n(s) = I_{n-2} \left( \prod_{i=1}^{n-1} 1_{I^i}(t_i) 1_{I^n}(s) 1_{[0,s]}^{\otimes n-2} \right); \quad \widehat{g}_m(s) = I_{m-2} \left( \prod_{j=1}^{m-1} 1_{I^j}(t_j) 1_{I^m}(s) 1_{[0,s]}^{\otimes m-2} \right).$$

Then it is easily seen as before that

$$\begin{aligned} & E \int_0^\infty I_{n-1}(f(\cdot, t)) I_{m-1}(g(\cdot, t)) d[M, M]_t \\ &= E \int_0^\infty (V_1(t) + V_2(t))(U_1(t) + U_2(t)) d[M, M]_t \\ &= E \int_0^\infty [V_1 U_1(t) + V_1(t) U_2(t) + V_2(t) U_1(t) + V_2(t) U_2(t)] d[M, M]_t. \end{aligned}$$

Using the same argument as that for (5.14), we have

$$E \int_0^\infty V_1(t) U_2(t) d[M, M]_t = E \int_0^\infty V_2(t) U_1(t) d[M, M]_t = 0.$$

By the predictability of  $V_1$  and  $U_1$ , we have

$$\begin{aligned} & E \int_0^\infty V_1(t) U_2(t) d[M, M]_t = E \int_0^\infty V_1(t) U_1(t) dt = \int_0^\infty E \{V_1(t) U_1(t)\} dt \\ &= (n-1)(m-1) \int_0^\infty 1_{I^{n+1}}(t) 1_{I^{m+1}}(t) E \left\{ \int_0^{t-} \widehat{f}(s) \widehat{g}(s) ds \right\} dt = 0. \end{aligned}$$

Finally, note that

$$(5.16) \quad E \int_0^\infty V_2(t)U_2(t)d[M, M]_t = \int_0^\infty E\{V_1(\tau_t)U_2(\tau_t)\}dt,$$

and

$$V_2(\tau_t) = \int_0^\infty 1_{I^{n+1}}(\tau_t)\widehat{f}(s)1_{s>\tau_t}dM_s; \quad U_2(\tau_t) = \int_0^\infty 1_{I^{m+1}}(\tau_t)\widehat{g}(s)1_{\{s>\tau_t\}}dM_s.$$

Since both integrands are predictable, we have

$$(5.17) \quad \begin{aligned} & \int_0^\infty E\{V_2(\tau_t)U_2(\tau_t)\}dt \\ &= \int_0^\infty E\left\{\int_0^\infty 1_{I^{n+1}}(\tau_t)\widehat{f}(s)1_{I^{m+1}}(\tau_t)\widehat{g}(s)1_{\{s>\tau_t\}}ds\right\}dt \\ &= E \int_0^\infty \int_0^\infty 1_{I^{n+1}}(t)\widehat{f}(s)1_{I^{m+1}}(t)\widehat{g}(s)1_{\{s>t\}}dsd[M, M]_t. \end{aligned}$$

Combining the facts  $f(\cdot, t, t) = g(\cdot, t, t) = 0$  and that

$$E \int_0^\infty \int_0^\infty 1_{I^{n+1}}(t)\widehat{f}(s)1_{I^{m+1}}(t)\widehat{g}(s)1_{\{s<t\}}dsd[M, M]_t = E\{V_1(t)U_1(t)\} = 0;$$

with the induction hypothesis we conclude

$$(5.18) \quad E \int_0^\infty 1_{I^{n+1}}(t)\widehat{f}(s)1_{I^{m+1}}(t)\widehat{g}(s)d[M, M]_t = 0, \quad \text{a.e. } s.$$

Finally we see from (5.16), (5.17) and (5.18) that

$$\begin{aligned} E \int_0^\infty V_2(t)U_2(t)d[M, M]_t &= E \int_0^\infty \int_0^\infty 1_{I^{n+1}}(\tau_t)\widehat{f}(s)1_{I^{m+1}}(\tau_t)\widehat{g}(s)dsd[M, M]_t \\ &= \int_0^\infty E\left\{\int_0^\infty 1_{I^{n+1}}(t)\widehat{f}(s)1_{I^{m+1}}(t)\widehat{g}(s)d[M, M]_t\right\}ds = 0. \end{aligned}$$

The proof is complete.  $\square$

Lemma 5.2 indicates an interesting fact, that is, if  $f_n \in S_{n+1}$ , then  $I_n(f_n(\cdot, t)) \in L^2(d\mu) \cap L^2(d\lambda)$ , where  $d\lambda = dt \times dP$ , and the sequence  $\{I_n(f_n(\cdot, t))\}$  is orthogonal in both  $L^2(d\mu)$  and  $L^2(d\lambda)$ , such that

$$\|I_n(f(\cdot, \cdot))\|_{L^2(d\mu)}^2 = \|I_n(f(\cdot, \cdot))\|_{L^2(d\lambda)}^2 = n!\|f\|_{n+1}^2.$$

Therefore, since  $S_{n+1}$  is dense in  $L_s^2(T^{n+1})$ , for any  $f \in L_s^2(T^{n+1})$ , we can take a sequence  $\{f^k\}$  converging to  $f$  in  $L^2(T^{n+1})$ , such that  $f^k \in S_{n+1}$ ,  $k = 1, 2, \dots$ . Hence we can define two limit processes:

$$\begin{aligned} F(t) &= \lim_{k \rightarrow \infty} I_n(f^k(\cdot, t)), & \text{in } L^2(d\mu); \\ G(t) &= \lim_{k \rightarrow \infty} I_n(f^k(\cdot, t)), & \text{in } L^2(d\lambda), \end{aligned}$$

such that  $\|F\|_{L^2(d\mu)}^2 = \|G\|_{L^2(d\lambda)}^2 = n! \|f\|_{n+1}^2$ . We now define a new measure on  $[0, 1] \times \Omega$  by

$$(5.19) \quad \nu([0, t] \times B) = \frac{\lambda([0, t] \times B) + \mu([0, t] \times B)}{2}, \quad t \in [0, 1], B \in \mathcal{F};$$

and

$$(5.20) \quad I_n^*(f(\cdot, t)) := \lim_{k \rightarrow \infty} I_n(f^{(k)}(\cdot, t)), \quad \text{in } L^2(d\nu).$$

Then  $I_n^*(f(\cdot, t))$  is well-defined, and it satisfies

$$I_n^*(f(\cdot, t)) = \begin{cases} F(t) & \text{a.e. } d\mu; \\ G(t) & \text{a.e. } d\lambda. \end{cases}$$

We now turn our attention to the *integration by parts* formula. Recall in the Brownian case we have, for any  $F, G \in \mathbb{D}_{1,2}$  and  $h \in H := L^2(T)$ ,

$$(5.21) \quad E\{G \langle DF, h \rangle_T + F \langle DG, h \rangle_T\} = E\left\{FG \int_0^1 h(t) dW_t\right\},$$

where  $W$  is the standard Wiener process. We will show an analogue of (5.21) in our case. First we give a theorem that is inspired by a result in Russo and Vallois (1994). (Note that the result of Russo and Vallois is slightly incorrect, since  $I_{n-1}$  replaces  $I_{n-1}^*$ .)

