# A NOTE ON THE MARCINKIEWICZ - ZYGMUND STRONG LAW

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**Abstract**

We prove the Marcinkiewicz-Zygmund strong law in a general set up.

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Suppose $(X_j,\ j \geq 1)$ are independent and identically distributed (i.i.d.) random variables and $E|X_1|^p < \infty$ for some $0 < p < 2$. Then the Marcinkiewicz-Zygmund strong law of large numbers (MZSLLN) states that as $n \to \infty$,

$$\sum_{j=1}^{n} X_j = nc + o(n^{\frac{1}{p}}) \text{ almost surely } \dots \dots \tag{1}$$

where $c = E(X_1)$ if $p \geq 1$, and $c$ may be taken to be 0 if $p < 1$. A proof using the Kolmogorov's three series theorem is given in Chow and Teicher (1978, page 122, Theorem 2). When the i.i.d. assumption is dropped, a similar result does not seem to appear in explicit form in the literature. This short note establishes such a result.

Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and $(\mathcal{F}_j,\ j \geq 0)$ is an increasing sequence of sub $\sigma$-fields of $\mathcal{F}$. Suppose $(X_j,\ U_j,\ j \geq 1)$ are random variables such that for each $j \geq 1$, $U_{j+1}$ and $X_j$ are $\mathcal{F}_j$ measurable and $U_j$ are positive. For any random variable $X$, define $X(c) = X\ I(|X| \leq c)$. All convergences are in the almost sure sense.

**Proposition 1.** Suppose there is a random variable $X$ and $c > 0$ such that that for all $j \geq 1$,

$$P(|X_j| \geq x) \leq C\ P\ (|X| \geq x) \dots \dots \tag{2}$$

where $E|X|^p < \infty$ for some $0 < p < 2$. Then

$$\sum_{j=1}^{n} X_j = c_n + o(n^{\frac{1}{p}}) \text{ almost surely } \dots \dots \tag{3}$$

where $c_n = \sum_{j=1}^{n} E(X_j|\mathcal{F}_{j-1})$ if $p > 1$, $c_n = 0$ if $p < 1$, and $c_n = \sum_{j=1}^{n} E\ X_j\ I(|X_j| \leq j|\mathcal{F}_{j-1})$ if $p = 1$. □

**Remark 1.** The result for $p = 1$ is given in Theorem 2.19 of Hall and Heyde (1980) where it is further shown that in this case if $c_n$ is replaced by $c_n^* = \sum_{j=1}^{n} E(X_j|\mathcal{F}_{j-1})$ then (1) holds only in probability. This probability convergence can be strengthened to almost sure convergence under any of the following: a) $E(|X| \log^+ |X|) < \infty$, b) $(X_j, j \geq 1)$ are independent, c) $(X_j,\ j \geq 1)$ and $(E(X_j|\mathcal{F}_{j-1}), j \geq 2)$ are stationary. □

Define the variables $(Y_j, j \geq 1)$ by

$$\begin{aligned} Y_j &= U_j^{-1}\ X_j\ I(|X_j| \leq U_j) \\ &= U_j^{-1}\ X_j\ (U_j) \end{aligned}$$

Note that for all $j \geq 1$, $Y_j$ is $\mathcal{F}_j$ measurable and $|Y_j| \leq 1$. Consider the following three series:

$$
\begin{aligned}
T_1 &= \sum_{j=1}^{\infty} (Y_j - E(Y_j|\mathcal{F}_{j-1})) \\
&= \sum_{j=1}^{\infty} U_j^{-1} \, (X_j(a_j) - E(X_j(a_j)|\mathcal{F}_{j-1})) \\
T_2 &= \sum_{j=1}^{\infty} I(U_j^{-1} \, X_j \neq Y_j) \\
&= \sum_{j=1}^{\infty} I(|X_j| > U_j) \\
T_3 &= \sum_{j=1}^{\infty} (U_j^{-1} \, X_j - E(Y_j|\mathcal{F}_{j-1})) \\
&= \sum_{j=1}^{\infty} U_j^{-1} \, (X_j - E(X_j(a_j)|\mathcal{F}_{j-1}))
\end{aligned}
$$

