# ON EMPIRICAL BAYES TWO-STAGE TEST IN A DISCRETE EXPONENTIAL FAMILY

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# ON EMPIRICAL BAYES TWO-STAGE TEST IN A DISCRETE EXPONENTIAL FAMILY

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## Abstract

The purpose of this article is to study the empirical Bayes theory of two-stage tests for the two-action problem in a discrete exponential family. An empirical Bayes two-stage test is constructed by resembling the behavior of a Bayes two-stage test. Asymptotic optimality of the empirical Bayes two-stage test is investigated and the rate of convergence of its associated regret Bayes risk is established. It is shown that the proposed empirical Bayes two-stage test is asymptotically optimal at a rate of convergence of order $O(\exp(-nc))$ for some positive constant $c$, where $n$ denotes the number of historical data at hand when the present decision problem is considered.

## 1. Introduction

The empirical Bayes theory, as pioneered by Robbins (1956, 1964), deals with a sequence of independent repetitions of a given statistical decision problem, where each component problem in the sequence involves the same unknown prior distribution $G$ on the parameter space. Since Robbins (1956, 1964), empirical Bayes procedures have been extensively investigated in the literature. To name a few, Johns and Van Ryzin (1971, 1972), Van Houwelingen (1976) and Liang (1988) have studied certain empirical Bayes tests; Lin (1972, 1975), Singh (1976, 1979) and Singh and Wei (1992) have investigated empirical Bayes estimation problems. Among all the works referenced precedingly, the empirical Bayes methods are dealing with components having one sample, fixed sample size and identical statistical decision problems. Recently, Karunamuni (1988, 1989, 1990) have developed empirical Bayes sequential procedures for certain statistical decision problems and investigated the associated asymptotic properties. However, due to the difficulty of determining Bayes stopping rules, the asymptotic optimality established in Karunamuni (1988, 1989, 1990) is with respect to, for example, a one-step-look-ahead Bayes procedures instead of the Bayes sequential procedure.

In this article, our purpose is to study the empirical Bayes theory of two-stage tests for the two-action problem in a discrete exponential family, particularly, the asymptotic optimality of an empirical Bayes two-stage test relative to a Bayes two-stage test. In Section 2, we introduce the notations and the statistical model of the two-stage test with a weighted linear error loss for the two-action problem and derive a Bayes two-stage test. The related empirical Bayes framework is described in Section 3. By resembling the behavior of the Bayes two-stage test, an empirical Bayes two-stage test is proposed in Section 3. We then investigate the asymptotic optimality of the empirical Bayes two-stage test in Section 4. Several useful lemmas are introduced in Section 5. The rate of convergence of the empirical Bayes two-stage test is established. It is shown that the empirical Bayes two-stage test is asymptotically optimal, with a rate of convergence of order $O(\exp(-nc))$ for some positive constant $c$, where $n$ denotes the number of historical data at hand when the present decision problem is considered. In Section 6, two examples are provided to demonstrate the asymptotic optimality of the empirical Bayes two-stage test.

## 2. Two-Stage Tests

### 2.1. An Introduction to Two-Stage Test

We let $(X, Y)$ be a pair of independent random variables belonging to a discrete exponential family with probability function

$$f(z|\theta) = a(z)\beta(\theta)\theta^z, \quad z = 0, 1, \ldots; \quad 0 < \theta < \mathcal{Q}, \tag{2.1}$$

where $a(z) > 0$ for all $z = 0, 1, \ldots$, and where $\mathcal{Q}$ may be finite or infinite. Here, $X$ denotes the random observation obtained at the first stage and $Y$ stands for the random observation observed at the second stage. We want to test the hypotheses $H_0$: $\theta \geq \theta_0$ against $H_1$: $\theta < \theta_0$ using a two-stage test where $\theta_0$ is a known constant such that $0 < \theta_0 < \mathcal{Q}$. For each $i = 0, 1$ let $i$ denote the action deciding in favor of the hypothesis $H_i$. Let $t$ denote the termination action: $t = 1$ means terminating sampling and taking an action immediately after observing $X$; $t = 0$ means going to second stage sampling, and then taking an action based on the observation $(X, Y)$. For the parameter $\theta$, action $i$ and termination action $t$, we assume the following weighted linear error loss

$$L(\theta, (i, t)) = L_1(\theta, i) + (2 - t)c, \tag{2.2}$$

and

$$L_1(\theta, i) = i\frac{(\theta - \theta_0)}{\beta(\theta)} I_{(\theta_0, \mathcal{Q})}(\theta) + (1 - i)\frac{(\theta_0 - \theta)}{\beta(\theta)} I_{(0, \theta_0)}(\theta), \tag{2.3}$$

where $I_A$ is the indicator function of the set $A$, and $c > 0$ is the cost per sampling. In (2.3), the first term is the loss due to wrongly accepting $H_1$ when $\theta \geq \theta_0$, and the second term is the loss of wrongly accepting $H_0$ when $\theta < \theta_0$. The second term in (2.2) is the cost of sampling.