**Theorem 5.3.** *Suppose that  $n \geq 1$ ,  $f \in L_s^2(T^n)$  and  $h \in L^2(T)$ , then it holds that*

$$(5.22) \quad I_n(f)I_1(h) = I_{n+1}(f \otimes h) + n \int_0^\infty I_{n-1}^*(f(\cdot, t))h(t)d[M, M]_t.$$

*Proof.* We follow the idea of Russo and Vallois. First assume that  $f \in S_n$  and  $h \in L^2(T)$ . By Lemma 5.1 we have

$$(5.23) \quad I_n(f)I_1(h) = I_{n+1}(f \otimes h) + n \int_0^\infty I_{n-1}(f(\cdot, t))h(t)d[M, M]_t.$$

Now let  $f \in L_s^2(T^n)$  and  $h \in L^2(T)$ , we can find a sequence  $\{f^k\} \subset S_n$  such that  $\lim_{k \rightarrow \infty} f^k = f$  in  $L_s^2(T^n)$ . By Lemma 5.2 and the discussion following it, we see that

$$\begin{cases} \lim_{k \rightarrow \infty} I_n(f^k) = I_n(f); & \text{in } L^2(\Omega); \\ \lim_{k \rightarrow \infty} I_{n+1}(f^k \otimes h) = I_{n+1}(f \otimes h); & \text{in } L^2(\Omega); \\ \lim_{k \rightarrow \infty} I_{n-1}(f^k(\cdot, \cdot)) = I_{n-1}^*(f(\cdot, \cdot)); & \text{in } L^2(d\mu). \end{cases}$$

Further, noting that  $E \int_0^1 |h(t)|^2 d[M, M]_t = \int_0^1 |h(t)|^2 dt = \|h\|_1^2$ , we have by the Cauchy-Schwartz inequality that

$$\begin{aligned} & E \left| \int_0^1 I_{n-1}(f^k(\cdot, t))h(t)d[M, M]_t - \int_0^1 I_{n-1}^*(f(\cdot, t))h(t)d[M, M]_t \right| \\ & \leq E \int_0^1 |I_{n-1}(f^k(\cdot, t)) - I_{n-1}^*(f(\cdot, t))| |h(t)| d[M, M]_t \\ & \leq \|I_{n-1}(f^k(\cdot, \cdot)) - I_{n-1}^*(f(\cdot, \cdot))\|_{L^2(d\mu)} \|h\|_1 \rightarrow 0. \end{aligned}$$

Therefore, taking limits on both sides of (5.23) (replacing  $f$  by  $f^k$ 's) in  $L^1(\Omega)$ , we obtain (5.22), proving the theorem.  $\square$

Let us now define an operator  $D^* : L^2(\Omega) \mapsto L^2(d\nu)$  by

$$D_t^* I_n(f) = n I_{n-1}^*(f(\cdot, t)), \quad \forall f \in L_s^2(T^n).$$

and for  $F = \sum_{n=0}^{\infty} I_n(f_n)$ , such that  $\sum_{n=0}^{\infty} n n! \|f_n\|_n^2 < \infty$ , we define

$$(5.24) \quad D_t^* F = \sum_{n=0}^{\infty} n I_{n-1}^*(f_n(\cdot, t)), \quad t \in [0, 1].$$

We have the following analogue of (5.21).

**Theorem 5.4.** *Suppose that  $G \in L^2(\Omega)$  has the chaos expansion  $G = \sum_{n=0}^{\infty} I_n(g_n)$ , where  $g_n \in L_s^2(T^n)$  and  $\sum_{n=0}^{\infty} \sqrt{n n!} \|g_n\|_n < \infty$ . Then for any  $F \in \mathbb{D}_{1,2}$  and  $h \in L^2(T)$ , it holds that*

$$(5.25) \quad E\{G \langle DF, h \rangle_T\} = E\left\{F\left(G \int_0^1 h(t) dM_t - \int_0^1 D_t^* G h(t) d[M, M]_t\right)\right\}.$$

*Proof.* Let  $n \geq 1$  be fixed. Since  $g_n \in L_s^2(T^n)$ , by Theorem 5.2 we have

$$I_n(g_n) I_1(h) = I_{n+1}(g_n \otimes h) + n \int_0^1 I_{n-1}^*(g_n(\cdot, t)) h(t) d[M, M]_t.$$

For any  $k, m \in \mathbb{N}$ , we have by the Cauchy-Schwartz inequality that

$$\begin{aligned}
& E \int_0^1 \left| \sum_{n=1}^m n I_{n-1}^*(g_n(\cdot, t)) - \sum_{n=1}^k n I_{n-1}^*(g_n(\cdot, t)) \right| |h(t)| d[M, M]_t \\
& \leq \sum_{n=k+1}^m E \int_0^1 |n I_{n-1}^*(g_n(\cdot, t))| |h(t)| d[M, M]_t \\
& \leq \sum_{n=k+1}^m n \sqrt{(n-1)!} \|g_n\|_n \|h\|_1 \rightarrow 0, \quad \text{as } k, m \rightarrow \infty,
\end{aligned}$$

by assumption (note that  $n\sqrt{(n-1)!} = \sqrt{nn!}$ ). Therefore,  $\sum_{n=0}^k n I_{n-1}^*(g_n(\cdot, \cdot))h(\cdot) \rightarrow D^*Gh(\cdot)$ , as  $k \rightarrow \infty$ , in  $L^1(d\mu)$ . Consequently,

$$\begin{aligned}
G \int_0^1 h(t) dM_t &= \sum_{n=0}^{\infty} I_n(g_n) I_1(h) \\
&= \sum_{n=0}^{\infty} I_{n+1}(g_n \otimes h) + \int_0^1 \sum_{n=1}^{\infty} n I_{n-1}^*(g_n(\cdot, t)) h(t) d[M, M]_t \\
&= \sum_{n=0}^{\infty} I_{n+1}(g_n \otimes h) + \int_0^1 D_t^* Gh(t) d[M, M]_t, \quad \text{in } L^1(\Omega).
\end{aligned}$$

or equivalently,

$$(5.26) \quad \sum_{n=0}^{\infty} I_{n+1}(g \otimes h) = G \int_0^1 h(t) dM_t - \int_0^1 D_t^* Gh(t) d[M, M]_t.$$