Define the following sets

$$
\begin{aligned}
D_1 &= \left\{ \sum_{j=1}^{\infty} P(|X_j| \geq U_j|\mathcal{F}_{j-1}) < \infty \right\} \\
D_2 &= \left\{ \sum_{j=1}^{\infty} E(Y_j^2|\mathcal{F}_{j-1}) < \infty \right\} \\
&= \left\{ \sum_{j=1}^{\infty} U_j^{-2} \, E(X_j^2(U_j)|\mathcal{F}_{j-1}) < \infty \right\}
\end{aligned}
$$

The following Lemma is an easy consequence of the conditional three series theorem. We omit its proof.

*Lemma 1.* *$T_2$ converges on $D_1$, and $T_1$ and $T_3$ converge on $D_1 \cap D_2$.* □

**Proof of the Proposition.** Let $p \neq 1$ and $U_j = j^{\frac{1}{p}}$ in Lemma 1.

By condition (2),

$$\sum_{j=1}^{\infty} P(|X_j| \geq j^{\frac{1}{p}}) \leq C \sum_{j=1}^{\infty} P(|X|^p \geq j) < \infty \quad \dots \quad \dots \tag{4}$$

$$\sum_{j=1}^{\infty} E(Y_j^2) = \sum_{j=1}^{\infty} j^{-2/p} E(X_j^2 \; I(|X_j| \leq j^{1/p})$$

$$= \sum_{j=1}^{\infty} j^{-2/p} \int_0^{j^{1/p}} P(X_j^2 \geq x) \, dx$$

$$\leq \sum_{j=1}^{\infty} j^{-2/p} \sum_{k=1}^{j} \int_{(k-1)^{1/p}}^{k^{1/p}} P(X^2 \geq x) \, dx$$

$$\leq \sum_{k=1}^{\infty} \int_{(k-1)^{1/p}}^{k^{1/p}} P(X^2 \geq x) dx \left( k^{-2/p} + \frac{p}{2-p} \, k^{-2/p+1} \right)$$

$$\leq 2/p \sum_{k=1}^{\infty} \left( k^{-2/p} + \left( \frac{p}{2-p} \right) k^{-2/p+1} \right) \int_{(k-1)^{1/2}}^{k^{1/2}} P(|X|^p \geq y) y^{2/p-1} \, dy$$

$$\leq 2/p \sum_{k=1}^{\infty} \left( k^{-1} + \frac{p}{(2-p)} \right) \int_{(k-1)^{1/2}}^{k^{1/2}} P(|X|^p \geq y) \, dy$$

$$\leq C_p \int_0^{\infty} P(|X|^p \geq y) \, dy < \infty \quad \dots \quad \dots \tag{5}$$

Since (4) and (5) are satisfied, by Lemma 1,

$$\sum_{j=1}^{\infty} j^{-1/p} (X_j - E(X_j \;\; I(|X_j| \leq j^{1/p}) | \mathcal{F}_{j-1})) < \infty \text{ almost surely } \dots \quad \dots \tag{6}$$

Now assume that $p > 1$.