Let $\mathcal{X}$ and $\mathcal{Y}$ denote the sample space generated by $X$ and $Y$, respectively. A two-stage test consists of three parts, say, $\delta = (d_1, d_2, \tau)$, where $d_i$ is the decision rule at stage $i$, $i = 1, 2$, and $\tau$ is the stopping rule. They are defined as follows:

(a) $\tau$: $\mathcal{X} \to [0, 1]$. For each $x \in \mathcal{X}$, $\tau(x)$ is the probability of stopping sampling and making a decision immediately when $X = x$ is observed.

(b) $d_1$: $\mathcal{X} \to [0, 1]$. For each $x \in \mathcal{X}$, $d_1(x)$ is the probability of accepting $H_0$ when $X = x$ is observed.

(c) $d_2: \mathcal{X} \times \mathcal{Y} \to [0,1]$. For $(x,y) \in \mathcal{X} \times \mathcal{Y}$, $d_2(x,y)$ is the probability of accepting $H_0$ when $(X,Y) = (x,y)$ are observed.

It is assumed that the parameter $\theta$ is a realization of a random variable $\Theta$ having an unknown prior distribution $G$ over $(0, Q)$. It is also assumed that $\int \frac{\theta}{\beta(\theta)} dG(\theta) < \infty$ so that the Bayes risk of a two-stage test $(d_1, d_2, \tau)$ is finite.

In the following, we introduce some notations. Let

$$f(x) = \int f(x|\theta) dG(\theta) = a(x) \int \beta(\theta)\theta^x dG(\theta): \text{ The marginal probability function of } X,$$

$$k(x) = \int \beta(\theta)\theta^x dG(\theta) = \frac{f(x)}{a(x)},$$

$$h(x) = \int \theta^x dG(\theta),$$

$G(\theta|x)$ : the posterior distribution of $\Theta$ given $X = x$,

$$W_1(x) = a(x)h(x)[\theta_0 - \frac{h(x+1)}{h(x)}],$$

$$W_2(x,y) = a(x)a(y)k(x+y)[\theta_0 - \frac{k(x+y+1)}{k(x+y)}],$$

$$\ell\ (x) = \int L_1(\theta, 1) dG(\theta|x).$$

Let $R(G, (d_1, d_2, \tau))$ denote the Bayes risk of the two-stage test $(d_1, d_2, \tau)$. Then, with the loss function (2.2)–(2.3) and by Fubini's theorem, we have

$$\begin{aligned} &R(G, (d_1, d_2, \tau)) \\ &= \sum_x \tau(x)[d_1(x)W_1(x) + \ell\ (x)f(x) + cf(x)] \\ &\quad + \sum_x [1 - \tau(x)]\{\sum_y d_2(x,y)W_2(x,y) + \ell\ (x)f(x) + 2cf(x)\} \\ &= \sum_x \tau(x)[d_1(x)W_1(x) - \sum_y d_2(x,y)W_2(x,y) - cf(x)] \\ &\quad + \sum_x \sum_y d_2(x,y)W_2(x,y) + \sum_x \ell\ (x)f(x) + 2c. \end{aligned} \tag{2.4}$$

## 2.2. A Bayes Two-Stage Test

Define

$$\Psi(x, d_1, d_2) = d_1(x)W_1(x) - \sum_y d_2(x,y)W_2(x,y) - cf(x). \tag{2.5}$$

We then consider a two-stage test $\delta_G = (d_{1G}, d_{2G}, \tau_G)$ which is defined as follows:

$$d_{1G}(x) = \begin{cases} 1 & \text{if } W_1(x) \le 0, \\ 0 & \text{otherwise;} \end{cases} \tag{2.6}$$

$$d_{2G}(x,y) = \begin{cases} 1 & \text{if } W_2(x,y) \le 0, \\ 0 & \text{otherwise;} \end{cases} \tag{2.7}$$

and

$$\tau_G(x) = \begin{cases} 1 & \text{if } \Psi(x, d_{1G}, d_{2G}) \le 0, \\ 0 & \text{otherwise,} \end{cases} \tag{2.8}$$

Since $a(x)$, $h(x)$ and $k(x)$ are positive for all $x = 0, 1, \ldots$, $d_{1G}$ and $d_{2G}$ can also be written as:

$$d_{1G}(x) = \begin{cases} 1 & \text{if } \frac{h(x+1)}{h(x)} \ge \theta_0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.6$'$}$$

$$d_{2G}(x,y) = \begin{cases} 1 & \text{if } \frac{k(x+y+1)}{k(x+y)} \ge \theta_0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.7$'$}$$

From (2.7′), the second stage decision rule $d_{2G}(x,y)$ depends on $(x,y)$ only through $x+y$.