Note that since the left side of (5.26) is in  $L^2(\Omega)$ , so also is the right-side. Thus for any  $F \in \mathbb{D}_{1,2} \subset L^2(\Omega)$ , we have

$$(5.27) \quad F \left\{ \sum_{n=0}^{\infty} I_{n+1}(g \otimes h) \right\} = F \left\{ G \int_0^1 h(t) dM_t - \int_0^1 D_t^* Gh(t) d[M, M]_t \right\},$$

and the equality holds in  $L^1(\Omega)$ . If we write  $F = \sum_{k=0}^{\infty} I_k(f_k)$  for its chaos expansion,



then  $\sum_{k=0}^{\infty} k k! \|f_k\|_k^2 < \infty$  since  $F \in \mathbb{D}_{1,2}$ , therefore

$$\begin{aligned}
(5.28) \quad E\left\{F \sum_{n=0}^{\infty} I_{n+1}(g_n \otimes h)\right\} &= E\left\{\sum_k I_k(f_k) \cdot \sum_n I_{n+1}(g_n \otimes h)\right\} \\
&= \sum_k E\{I_k(f_k) I_k(g_{k-1} \otimes h)\} \\
&= \sum_k k! \langle f_k, g_{k-1} \otimes h \rangle_{T^k} \\
&= \sum_k (k-1)! \int_0^1 \langle k f_k(\cdot, t), g_{k-1} \rangle_{T^{k-1}} h(t) dt \\
&= \int_0^1 \sum_k E(k I_{k-1}(f_k(\cdot, t)), I_{k-1}(g_{k-1})) h(t) dt \\
&= \int_0^1 E\left\{\sum_k k I_{k-1}(f_k(\cdot, t)) \cdot \sum_n I_n(g_n)\right\} h(t) dt \\
&= E \int_0^1 D_t F G h(t) dt = E\{G \langle D_t F, h \rangle_T\}.
\end{aligned}$$

Taking expectations on both sides of (5.27), and replacing its left side by (5.28), we obtain (5.25). The Theorem is proved.  $\square$

**Corollary 5.5.** Suppose that  $F, G \in L^2(\Omega)$ ;  $F = \sum_n I_n(f_n)$ ,  $G = \sum_k I_k(g_k)$ , such that

$$(5.29) \quad \sum_n \sqrt{nn!} \|f_n\|_n < \infty; \quad \sum_k \sqrt{kk!} \|g_k\|_k < \infty.$$

Assume that  $h \in L^2(T)$  and  $E|FG \int_0^1 h(t) dM_t| < \infty$ , then

$$(5.30) \quad E\left\{FG \int_0^1 h(t) dM_t\right\} = \int_{[0,1] \times \Omega} \{FD_t^* G + GD_t^* F\} h(t) d\nu.$$

In particular, if  $M$  is a Brownian motion, then (5.30) becomes (5.21).

*Proof.* First note that if  $F = \sum_n I_n(f_n)$  and (5.29) holds, then  $F \in \mathbb{D}_{1,2}$ . Indeed, for  $n$  large enough, one has  $nn! \|f_n\|_n^2 \leq \sqrt{nn!} \|f_n\|_n < 1$ . Thus  $\sum_n nn! \|f_n\|_n^2 < \infty$  and  $F \in \mathbb{D}_{1,2}$ . In other words, under condition (5.29),  $F, G \in \mathbb{D}_{1,2}$ .

We can now apply Theorem 5.4 twice with the positions of  $F$  and  $G$  being switched, and add the resulting equalities together to get

$$\begin{aligned}
(5.31) \quad E\{G \langle DF, h \rangle_T + F \langle DG, h \rangle_T\} \\
= E\left\{2FG \int_0^1 h(t) dM_t - \int_0^1 [FD_t^* G + GD_t^* F] h(t) d[M, M]_t\right\}.
\end{aligned}$$

Since  $FG \int_0^1 h(t) dM_t \in L^1(\Omega)$  by assumption, we obtain from (5.31) that

$$\int_0^1 [FD_t^*G + GD_t^*F]h(t)d[M, M]_t \in L^1(\Omega).$$

Further, notice that  $D^*F = D.F$ ;  $D^*G = D.G$ , a.e.  $dt \times dP$ , we deduce from (5.31) that

$$\begin{aligned} E\left\{FG \int_0^1 h(t)dM_t\right\} &= \frac{1}{2}E\left\{\int_0^1 [FD_t^*G + GD_t^*F]h(t)(d[M, M]_t + dt)\right\} \\ &= \int_{[0,1] \times \Omega} [FD_t^*G + GD_t^*F]h(t)d\nu, \end{aligned}$$

proving the first assertion. In the case when  $M$  is a Brownian motion, we have  $[M, M]_t = \langle M, M \rangle_t = t$ . Hence  $d\nu = dt \times dP$ ,  $D_t^*F = D_tF$  and  $D_t^*G = D_tG$ . Consequently,

$$\begin{aligned} \int_{[0,1] \times \Omega} [FD_t^*G + GD_t^*F]h(t)d\nu &= E\left\{\int_0^1 [FD_tG + GD_tF]h(t)dt\right\} \\ &= E\{F < DG, h >_T + G < DF, h >_T\}. \end{aligned}$$

The proof is complete. □

We remark that in Theorem 5.4 and Corollary 5.5 no anticipating integral is involved. To conclude this section we shall present another formula that does involve the “anticipating integrals”. In the Brownian case, this formula is nothing but the definition of an “adjoint relation” (see (4.3)). Our formula involves the measure  $d[M, M]_t$ , therefore the anticipating integral for a special class of processes involving  $I_n^*$ ’s has to be considered.

Let us consider a class of processes of the form  $u_t = \sum_{m=0}^{\infty} I_m^*(f_m(\cdot, t))$ , where  $f_m \in L_s^2(T^{m+1})$  and the convergence of the series is in the sense of  $L^2(d\nu)$ . We say that a process of this kind is of class  $\mathcal{L}^*$ , if

$$\sum_{m=0}^{\infty} (m+1)! \|f_m\|_{L^2(T^{m+1})}^2 < \infty.$$

Denote by  $\text{Dom}(\delta^*)$  the set of all processes of class  $\mathcal{L}^*$ . We consider the anticipating integral for processes in  $\text{Dom}(\delta^*)$ . First note that for each  $n$  we can find a sequence  $\{f_m^k\} \subset S_{m+1}$  such that  $\lim_{k \rightarrow \infty} f_m^k = f_m$  in  $L^2(T^{m+1})$ . Define for every pair of integers  $N$  and  $k$  a process

$$u_t^{N,k} = \sum_{m=0}^N I_m^*(f_m^k(\cdot, t)) = \sum_{m=0}^N I_m(f_m^k(\cdot, t)), \quad \forall t \in [0, 1].$$

Then by definition (5.20), we see that  $\lim_{N \rightarrow \infty} \lim_{k \rightarrow \infty} u^{N,k} = u.$ , in  $L^2(d\nu)$ . On the other hand, for fixed  $N$  and  $k$  we have  $u^{N,k} \in \text{Dom}(\delta)$  and  $\delta(u^{N,k}) = \sum_{m=0}^N I_{m+1}(f_m^k)$  by definition (note that  $f_m^k$ 's are symmetric), thus if  $u$  is of class  $\mathcal{L}^*$ , then

$$\lim_{N \rightarrow \infty} \lim_{k \rightarrow \infty} \delta(u^{N,k}) = \lim_{N \rightarrow \infty} \lim_{k \rightarrow \infty} \sum_{m=0}^N I_{m+1}(f_m^k) = \sum_{m=0}^{\infty} I_{m+1}(f_m), \quad \text{in } L^2(\Omega).$$

We define the anticipating integral of  $u \in \mathcal{L}^*$  to be

$$(5.32) \quad \delta^*(u) := \sum_{m=0}^{\infty} I_{m+1}(f_m).$$

Clearly, the value of  $\delta^*(u)$  is independent of the choice of the approximating sequence, and it coincides with  $\delta(u)$  if in the expansion of  $u$ , all  $f_m \in S_{m+1}$ . Using the notions  $\delta^*$  and  $D^*$ , we now give a new integration by parts formula.