$$E \sum_{j=1}^{\infty} j^{-1/p} \; |E\; X_j I(|X_j| > j^{1/p}) | \mathcal{F}_{j-1}|$$

$$\leq \sum_{j=1}^{\infty} j^{-1/p} \;\; E[|X_j| I(|X_j| > j^{1/p})]$$

$$\leq \sum_{j=1}^{\infty} j^{-1/p} \sum_{k=j+1}^{\infty} \int_{(k-1)^{1/p}}^{k^{1/p}} P(|X_j| \geq x)\; dx$$

$$\leq C \sum_{k=2}^{\infty} \sum_{j=1}^{k-1} j^{-1/p} \int_{(k-1)^{1/p}}^{k^{1/p}} P(|X| \geq x)\; dx$$

$$\leq \frac{Cp}{(p-1)} \sum_{k=2}^{\infty} (k-1)^{(p-1)/p} \int_{(k-1)^{1/p}}^{k^{1/p}} P(|X| \geq x)\; dx$$

$$\leq C(p) \sum_{k=2}^{\infty} (k-1)^{(p-1)/p} \int_{(k-1)}^{k} P(|X|^p \geq y)\; y^{(\frac{1}{p}-1)}\; dy$$

$$= C(p) \sum_{k=2}^{\infty} \int_{(k-1)}^{k} P(|X|^p \geq y)\; dy < \infty$$

Thus if $p > 1$,

$$\sum_{j=1}^{\infty} j^{-1/p}\; EX_j\; I(|X_j| > j^{1/p} | \mathcal{F}_{j-1}) < \infty \text{ almost surely } \ldots \quad \ldots \qquad (7)$$

Hence from (7) and (6), if $p > 1$, then

$$\sum_{j=1}^{\infty} \frac{X_j - E(X_j|\mathcal{F}_{j-1})}{j^{1/p}} < \infty \text{ almost surely } \ldots \quad \ldots \qquad (8)$$

If $p < 1$,

$$\sum_{j=1}^{\infty} j^{-1/p} \quad |EX_j I(|X_j| \leq j^{1/p} | \mathcal{F}_{j-1})|$$

$$\leq \sum_{j=1}^{\infty} j^{-2/p} \quad E(X_j^2 J(|X_j| \leq j^{1/p} | \mathcal{F}_{j-1}))$$

$$< \infty \text{ by (5).}$$

Hence if $p < 1$,

$$\sum_{j=1}^{\infty} j^{-1/p} \quad X_j < \infty \text{ almost surely} \quad \cdots \quad \cdots \tag{9}$$

Combining (8) and (9) proves the Proposition after an application of Kronecker's Lemma. □

**Remark 2.** It is clear from the above proof that if $(X_j, \mathcal{F}_j, \ j \geq 1)$ is a martingale difference sequence, then for $U_n \uparrow \infty$, (in particular for $U_j = j^{1/p}$),

$$\sum_{k=1}^{n} X_k = o\,(U_n) \text{ almost surely} \quad \cdots \quad \cdots \tag{10}$$

if the following conditions hold:

(C1) $\displaystyle\sum_{j=1}^{\infty} P(|X_j| \geq U_j | \mathcal{F}_{j-1}) < \infty$ almost surely

(C2) $\displaystyle\sum_{j=1}^{\infty} U_j^{-2} \quad EX_j^2 \ I(|X_j| \leq U_j | \mathcal{F}_{j-1}) < \infty$ almost surely

(C3) $\displaystyle\sum_{j=1}^{\infty} U_j^{-1} \quad EX_j \ I(|X_j| \leq U_j | \mathcal{F}_{j-1}) < \infty$ almost surely. □

One may compare this with Theorem 2.18 given in Hall and Heyde (1980) where it is shown that (10) holds if

$$\sum_{j=1}^{\infty} U_j^{-p} E(|X_j|^p | \mathcal{F}_{j-1}) < \infty \text{ almost surely} \quad \cdots \quad \cdots \tag{10}$$

for some $1 \leq p \leq 2$.

This is clearly not enough to establish Proposition 1 since $U_j$ must be chosen to equal $j^{1/p}$ and in that case the series in (10) is $\sum_{j=1}^{\infty} j^{-1} E(|X_j|^p | \mathcal{F}_{j-1})$ whose convergence is not guaranteed under the conditions given. On the other hand it is easy to see that if (10) holds for any $(U_j, j \geq 1)$, then (C1), (C2) and (C3) hold. □

## References


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