It should be noted that the two-stage test $(d_{1G}, D_{2G}, \tau_G)$ is the Bayes one-step look ahead procedure truncated at two. Thus $(d_{1G}, D_{2G}, \tau_G)$ is a Bayes two-stage test. For detailed discussion about one-step look ahead procedure, see Berger (1985).

## 2.3. Alternative form of $(d_{1G}, d_{2G}, \tau_G)$

For each $z = 0, 1, \ldots$, let

$$\varphi_1(z) = \frac{h(z+1)}{h(z)} = \frac{\int \theta^{z+1} dG(\theta)}{\int \theta^z dG(\theta)},$$

and

$$\varphi_2(z) = \frac{k(z+1)}{k(z)} = \frac{\int \beta(\theta)\theta^{z+1} dG(\theta)}{\int \beta(\theta)\theta^z dG(\theta)}.$$

It is easy to see that both $\varphi_1(z)$ and $\varphi_2(z)$ are increasing function of $z$. Since $\frac{1}{\beta(\theta)} = \sum_{z=0}^{\infty} \theta^z a(z)$, $\beta(\theta)$ is decreasing in $\theta$ for $\theta > 0$. By Theorem 2 of Wijsman (1985),

$$\varphi_1(z) > \varphi_2(z) \text{ for all } z = 0, 1, 2, \ldots,. \tag{2.9}$$

For each $i = 1, 2$, let

$$A_i(\theta_0) = \{z | \varphi_i(z) < \theta_0\} \quad \text{and} \quad B_i(\theta_0) = \{z | \varphi_i(z) > \theta_0\}.$$

Define

$$m_i = \begin{cases} \sup A_i(\theta_0) & \text{if } A_i(\theta_0) \neq \phi, \\ -1 & \text{if } A_i(\theta) = \phi; \end{cases} \tag{2.10}$$

$$M_i = \begin{cases} \inf B_i(\theta_0) & \text{if } B_i(\theta_0) \neq \phi, \\ \infty & \text{if } B_i(\theta_0) = \phi. \end{cases} \tag{2.11}$$

Note that $m_i \leq M_i \leq m_i + 2$, $i = 1, 2$, and when $B_i(\theta_0) \neq \phi$, $m_i < M_i < \infty$. By the inequality (2.9), $m_1 \leq m_2$ and $M_1 \leq M_2$. In the following, it is assumed that $B_i(\theta_0) \neq \phi$, $i = 1, 2$.

Note that

$$\begin{array}{llll} x \leq m_1 & \text{iff} \quad \theta_0 h(x) - h(x+1) > 0 & \text{iff} & W_1(x) > 0; \\ x \geq M_1 & \text{iff} \quad \theta_0 h(x) - h(x+1) < 0 & \text{iff} & W_1(x) < 0; \\ m_1 < x < M_1 & \text{iff} \quad \theta_0 h(x) - h(x+1) = 0 & \text{iff} & W_1(x) = 0; \end{array}$$

and

$$\begin{array}{llll} x + y \leq m_2 & \text{iff} \quad \theta_0 k(x+y) - k(x+y+1) > 0 & \text{iff} & W_2(x,y) > 0; \\ x + y \geq M_2 & \text{iff} \quad \theta_0 k(x+y) - k(x+y+1) < 0 & \text{iff} & W_2(x,y) < 0; \\ m_2 < x + y < M_2 & \text{iff} \quad \theta_0 k(x+y) - k(x+y+1) = 0 & \text{iff} & W_2(x,y) = 0. \end{array}$$

It should be pointed out that the set $\{z | m_1 < z < M_1\} = \{z | \varphi_1(z) = \theta_0\}$ may be an empty set if $\varphi_1(z) \neq \theta_0$ for all $z = 0, 1, \ldots$; or it may not be an empty set which occurs only when $\varphi_1(m_1 + 1) = \theta_0$ in such case $M_1 = m_1 + 2$. Similarly, the set $\{z | m_2 < z < M_2\}$ may be an empty set if $\varphi_2(z) \neq \theta_0$ for all $z = 0, 1, \ldots$; or it may not be an empty set which occurs when $\varphi_2(m_2 + 1) = \theta_0$ in such case $M_2 = m_2 + 2$.