**Theorem 5.6.** *Suppose that  $u \in \text{Dom}(\delta^*)$ , and  $G \in \mathbb{D}_{1,2}$ . Then it holds that*

$$(5.33) \quad E\{\delta^*(u)G\} = E\left\{\int_0^1 u_t(D_t^*G)d[M, M]_t\right\} = E\left\{\int_0^1 u_t(D_t^*G)dt\right\}.$$

Consequently,

$$(5.34) \quad E\{\delta^*(u)G\} = \int_{[0,1] \times \Omega} u_t(\omega)(D_t^*G)(\omega)\nu(dt \times d\omega).$$

*Proof.* First we note that (5.34) follows from (5.33) by the definition (5.19), therefore we need to show only (5.33). To begin with, let us choose for each  $n$  and  $m$  two sequences  $\{f_m^k\}_{k=1}^{\infty} \subset S_{m+1}$ , and  $\{g_n^\ell\} \subset S_n$ , such that  $\lim_{k \rightarrow \infty} f_m^k = f_m$  in  $L^2(T^{m+1})$  and  $\lim_{\ell \rightarrow \infty} g_n^\ell = g_n$  in  $L^2(T^n)$ . Now for given integers  $N, k$  and  $\ell$  we define

$$u_t^{N,k} = \sum_{m=0}^N I_m(f_m^k(\cdot, t)); \quad G^{N,\ell} = \sum_{n=0}^N I_n(g_n^\ell).$$

It is easily seen by definitions (5.32) (for  $\delta^*$ ), (5.24) (for  $D^*$ ) and (5.20) (for  $I_n^*$ 's) that

$$\left\{ \begin{array}{l} \lim_{N \rightarrow \infty} \lim_{k \rightarrow \infty} u^{N,k} = \lim_{N \rightarrow \infty} \sum_{m=0}^N I_m^*(f_m(\cdot, \cdot)) = u., \quad \text{in } L^2(d\nu); \\ \lim_{N \rightarrow \infty} \lim_{k \rightarrow \infty} \delta(u^{N,k}) = \delta^*(u), \quad \text{in } L^2(\Omega); \\ \lim_{N \rightarrow \infty} \lim_{\ell \rightarrow \infty} D.G^{N,\ell} = \lim_{N \rightarrow \infty} \sum_{n=0}^N nI_{n-1}^*(g_n(\cdot, \cdot)) = D^*G, \quad \text{in } L^2(\nu); \\ \lim_{N \rightarrow \infty} \lim_{\ell \rightarrow \infty} G^{N,\ell} = G, \quad \text{in } L^2(\Omega). \end{array} \right.$$

Since  $d\mu, d\lambda \ll d\nu$ , it follows that

$$\begin{aligned} \lim_{N \rightarrow \infty} \left\{ \lim_{k \rightarrow \infty} \lim_{\ell \rightarrow \infty} E\{\delta(u^{N,k})G^{N,\ell}\} \right\} &= E\{\delta^*(u)G\}; \\ \lim_{N \rightarrow \infty} \left\{ \lim_{k \rightarrow \infty} \lim_{\ell \rightarrow \infty} E\left\{ \int_0^1 u_t^{N,k}(D_t G^{N,\ell})d[M, M]_t \right\} \right\} &= E\left\{ \int_0^1 u_t(D_t^* G)d[M, M]_t \right\}; \\ \lim_{N \rightarrow \infty} \left\{ \lim_{k \rightarrow \infty} \lim_{\ell \rightarrow \infty} E\left\{ \int_0^1 u_t^{N,k}(D_t G^{N,\ell})dt \right\} \right\} &= E\left\{ \int_0^1 u_t(D_t^* G)dt \right\}. \end{aligned}$$

Therefore it suffices to prove (5.33) for  $u = u^{N,k}$  and  $G = G^{N,\ell}$ . Note that in this case  $\delta^*(u) = \delta(u)$ , and  $D^*G = D.G$ , so by the definition of  $\delta(u)$  (4.3) we have

$$E\{\delta^*(u)G\} = E\{\delta(u^{N,k})G^{N,\ell}\} = E\left\{ \int_0^1 u_t^{N,k}(D_t G^{N,\ell})dt \right\} = E\left\{ \int_0^1 u_t D_t^* G dt \right\}.$$

On the other hand, since all  $f_m^k$ 's and  $g_n^\ell$ 's are symmetric, we have by Lemma 5.2 that

$$\begin{aligned} E\left\{ \int_0^1 u_t(D_t^* G)d[M, M]_t \right\} &= E\left\{ \sum_{m,n=0}^N \int_0^1 I_m(f_m^k(\cdot, t))nI_{n-1}(g_n^\ell(\cdot, t))d[M, M]_t \right\} \\ &= \sum_{m,n=0}^N n(f_m^k, g_n^\ell)_{m+1,n} = \sum_{n=1}^N n(n-1)! \langle f_{n-1}^k, g_n^\ell \rangle_n \\ &= \sum_{m,n=0}^N E\{I_{m+1}(f_m^k)I_n(g_n^\ell)\} = E\{\delta^*(u)G\}. \end{aligned}$$

Therefore (5.33) holds, proving the theorem.  $\square$

## §6 A Class of Stochastic Differential Equations.

In this section we study a class of stochastic differential equations based on the anticipating stochastic integrals discussed in the previous sections. Notice that since we have not yet derived any path regularity for the anticipating integral, the traditional ways of dealing with SDEs will not apply here. The method that we are going to use relies solely on the structure of CRP of the  $L^2$ -processes and the definition of anticipating integrals, hence it will also be valid for the Brownian case. However in the Brownian case much better results are available; see for example Buckdahn (1994).

We consider the following linear SDE for  $0 \leq t \leq 1$ :

$$(6.1) \quad X_t = H_t + \int_0^t \alpha(s) X_s \delta M_s + \int_0^t \beta(s) X_s ds,$$

where  $H_t$  is a square-integrable process,  $\alpha$  and  $\beta$  are deterministic functions. Since we have no knowledge of the path regularity of the anticipating integral, the equation (6.1) is understood to hold in  $L^2(dt \times dP)$ .