Therefore, the Bayes decision rules $d_{1G}$ and $d_{2G}$ can be represented as:

$$d_{1G}(x) = \begin{cases} 0 & \text{if } x \le m_1, \\ \gamma_1 & \text{if } m_1 < x < M_1, \\ 1 & \text{if } x \ge M_1; \end{cases}$$
$$= \begin{cases} 0 & \text{if } \theta_0 h(x) - h(x+1) > 0, \\ \gamma_1 & \text{if } \theta_0 h(x) - h(x+1) = 0, \\ 1 & \text{if } \theta_0 h(x) - h(x+1) < 0; \end{cases} \tag{2.12}$$

and

$$d_{2G}(x,y) = \begin{cases} 0 & \text{if } x+y \le m_2, \\ \gamma_2 & \text{if } m_2 < x+y < M_2, \\ 1 & \text{if } x+y \ge M_2; \end{cases}$$
$$= \begin{cases} 0 & \text{if } \theta_0 k(x+y) - k(x+y+1) > 0, \\ \gamma_2 & \text{if } \theta_0 k(x+y) - k(x+y+1) = 0, \\ 1 & \text{if } \theta_0 k(x+y) - k(x+y+1) < 0, \end{cases} \tag{2.13}$$

where $\gamma_1$ and $\gamma_2$ are any values between (including) 0 and 1.

Now consider the function

$$\begin{aligned} \Psi(x, d_{1G}, d_{2G}) &= d_{1G}(x)W_1(x) - \sum_y d_{2G}(x,y)W_2(x,y) - cf(x) \\ &= d_{1G}(x)a(x)h(x)[\theta_0 - \frac{h(x+1)}{h(x)}] \\ &\quad - \sum_y d_{2G}(x,y)a(x)a(y)k(x+y)[\theta_0 - \frac{k(x+y+1)}{k(x+y)}] \\ &\quad - cf(x). \end{aligned}$$

**(a) As $x > m_2$.**

By the inequality (2.9) and the definitions of $m_1$ and $m_2$, $m_2 + 1 \ge M_1$. Thus, as $x > m_2$, $d_{1G}(x) = 1$ and $d_2(x,y) = 1$ for $x+y \ge M_2$. As $x+y = m_2+1$, either $d_2(x,y) = 1$ or $\theta_0 k(m_2+1) - k(m_2+2) = 0$. In either cases, we always have

$$\Psi(x, d_{1G}, d_{2G}) = -cf(x) < 0. \tag{2.14}$$

Thus, $\tau_G(x) = 1$.

**(b) As $0 \le x \le m_1$.**

As $0 \le x \le m_1$, $d_{1G}(x) = 0$ and

$$d_{2G}(x,y) = \begin{cases} 0 & \text{if } x+y \le m_2, \\ \gamma_2 & \text{if } m_2 < x+y < M_2, \\ 1 & \text{if } x+y \ge M_2. \end{cases}$$

It should be noted that if there is any pair $(x, y)$ such that $m_2 < x + y < M_2$, it must be that $M_2 = m_2 + 2$, $x + y = m_2 + 1$ and $\theta_0 k(m_2 + 1) - k(m_2 + 2) = 0$. With the preceding results, a straightforward computation leads to

$$\begin{aligned}
&\Psi(x, d_{1G}, d_{2G}) \\
&= a(x)[h(x+1) - \theta_0 h(x)] + \sum_{y=0}^{m_2 - x} a(x)a(y)[\theta_0 k(x+y) - k(x+y+1)] - cf(x), \\
&\equiv \Psi_1(x, m_2). \qquad (2.15)
\end{aligned}$$

It can be shown that $\Psi(x, d_{1G}, d_{2G})/f(x)$ is increasing in $x$. However, it is not known whether $\Psi(x, d_{1G}, d_{2G})$ is positive or negative when the prior distribution $G$ is unknown.

**(c) As $m_1 + 1 \leq x \leq m_2$.**

It is possible that the set $\{x | m_1 + 1 \leq x \leq m_2\}$ be empty. When it is not empty, as $m_1 + 1 \leq x \leq m_2$, by the definitions of $d_{1G}$, $d_{2G}$, $m_1$ and $m_2$, a straightforward computation leads to

$$\begin{aligned}
&\Psi(x, d_{1G}, d_{2G}) \\
&= \sum_{y=0}^{m_2 - x} a(x)a(y)[\theta_0 k(x+y) - k(x+y+1)] - cf(x) \qquad (2.16) \\
&\equiv \Psi_2(x, m_2).
\end{aligned}$$

It is not known whether $\Psi(x, d_{1G}, d_{2G})$ is positive or negative when the prior distribution $G$ is unknown.