We shall make use of the following assumption on the process  $H$ :

(A1) The process  $H$  is decomposable as follows:  $H = G + J$ , where  $J$  is an adapted, square-integrable process, and  $G$  belongs to a subspace of  $L^2([0, 1] \times \Omega)$  consisting of a finite number of chaoses. In other words,  $G = \sum_{n=1}^N I_n(g_n(\cdot, t))$ .

(A2) The functions  $\alpha$  and  $\beta$  are bounded, measurable, and non-random.

We shall denote  $\|\alpha\| := \|\alpha\|_\infty$ , and similarly for  $\beta$ .

Observe that by linearity of the equation, every solution of (6.1) can be written as the sum  $X = X^0 + X^1$ , where  $X^0$  and  $X^1$  are the solutions of the following SDEs respectively:

$$(6.2) \quad X_t^0 = J_t + \int_0^t \alpha(s) X_s^0 dM_s + \int_0^t \beta(s) X_s^0 ds;$$

and

$$(6.3) \quad X_t^1 = G_t + \int_0^t \alpha(s) X_s^1 \delta M_s + \int_0^t \beta(s) X_s^1 ds.$$

Note that by the semimartingale theory (cf. e.g., Protter (1990)), (6.2) always possesses a unique adapted solution that is square-integrable, so we need pay attention only to equation (6.3). To simplify the representation, let us first assume that  $\beta \equiv 0$ . Namely, we first consider the simpler form of (6.3):

$$(6.4) \quad X_t = G_t + \int_0^t \alpha(s) X_s \delta M_s.$$

Assume that  $X$  is a solution of (6.4), and write  $X_t = \sum_{n=0}^{\infty} I_n(f_n(\cdot, t))$ ,  $G_t = \sum_{n=0}^N I_n(g_n(\cdot, t))$ , where  $f_n, g_n \in L^2(T^{n+1})$  and for a.a.  $t \in [0, 1]$ ,  $f_n(\cdot, t), g_n(\cdot, t) \in L_s^2(T^n)$ ,  $n = 1, 2, \dots$ . Then it is necessary that

$$(6.5) \quad \sum_{n=0}^{\infty} I_n(f_n(\cdot, t)) = \sum_{n=0}^N I_n(g_n(\cdot, t)) + \sum_{n=0}^{\infty} I_{n+1}(\overline{(f_n(\cdot, *) \alpha(*) 1_{[0,t]}(*)^s)}),$$

by the definition of the anticipating stochastic integral. Comparing both sides of (6.5) term by term, we see that

$$(6.6) \quad f_n(\cdot, *, t) = \begin{cases} g_n(\cdot, *, t) + \overline{f_{n-1}(\cdot, *)\alpha(*)1_{[0,t]}(*)^s}, & 1 \leq n \leq N; \\ \overline{f_{n-1}(\cdot, *)\alpha(*)1_{[0,t]}(*)^s}, & n > N; \end{cases}$$

$$f_0(t) = g_0(t),$$

where, the symbol “ $\cdot$ ” represents  $n - 2$  variables, while “ $*$ ” represents one variable. Therefore, we have reduced the problem to finding a sequence  $\{f_n\}$  such that (6.6) is satisfied and every term in (6.4) defined by using  $\{f_n\}$  makes sense. Note that (6.6) is a recursive procedure, so the sequence  $\{f_n\}$  is uniquely determined. Since any solution to (6.4) must satisfy (6.6), we conclude that the equation (6.4) possesses at most one solution.

It now remains to show the existence of the solution to (6.4). By the preceding argument, we assume that the sequence  $\{f_n\}$  is defined via (6.6). Since  $N > 0$  is a fixed finite integer, the values of the terms  $f_0, f_1, \dots, f_N$  will not affect the convergence of the sums appearing in (6.5). We shall only analyze the values of  $\{f_n\}_{n \geq N+1}$ . Let us first define some notation. For each  $p \geq 1$ , let  $\mathbb{P}^{N+p}$  denote all permutations of  $\{1, \dots, N+p\}$ , and

$$(6.7) \quad \mathcal{A}_p := \{\sigma \in \mathbb{P}^{N+p} \mid \sigma(i) = i, i \notin \{\sigma(N+k), N+k : k = 1, \dots, p\}\}.$$

Namely,  $\mathcal{A}_p$  consists of those permutations  $\sigma$  that keep the indices  $i$  unchanged unless  $i \geq N+1$  or it is the image of  $N+k$  under  $\sigma$ , for some  $p \geq k \geq 1$ . Define for each  $p \geq 1$ ,

$$(6.8) \quad \mathcal{B}_p := \{\sigma \in \mathcal{A}_p \mid \sigma(N+1) < \sigma(N+2) < \dots < \sigma(N+p)\}.$$

Denote, for any finite set  $\mathcal{S}$ , by  $|\mathcal{S}|$  the cardinality of  $\mathcal{S}$ .

**Lemma 6.1.** *If  $\mathcal{A}_p, \mathcal{B}_p, p = 1, 2, \dots$  are defined by (6.7) and (6.8), then it holds that*

$$(6.9) \quad |\mathcal{A}_p| = \prod_{k=1}^p (N+k), \quad |\mathcal{B}_p| = \binom{N+p}{p}.$$

Consequently, one has  $|\mathcal{A}_p|/|\mathcal{B}_p| = p!$ .

*Proof.* We proceed by induction. For  $p = 1$ , note that the typical element in  $\mathcal{A}_1$  is of the form:

$$\begin{pmatrix} 1 & 2 & \dots & j-1 & j & j+1 & \dots & N+1 \\ 1 & 2 & \dots & j-1 & N+1 & j+1 & \dots & j \end{pmatrix}, \quad j = 1, \dots, N+1.$$

Therefore it is easily seen that  $|\mathcal{A}_1| = N+1$ . Also, in this case, we have  $|\mathcal{B}_1| = |\mathcal{A}_1| = N+1$ , so (6.9) is proved.

Next, assume that  $\mathcal{A}_1, \dots, \mathcal{A}_k$  are constructed so that (6.9) holds, consider the case when  $p = k+1$ . Define for each  $j = 1, \dots, N+k+1$ , and  $\sigma \in \mathcal{A}_k$ , a permutation  $\sigma^j$  by

$$(6.10) \quad \sigma^j(i) = \begin{cases} j, & i = N+k+1; \\ N+k+1, & \sigma(i) = j; \\ \sigma(i), & \text{otherwise.} \end{cases}$$

then we have  $\mathcal{A}_{k+1} = \bigcup_{j=1}^{N+k+1} \mathcal{A}_k^j$ , where  $\mathcal{A}_k^j := \{\sigma^j \mid \sigma \in \mathcal{A}_k\}$ . Note that for every  $j$ , one has  $|\mathcal{A}_k^j| = |\mathcal{A}_k|$  and  $\mathcal{A}_k^j \cap \mathcal{A}_k^\ell = \emptyset$  for  $j \neq \ell$ , thus

$$|\mathcal{A}_{k+1}| = \sum_{j=1}^{N+k+1} |\mathcal{A}_k^j| = (N+k+1)|\mathcal{A}_k| = (N+k+1) \prod_{j=1}^{N+k} (N+j) = \prod_{j=1}^{N+k+1} (N+j),$$

proving the lemma.  $\square$

Our next goal is to take a closer look at the sequence  $\{f_n\}$  obtained by solving (6.6), so that an estimation can be made to prove the  $L^2$ -convergence of the series appearing in (6.5).