According to the preceding analysis, the Bayes stopping rule $\tau_G$ can be represented as:

$$\tau_G(x) = \begin{cases} 1 & \text{if } (x > m_2) \text{ or } (x \leq m_2 \text{ and } \Psi(x, d_{1G}, d_{2G}) \leq 0), \\ 0 & \text{if } (x \leq m_2 \text{ and } \Psi(x, d_{1G}, d_{2G}) > 0). \end{cases} \qquad (2.17)$$

## 3. Construction of Empirical Bayes Two-Stage Test

### 3.1. Empirical Bayes Framework

In the empirical Bayes framework, we let $(X_j, Y_j, \Theta_j)$ denote the random vector occurring at time $j$, $j = 1, 2, \ldots$, where $\Theta_j$ is a random parameter following the unknown prior distribution $G$; $X_j$ and $Y_j$ denote the random observations obtained at the first and second stage sampling, respectively, and conditioning on $\Theta_j = \theta_j$, $X_j$ and $Y_j$ are mutually independent, with the probability function $f(\cdot|\theta_j)$ of (2.1). It should be noted that the random parameter is unobservable while $X_j$ is observable. Also, $Y_j$ may or may not be observed depending on at time $j$ the second-stage sampling is made or not. It is assumed that $(X_j, Y_j, \Theta_j)$ $(j = 1, 2, \ldots,)$ are mutually independent. Therefore, $(X_j, Y_j, \Theta_j)$, $j = 1, 2, \ldots$, are iid. At the present time $n+1$, let $(X_{n+1},\ Y_{n+1},\ \Theta_{n+1}) = (X, Y, \Theta)$. We are interested in testing $H_0\colon \theta_{n+1} \geq \theta_0$ against $H_1\colon \theta_{n+1} < \theta_0$ with the loss function $L(\theta_{n+1}, (i, t))$ of (2.2), where $\theta_{n+1}$ is a realization of the random parameter $\Theta_{n+1}$. We let $\underset{\sim}{X}(n) = (X_1, \ldots, X_n)$ denote the accumulated historical data obtained at each first-stage sampling during the previous time.

Since the prior distribution $G$ is unknown, it is not possible to implement the Bayes two-stage test $(d_{1G}, d_{2G}, \tau_G)$ for the testing problem. The empirical Bayes approach is adopted. We attempt to incorporate useful information from the accumulated historical date $\underset{\sim}{X}(n)$ to improve the decision for the current testing problem. A two-stage test $(d_{1n}, d_{2n}, \tau_n)$, called as an empirical Bayes two-stage test, is a function of the present observation $(X_{n+1}, Y_{n+1}) = (x, y)$ and the past data $\underset{\sim}{X}(n)$ such that $\tau_n(x, \underset{\sim}{X}(n))$ is the probability of stopping sampling and making a decision immediately when $X_{n+1} = x$ and $\underset{\sim}{X}(n)$ are observed. When $X_{n+1} = x$ and $\underset{\sim}{X}(n)$ are observed and the decision to stop sampling is made, $d_{1n}(x, \underset{\sim}{X}(n))$ is the probability of accepting $H_0$. When the decision of taking the second-stage sampling is made and $Y_{n+1} = y$ is observed, $d_{2n}(x, y, \underset{\sim}{X}(n))$ is the probability of accepting $H_0$.

Let $R(G, (d_{1n}, d_{2n}, \tau_n)|\underset{\sim}{X}(n))$ be the Bayes risk of the empirical Bayes two-stage test $(d_{1n}, d_{2n}, \tau_n)$ conditioning on $\underset{\sim}{X}(n)$. Also, let $R(G, (d_{1n}, d_{2n}, \tau_n)) = E_{\underset{\sim}{X}(n)} R(G, (d_{1n}, d_{2n}, \tau_n)|\underset{\sim}{X}(n))$ denote the overall Bayes risk of the empirical Bayes two-stage test $(d_{1n}, d_{2n}, \tau_n)$, where the expectation $E_{\underset{\sim}{X}(n)}$ is taken with respect to the probability measure generated by $\underset{\sim}{X}(n)$. Since $R(G, (d_{1G}, d_{2G}, \tau_G))$ is the minimum Bayes risk, $R(G, (d_{1n}, d_{2n}, \tau_n)|\underset{\sim}{X}(n)) \geq R(G, (d_{1G}, d_{2G}, \tau_G))$ for all $\underset{\sim}{X}(n)$ and for all $n = 1, 2, \ldots$. Hence, $R(G, (d_{1n}, d_{2n}, \tau_n)) \geq R(G, (d_{1G}, d_{2G}, \tau_G))$ for all $n = 1, 2, \ldots$. The nonnegative regret Bayes risk $R(G, (d_{1n}, d_{2n}, \tau_n)) - R(G, (d_{1G}, d_{2G}, \tau_G))$ is used as a measure of performance of the empirical Bayes two-stage test $(d_{1n}, d_{2n}, \tau_n)$. A sequence of empirical Bayes two-stage tests $\{(d_{1n}, d_{2n}, \tau_n)\}_{n=1}^{\infty}$ is said to be asymptotically optimal relative to the prior distribution $G$ at a rate of convergence of order $O(\alpha_n)$ if $R(G,(d_{1n},d_{2n},\tau_n)) - R(G,(d_{1G},d_{2G},\tau_G)) = O(\alpha_n)$ where $\{\alpha_n\}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \alpha_n = 0$.