**Lemma 6.2.** *Let  $\{f_n\}$  be the sequence defined by (6.6). Then for any  $p \geq 1$ , it holds that*

$$(6.11) \quad f_{N+p}(t_1, \dots, t_{N+p}, t) = \frac{1}{|\mathcal{A}_p|} \sum_{\sigma \in \mathcal{A}_p} f_N(t_{\sigma(1)}, \dots, t_{\sigma(N+1)}) \times \\ \times \left\{ \prod_{i=1}^{p-1} \alpha(t_{\sigma(N+i)}) \mathbf{1}_{[0, t_{\sigma(N+i+1)}]}(t_{\sigma(N+i)}) \right\} \alpha(t_{\sigma(N+p)}) \mathbf{1}_{[0, t]}(t_{\sigma(N+p)}).$$

*Proof.* We again proceed by induction. For  $p = 1$ , using Lemma 6.1 we have

$$\begin{aligned} f_{N+1}(t_1, \dots, t_{N+1}, t) &= \overline{f_N(t_1, \dots, t_{N+1}) \alpha(t_{N+1}) \mathbf{1}_{[0, t]}(t_{N+1})}^s \\ &= \frac{1}{N+1} \sum_{i=1}^{N+1} f_N(t_1, \dots, t_{i-1}, t_{N+1}, t_{i+1}, \dots, t_N, t_i) \alpha(t_i) \mathbf{1}_{[0, t]}(t_i) \\ &= \frac{1}{|\mathcal{A}_1|} \sum_{\sigma \in \mathcal{A}_1} f_N(t_{\sigma(1)}, \dots, t_{\sigma(N+1)}) \alpha(t_{\sigma(N+1)}) \mathbf{1}_{[0, t]}(t_{\sigma(N+1)}). \end{aligned}$$

Thus (6.11) is true. Now suppose that (6.11) is true for  $N + p$ . We have by definition of the symmetrization and (6.6) that

$$(6.12) \quad \begin{aligned} & f_{N+p+1}(t_1, \dots, t_{N+p+1}, t) \\ &= \frac{1}{N+p+1} \sum_{j=1}^{N+p+1} f_{N+p}(t_1, \dots, t_{j-1}, t_{N+p+1}, t_{j+1}, \dots, t_{N+p}, t_j) \alpha(t_j) 1_{[0,t]}(t_j). \end{aligned}$$

Let us define

$$(6.13) \quad \bar{t}_i^j := \begin{cases} t_i, & i \neq j, N+p+1; \\ t_{N+p+1}, & i = j; \\ t_j, & i = N+p+1. \end{cases}$$

This, together with the induction hypothesis, enables us to write the summands in the right side of (6.12) as

$$(6.14) \quad \begin{aligned} f_{N+p}(\bar{t}_1^j, \dots, \bar{t}_{N+p}^j) &= \frac{1}{|\mathcal{A}_p|} \sum_{\sigma \in \mathcal{A}_p} f_N(\bar{t}_{\sigma(1)}^j, \dots, \bar{t}_{\sigma(N+1)}^j) \times \\ &\times \prod_{k=1}^p \alpha(\bar{t}_{\sigma(N+k)}^j) 1_{[0, \bar{t}_{\sigma(N+k+1)}^j]}(\bar{t}_{\sigma(N+p)}^j). \end{aligned}$$

We show that for any  $\sigma \in \mathcal{A}_p$ , it holds that

$$(6.15) \quad \bar{t}_{\sigma(k)}^j = t_{\sigma^j(k)}, \quad k = 1, \dots, N+p,$$

where  $\sigma^j$  is defined by (6.10). In fact, by (6.10) and (6.13) we see that for any  $j$ , we have  $t_j = t_{\sigma^j(N+p+1)}$ ; and for those  $k$  such that  $\sigma(k) = j$  we have  $\bar{t}_{\sigma(k)}^j = \bar{t}_j^j = t_{N+p+1} = t_{\sigma^j(k)}$ . Finally, for those  $k$  such that  $\sigma(k) \neq j$  we have  $\bar{t}_{\sigma(k)}^j = t_{\sigma(k)}$ . Hence (6.15) is proven.

Now note that  $|\mathcal{A}_{p+1}| = (N+p+1)|\mathcal{A}_p|$ , we have from (6.11), (6.12) (6.14) and (6.15) that

$$\begin{aligned} & f_{N+p+1}(t_1, \dots, t_{N+p+1}, t) \\ &= \frac{1}{N+p+1} \sum_{j=1}^{N+p+1} \frac{1}{|\mathcal{A}_p|} \sum_{\sigma \in \mathcal{A}_p} f_N(\bar{t}_{\sigma(1)}^j, \dots, \bar{t}_{\sigma(N+1)}^j) \prod_{k=1}^p \alpha(\bar{t}_{\sigma(N+k)}^j) 1_{[0, \bar{t}_{\sigma(N+k+1)}^j]}(\bar{t}_{\sigma(N+p)}^j) \\ &\quad \times \alpha(t_j) 1_{[0,t]}(t_j) \\ &= \frac{1}{|\mathcal{A}_{p+1}|} \sum_{j=1}^{N+p+1} \sum_{\sigma^j \in \mathcal{A}_p^j} f_N(t_{\sigma^j(1)}, \dots, t_{\sigma^j(N+1)}) \prod_{k=1}^p \alpha(t_{\sigma^j(N+k)}) 1_{[0, t_{\sigma^j(N+k+1)}]}(t_{\sigma^j(N+p)}) \\ &\quad \times \alpha(t_{\sigma^j(N+p+1)}) 1_{[0,t]}(t_{\sigma^j(N+p+1)}) \\ &= \frac{1}{|\mathcal{A}_{p+1}|} \sum_{\sigma \in \mathcal{A}_{p+1}} f_N(t_{\sigma(1)}, \dots, t_{\sigma(N+1)}) \prod_{k=1}^p \alpha(t_{\sigma(N+k)}) 1_{[0, t_{\sigma(N+k+1)}]}(t_{\sigma(N+p)}) \\ &\quad \times \alpha(t_{\sigma(N+p+1)}) 1_{[0,t]}(t_{\sigma(N+p+1)}), \end{aligned}$$



since  $\mathcal{A}_{p+1} = \cup_{j=1}^{N+p+1} \mathcal{A}_p^j$ . Therefore (6.11) holds for  $N + p + 1$  and the proof is now complete.  $\square$