### 3.2. Estimation of $m_1, m_2$ and $\Psi(x, d_{1G}, d_{2G})$

In view of (2.12), (2.13) and (2.16), one can see that the Bayes two-stage test $(d_{1G}, d_{2G}, \tau_G)$ is characterized by the values of $m_1$, $m_2$ and $\Psi(x, d_{1G}, d_{2G})$ for $0 \leq x \leq m_2$. Hence, in order to construct an empirical Bayes two-stage test, we first construct empirical Bayes estimators for $m_1$, $m_2$, and $\Psi(x, d_{1G}, d_{2G})$. Before doing so, we investigate certain properties related to $m_1$ and $m_2$.

For each $z = 0, 1, \ldots$, let

$$b_1(z) = \theta_0 h(z) - h(z+1),$$
$$b_2(z) = \theta_0 k(z) - k(z+1),$$

and let $\{w_i(z)\}_{z=0}^{\infty}$, $i = 1, 2$, be positive weight functions. For each $i = 1, 2$, define

$$B_i(t) = \sum_{z=0}^{t} w_i(z) b_i(z). \tag{3.1}$$

Since $b_i(z) > (= \text{ or } <)\ 0$ if $z \leq m_i($ if $m_i < z < M_i$ or if $z \geq M_i)$, we have: $B_i(t)$ is increasing in $t$ and $B_i(t) > 0$ for $0 \leq t \leq m_i$; $B_i(t) = B_i(m_i)$ for $m_i < t < M_i$; and $B_i(t)$ is decreasing in $t$ for $t \geq M_i$. Thus,

$$m_i = \inf\{m \geq 0 | B_i(m) = \max_{t \geq 0} B_i(t)\}. \tag{3.2}$$

The property (3.2) will be used to construct estimators for $m_1$ and $m_2$, respectively.

For each $n = 1, 2, \ldots$, and $z = 0, 1, \ldots$, let

$$f_n(z) = \frac{1}{n} \sum_{j=1}^{n} I_{\{z\}}(X_j) \text{ and } k_n(z) = f_n(z)/a(z).$$

Then, $E_{\underset{\sim}{X}(n)} f_n(z) = f(z)$ and $E_{\underset{\sim}{X}(n)} k_n(z) = k(z)$.

Define $a(z) = 0$ if $z < 0$. Note that for each $z = 0, 1, \ldots$,

$$E_X[a(X-z)/a(X)] = \sum_{x=z}^{\infty} \frac{a(x-z)}{a(x)} f(x)$$

$$
\begin{aligned}
&= \sum_{x=z}^{\infty} a(x-z) \int \beta(\theta)\theta^{x} dG(\theta) \\
&= \int \theta^{z} \sum_{x=z}^{\infty} a(x-z)\beta(\theta)\theta^{x-z} dG(\theta) \qquad (3.3)\\
&= \int \theta^{z} dG(\theta) \\
&= h(z)
\end{aligned}
$$

since $\sum_{x=z}^{\infty} a(x-z)\beta(\theta)\ \theta^{x-z} = \sum_{t=0}^{\infty} a(t)\beta(\theta)\theta^{t} = 1.$

Based on (3.3), we define, for each $n = 1, 2, \ldots,$ and $z = 0, 1, \ldots$

$$h_n(z) = \frac{1}{n} \sum_{j=1}^{n} a(X_j - z)/a(X_j).$$

Note that $E_{\underset{\sim}{X}(n)} h_n(z) = h(z)$.

According to the form of $B_2(t)$, we define

$$B_{2n}(t) = \sum_{z=0}^{t} w_2(z) b_{2n}(z) \qquad (3.4)$$

where $b_{2n}(z) = \theta_0 k_n(z) - k_n(z+1)$. Let

$$m_{2n}^{*} = \inf\{m \geq 0 | B_{2n}(m) = \max_{t \geq 0} B_{2n}(t)\}. \qquad (3.5)$$

Next, define

$$B_{1n}(t) = \sum_{z=0}^{t} w_1(z) b_{1n}(z) \qquad (3.6)$$

where $b_{1n}(z) = \theta_0 h_n(z) - h_n(z+1)$. Since $m_1 \leq m_2$, therefore, we define

$$m_{1n}^{*} = \inf\{0 \leq m \leq m_{2n}^{*} | B_{1n}(m) = \max_{0 \leq t \leq m_{2n}^{*}} B_{1n}(t)\}. \qquad (3.7)$$

By the definition of $m_{1n}^{*}$, $m_{1n}^{*} \leq m_{2n}^{*}$. We use $m_{1n}^{*}$ and $m_{2n}^{*}$ to estimate $m_1$ and $m_2$, respectively.