In order to prove the  $L^2$ -convergence of the series  $X_t = \sum_n I_n(f_n(\cdot, t))$ , we first evaluate the  $L^2$  norms of the  $f_n$ 's on the set  $\Sigma_n$ . Again, we consider only the terms  $f_n$ ,  $n > N$ . Suppose that  $(t_1, \dots, t_{N+p}) \in \Sigma_{N+p}$ . Since  $t_1 < t_2 < \dots < t_{N+p}$ , we have by definition of the set  $\mathcal{B}_p$  and some simple computations that

$$f_{N+p}(t_1, \dots, t_{N+p}) = \frac{1}{|\mathcal{A}_p|} \sum_{\sigma \in \mathcal{B}_p} f_N(t_{\sigma(1)}, \dots, t_{\sigma(N+1)}) \left\{ \prod_{i=1}^p \alpha(t_{\sigma(N+i)}) \right\} 1_{[0,t]}(t_{\sigma(N+p)}).$$

Define, for each  $p \geq 1$  and  $\sigma \in \mathcal{B}_p$ ,

$$f_{N,p}^\sigma = f_N(t_{\sigma(1)}, \dots, t_{\sigma(N+1)}) \left\{ \prod_{i=1}^p \alpha(t_{\sigma(N+i)}) \right\} 1_{[0,t]}(t_{\sigma(N+p)}).$$

Then under condition (A2) we have

$$\begin{aligned} \|f_{N,p}^\sigma\|_{L^2(\Sigma_{N+p})}^2 &= \int_{\Sigma_{N+p}} |f_{N,p}^\sigma|^2 dt_1 \cdots dt_{N+p} \\ (6.16) \quad &\leq \|\alpha\|^{2p} \|f_N\|_N^2 \int_{t_{\sigma(N+1)} < \dots < t_{\sigma(N+p)} < t} dt_{\sigma(N+1)} \cdots dt_{\sigma(N+p)} \\ &= \frac{\|\alpha\|^{2p} \|f_N\|_N^2 \cdot t^p}{p!}. \end{aligned}$$

Therefore by (6.15) we have

$$\begin{aligned} \|f_{N+p}(\cdot, t)\|_{N+p}^2 &= (N+p)! \|f_{N+p}(\cdot, t)\|_{L^2(\Sigma_{N+p})}^2 \\ (6.17) \quad &\leq (N+p)! \left\{ \frac{1}{|\mathcal{A}_p|} \sum_{\sigma \in \mathcal{B}_p} \|f_{N,p}^\sigma\|_{L^2(\Sigma_{N+p})} \right\}^2 \\ &\leq (N+p)! \left( \frac{|\mathcal{B}_p|}{|\mathcal{A}_p|} \right)^2 \|f_{N,p}^\sigma\|_{L^2(\Sigma_{N+p})}^2 \\ &\leq \frac{(N+p)!}{(p!)^3} \|\alpha\|^{2p} \|f_N\|_N^2 \cdot t^p \end{aligned}$$

Now let us define, for each  $n \in \mathbb{N}$ ,  $M_n := \sum_{k=0}^n \|I_k(f_k(\cdot, \cdot))\|_{L^2(T \times \Omega)}^2$ ;  $C := \|f_N\|_{L^2(T^N)}^2$ ; and note that

$$\frac{(N+p)!}{p!} = \left(1 + \frac{N}{p}\right) \left(1 + \frac{N}{p-1}\right) \cdots (1+N) N! \leq (1+N)^p N!,$$

we deduce that

$$\begin{aligned}
E \int_0^1 |X_t|^2 dt &= \int_0^1 \sum_{p=1}^{\infty} \|I_{N+p}(f_{N+p}(\cdot, t))\|_{L^2(\Omega)}^2 dt + \int_0^1 \sum_{k=0}^N \|I_k(f_k(\cdot, t))\|_{L^2(\Omega)}^2 dt \\
&= \int_0^1 \sum_{p=1}^{\infty} (N+p)! \|f_{N+p}(\cdot, t)\|_{N+p}^2 dt + M_N \\
&\leq C \sum_{p=1}^{\infty} \frac{((N+p)!)^2}{(p!)^3} \|\alpha\|^{2p} \int_0^1 t^p dt + M_N \\
&\leq C(N!)^2 \sum_{p=1}^{\infty} \frac{(1+N)^{2p}}{(p+1)!} \|\alpha\|^{2p} + M_N \\
&\leq C(N!)^2 e^{(1+N)^2 \|\alpha\|^2} + M_N < \infty.
\end{aligned}$$

Therefore,  $X \in L^2(dt \times dP)$ . It remains to check that for each  $t \in [0, 1]$  the process  $X.\alpha(\cdot)1_{[0,t]}(\cdot) \in \text{Dom}(\delta)$ . By Lemma 4.6, it suffices to show that

$$(6.18) \quad \sum_n \|I_{n+1}(f_n(\cdot, *)\alpha(*))\|_{L^2(\Omega)}^2 < \infty.$$

Since for each  $p \geq 1$  we have by (6.17) that

$$\begin{aligned}
&\|I_{N+p+1}(f_{N+p}(\cdot, *)\alpha(*))\|_{L^2(\Omega)}^2 \\
&\leq (N+p+1)! \int_0^1 \|f_{N+p}(\cdot, s)\|_{N+p}^2 \alpha^2(s) ds \\
&\leq C(N+p+1)! \|\alpha\|^{2(p+1)} \left\{ \frac{(N+p)!}{(p!)^3} \int_0^1 s^p ds \right\} \\
&\leq C \frac{(N+p+1)!(N+p)!}{(p!)^2(p+1)!} \|\alpha\|^{2(p+1)}.
\end{aligned}$$

Now a similar estimation as before shows that

$$\begin{aligned}
\sum_{n=0}^{\infty} \|I_{n+1}(f_n(\cdot, *)\alpha(*))\|_{L^2(\Omega)}^2 &= \sum_{p=1}^{\infty} \|I_{N+p+1}(f_{N+p+1}(\cdot, *)\alpha(*))\|_{L^2(\Omega)}^2 + M_{N+1} \\
&\leq C(N+1) \|\alpha\|^2 (N!)^2 e^{\|\alpha\|^2 (N+1)^2} + M_{N+1} < \infty.
\end{aligned}$$

Thus (6.18) holds, and we have proved the following theorem:

**Theorem 6.3.** *Suppose that (A1) and (A2) hold. Then the SDE (6.4) has a unique solution  $X_t = \sum_{n=0}^{\infty} I_n(f_n(\cdot, t))$ , where  $\{f_n\}_{n=0}^{\infty}$  satisfies the recursive equations (6.6).*