Since

$$
\Psi(x, d_{1G}, d_{2G})
= \begin{cases}
-a(x)b_1(x) + \sum_{y=0}^{m_2-x} a(x)a(y)b_2(x+y) - cf(x) & \text{if } 0 \leq x \leq m_1 \\
\sum_{y=0}^{m_2-x} a(x)a(y)b_2(x+y) - cf(x) & \text{if } m_1 + 1 \leq x \leq m_2, \\
-cf(x) & \text{if } x > m_2;
\end{cases} \qquad (3.8)
$$

we define an empirical Bayes estimate $\Psi_n^*(x)$ of $\Psi(x, d_{1G}, d_{2G})$ as follows:

$$\Psi_n^*(x) = \begin{cases} -a(x)b_{1n}(x) + \sum_{y=0}^{m_{2n}^*-x} a(x)a(y)b_{2n}(x+y) - cf_n(x) & \text{if } 0 \le x \le m_{1n}^*, \\ \sum_{y=0}^{m_{2n}^*-x} a(x)a(y)b_{2n}(x+y) - cf_n(x) & \text{if } m_{1n}^*+1 \le x \le m_{2n}^*, \\ -cf_n(x) & \text{if } x > m_{2n}^*. \end{cases} \tag{3.9}$$

### 3.3. The Proposed Empirical Two-Stage Test

We propose an empirical Bayes two-stage test $(d_{1n}^*, d_{2n}^*, \tau_n^*)$ as follows:

(1) For each $x = 0, 1, \ldots$

$$d_{1n}^*(x) = \begin{cases} 0 & \text{if } x \le m_{1n}^*, \\ 1 & \text{if } x > m_{1n}^*. \end{cases} \tag{3.10}$$

(2) For each pair $(x, y)$, $x, y = 0, 1, \ldots$

$$d_{2n}^*(x,y) = \begin{cases} 0 & \text{if } x+y \le m_{2n}^*, \\ 1 & \text{if } x+y > m_{2n}^*. \end{cases} \tag{3.11}$$

(3) For each $x = 0, 1, \ldots$,

$$\tau_n^*(x) = \begin{cases} 1 & \text{if } (x > m_{2n}^*) \text{ or } (0 \le x \le m_{2n}^* \text{ and } \Psi_n^*(x) \le 0). \\ 0 & \text{if } (0 \le x \le m_{2n}^* \text{ and } \Psi_n^*(x) > 0). \end{cases} \tag{3.12}$$

The conditional Bayes risk of the empirical Bayes two-stage test $(d_{1n}^*, d_{2n}^*, \tau_n^*)$ given $\underset{\sim}{X}(n)$ is

$$\begin{aligned} &R(G, (d_{1n}^*, d_{2n}^*, \tau_n^*) | \underset{\sim}{X}(n)) \\ &= \sum_x \tau_n^*(x)[d_{1n}^*(x)W_1(x) + \ell\ (x)f(x) + cf(x)] \\ &\quad + \sum_x [1 - \tau_n^*(x)]\{\sum_y d_{2n}^*(x,y)W_2(x,y) + \ell\ (x)f(x) + 2cf(x)\} \end{aligned} \tag{3.14}$$

and its associated Bayes risk is

$$R(G, (d_{1n}^*, d_{2n}^*, \tau_n^*)) = E_{\underset{\sim}{X}(n)} R(G, (d_{1n}^*, d_{2n}^*, \tau_n^*) | \underset{\sim}{X}(n)).$$

## 4. Asymptotic Optimality

### 4.1. Analysis of Regret Bayes Risk

Define events $E_{in} = \{m^*_{in} = m_i\}$, $E^c_{in} = \{m^*_{in} \neq m_i\}$, $E^c_{in}(1) = \{m^*_{in} < m_i\}$, $E^c_{in}(2) = \{m_i < m^*_{in} < M_i\}$ and $E^c_{in}(3) = \{m^*_{in} \geq M_i\}$ for $i = 1, 2$. Also, let $J(A)$ denote the indicator function of an event $A$. Note that $E^c_{in}(2)$ may or may not be an empty set. When $\theta_0 h(z) - h(z+1) \neq 0$ for all $z = 0, 1, \ldots$, then $E^*_{1n}(2) = \phi$. When $\theta_0 h(m_1+1) - h(m_1+2) = 0$, then $E^c_{1n}(2) = \{m^*_{1n} = m_1 + 1\}$. Similarly, when $\theta_0 k(z) - k(z+1) \neq 0$ for all $z = 0, 1, \ldots$, $E^c_{2n}(2) = \phi$. When $\theta_0 k(m_2+1) - k(m_2+2) = 0$, $E^c_{2n}(2) = \{m^*_{2n} = m_2 + 1\}$.