We now consider the general case when  $\beta \neq 0$ . Since the argument is virtually the same, we give only an outline. Assume (A1) and (A2). It is easy to check that in this case the recursive equation (6.6) should be replaced by

$$(6.19) \quad f_n(\cdot, *, t) = \begin{cases} g_n(\cdot, *, t) + \overline{f_{n-1}(\cdot, *)\alpha(*)1_{[0,t]}(*)}^s + \int_0^t f_n(\cdot, *, s)\beta(s)ds & n \leq N; \\ \overline{f_{n-1}(\cdot, *)\alpha(*)1_{[0,t]}(*)}^s + \int_0^t f_n(\cdot, *, s)\beta(s)ds & n > N; \end{cases}$$

$$f_0(t) = g_0(t) + \int_0^t f_0(s)\beta(s)ds.$$

The integral equation:  $x(t) = \eta(t) + \int_0^t x(s)\beta(s)ds$  has the unique solution

$$x(t) = \eta(t) - \int_0^t e^{-\int_s^t \beta(u)du} \beta(s)\eta(s)ds.$$

Therefore, if we denote  $\gamma(s, t) = e^{-\int_s^t \beta(u)du} \beta(s)$ , and

$$h_n(\cdot, *, t) = \begin{cases} g_n(\cdot, *, t) + \overline{f_{n-1}(\cdot, *)\alpha(*)1_{[0,t]}(*)}^s & n \leq N; \\ \overline{f_{n-1}(\cdot, *)\alpha(*)1_{[0,t]}(*)}^s & n > N, \end{cases}$$

Then from (6.19) we have

$$\begin{cases} f_0(t) = g_0(t) - \int_0^t \gamma(s, t)g_0(s)ds, \\ f_n(\cdot, *, t) = h_n(\cdot, *, t) - \int_0^t \gamma(s, t)h_n(\cdot, *, s)ds. \end{cases}$$

Again we need only content ourselves to the terms when  $n > N$ . Denoting  $m(s, t) := \alpha(s) \left(1 - \int_0^s \gamma(u, t)du\right)$ , we show by induction that the sequence  $\{f_n\}$  defined by (6.19) satisfies, for any  $p \geq 1$ ,

$$(6.20) \quad f_{N+p}(t_1, \dots, t_{N+p}, t) = \frac{1}{|\mathcal{A}_p|} \sum_{\sigma \in \mathcal{A}_p} f_N(t_{\sigma(1)}, \dots, t_{\sigma(N+1)}) \times \\ \times \left\{ \prod_{i=1}^{p-1} m(t_{\sigma(N+i)}, t_{\sigma(N+i+1)}) 1_{[0, t_{\sigma(N+i+1)}]}(t_{\sigma(N+i)}) \right\} m(t_{\sigma(N+p)}, t) 1_{[0, t]}(t_{\sigma(N+p)}).$$

Indeed, for  $p = 1$ , using Lemma 6.1 and the similar computation in Lemma 6.2, we have

$$\begin{aligned} & f_{N+1}(t_1, \dots, t_{N+1}, t) \\ &= \frac{1}{N+1} \sum_{i=1}^{N+1} f_N \left( \begin{matrix} t_i \\ t_{N+1} \end{matrix} \right) \alpha(t_i) 1_{[0, t]}(t_i) \left\{ 1 - \int_0^t \gamma(s, t) 1_{[0, s]}(t_i) ds \right\} \\ &= \frac{1}{|\mathcal{A}_1|} \sum_{\sigma \in \mathcal{A}_1} f_N(t_{\sigma(1)}, \dots, t_{\sigma(N+1)}) m(t_{\sigma(N+1)}, t) 1_{[0, t]}(t_{\sigma(N+1)}), \end{aligned}$$

where  $f_N \begin{pmatrix} t_i \\ t_{N+1} \end{pmatrix} := f_N(t_1, \dots, t_{i-1}, t_{N+1}, t_{i+1}, \dots, t_N, t_i)$ , Therefore (6.20) holds for  $p =$   
1. Now suppose that (6.20) is true for  $N + p$ . We have

$$\begin{aligned}
& f_{N+p+1}(t_1, \dots, t_{N+p+1}, t) \\
&= \frac{1}{N+p+1} \sum_{j=1}^{N+p+1} f_{N+p}(t_1, \dots, t_{j-1}, t_{N+p+1}, t_{j+1}, \dots, t_{N+p}, t_j) m(t_j, t) 1_{[0,t]}(t_j) \\
&= \frac{1}{N+p+1} \sum_{j=1}^{N+p+1} \left\{ \frac{1}{|\mathcal{A}_p|} \sum_{\sigma \in \mathcal{A}_p} f_N(\bar{t}_{\sigma(1)}^j, \dots, \bar{t}_{\sigma(N+1)}^j) \times \right. \\
&\quad \left. \times \prod_{k=1}^p m(\bar{t}_{\sigma(N+k)}^j, \bar{t}_{\sigma(N+k+1)}^j) 1_{[0, \bar{t}_{\sigma(N+k+1)}^j]}(\bar{t}_{\sigma(N+k)}^j) \cdot m(\bar{t}_{\sigma(N+p+1)}^j, t) 1_{[0,t]}(\bar{t}_{\sigma(N+p+1)}^j) \right\} \\
&= \frac{1}{|\mathcal{A}_{p+1}|} \sum_{\sigma \in \mathcal{A}_{p+1}} f_N(t_{\sigma(1)}, \dots, t_{\sigma(N+1)}) \times \\
&\quad \times \prod_{k=1}^{p+1} m(t_{\sigma(N+k)}, t_{\sigma(N+k+1)}) 1_{[0, t_{\sigma(N+p+2)}]}(t_{\sigma(N+p+1)}),
\end{aligned}$$

where  $t_{\sigma(N+p+2)} = t$ , proving (6.20).

Note that the boundedness of  $\alpha$  and  $\beta$  implies that  $m$  is also bounded, so by replacing  $\|\alpha\|_\infty$  by  $\|m\|_\infty$  in the estimations in the case  $\beta = 0$ , we obtain the main result of this section.

**Theorem 6.4.** *Suppose that (A1) and (A2) hold. Then the SDE (6.3) has a unique solution  $X_t = \sum_{n=0}^{\infty} I_n(f_n(\cdot, t))$ , where  $\{f_n\}_{n=0}^{\infty}$  satisfies the recursive equations (6.19).*

We can combine Theorem 6.4 with the semimartingale theory to obtain

**Theorem 6.5.** *Suppose that (A1) and (A2) hold. Then equation (6.1) has a solution and it is unique in  $L^2(dt \times dP)$ .*

*Proof.* We only show the uniqueness. Let  $X^0$  be the unique semimartingale solution of (6.2); and let  $Y$  be another solution of (6.1). Let

$$\mathcal{L}(Z) := \int_0^t \alpha(s) Z_s \delta M_s + \int_0^t \beta(s) Z_s ds, \quad \forall Z \in L^2(dt \times dP).$$

Thus we have  $Y = H + \mathcal{L}(Y)$ , and

$$Y - X^0 = H + \mathcal{L}(Y) - (J + \mathcal{L}(X^0)) = H - J + \mathcal{L}(Y - X^0) = G + \mathcal{L}(Y - X^0).$$

and by Theorem 6.4 we have  $Y - X^0 = X^1$ , the unique solution of (6.3). So  $Y = X^0 + X^1$  in  $L^2(dt \times dP)$  and we have uniqueness.  $\square$

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