Given $\underset{\sim}{X}(n)$, the conditional regret Bayes risk of $(d^*_{1n}, d^*_{2n}, \tau^*_n)$ can be written as:

$$\begin{aligned} &R(G, (d^*_{1n}, d^*_{2n}, \tau^*_n)|\underset{\sim}{X}(n)) - R(G, (d_{1G}, d_{2G}, \tau_G)) \\ &= [R(G, (d^*_{1n}, d^*_{2n}, \tau^*_n)|\underset{\sim}{X}(n)) - R(G, (d_{1G}, d_{2G}, \tau^*_n)|\underset{\sim}{X}(n))] \\ &\qquad + [R(G, (d_{1G}, d_{2G}, \tau^*_n)|\underset{\sim}{X}(n)) - R(G, (d_{1G}, d_{2G}, \tau_G))] \\ &= I(n) + II(n), \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} I(n) &= R(G, (d^*_{1n}, d^*_{2n}, \tau^*_n)|\underset{\sim}{X}(n)) - R(G, (d_{1G}, d_{2G}, \tau^*_n)|\underset{\sim}{X}(n)) \\ &= \sum_x \tau^*_n(x)[d^*_{1n}(x) - d_{1G}(x)]W_1(x) \\ &\qquad + \sum_x [1 - \tau^*_n(x)]\{\sum_y [d^*_{2n}(x,y) - d_{2G}(x,y)]W_2(x,y)\}, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} II(n) &= R(G, (d_{1G}, d_{2G}, \tau^*_n)|\underset{\sim}{X}(n)) - R(G, (d_{1G}, d_{2G}, \tau_G)) \\ &= \sum_x [\tau^*_n(x) - \tau_G(x)]\Psi(x, d_{1G}, d_{2G}). \end{aligned} \tag{4.3}$$

#### 4.1.1. Analysis of $I(n)$

Since on $E_{1n}$, $[d^*_{1n}(x) - d_{1G}(x)]W_1(x)J(E_{1n}) = 0$ for all $x$, and on $E_{2n}$, $[d^*_{2n}(x,y) - d_{2G}(x,y)]W_2(x,y)J(E_{2n}) = 0$ for all $(x,y)$, therefore,

$$
\begin{aligned}
I(n) &= \sum_x \tau_n^*(x)[d_{1n}^*(x) - d_{1G}(x)]W_1(x)J(E_{1n}^c) \\
&\quad + \sum_x [1-\tau_n^*(x)]\{\sum_y [d_{2n}^*(x,y) - d_{2G}(x,y)]W_2(x,y)\}J(E_{2n}^c) \\
&= \sum_x \tau_n^*(x)[d_{1n}^*(x) - d_{1G}(x)]W_1(x)J(E_{2n}E_{1n}^c) \\
&\quad + \sum_x \tau_n^*(x)[d_{1n}^*(x) - d_{1G}(x)]W_1(x)J(E_{2n}^c(1)E_{1n}^c) \\
&\quad + \sum_x \tau_n^*(x)[d_{1n}^*(x) - d_{1G}(x)]W_1(x)J(E_{2n}^c(2)E_{1n}^c) \\
&\quad + \sum_x \tau_n^*(x)[d_{1n}^*(x) - d_{1G}(x)]W_1(x)J(E_{2n}^c(3)E_{1n}^c) \\
&\quad + \sum_x [1-\tau_n^*(x)]\{\sum_y [d_{2n}^*(x,y) - d_{2G}(x,y)]W_2(x,y)\}J(E_{2n}^c(1)) \\
&\quad + \sum_x [1-\tau_n^*(x)]\{\sum_y [d_{2n}^*(x,y) - d_{2G}(x,y)]W_2(x,y)\}J(E_{2n}^c(2)) \\
&\quad + \sum_x [1-\tau_n^*(x)]\{\sum_y [d_{2n}^*(x,y) - d_{2G}(x,y)]W_2(x,y)\}J(E_{2n}^c(3)) \\
&\equiv I_1(n) + I_2(n) + I_3(n) + I_4(n) + I_5(n) + I_6(n) + I_7(n).
\end{aligned} \tag{4.4}
$$

Since $0 \le \tau_n^*(x) \le 1$, $[d_{1n}^*(x) - d_{1G}(x)]\ W_1(x) \ge 0$, $[d_{2n}^*(x,y) - d_{2G}(x,y)]W_2(x,y) \ge 0$ for all $x$ and $y$, $I_i(n) \ge 0$ for all $i = 1, \ldots, 7$.

(I.1) Note on $E_{1n}^c(2)$, $[d_{1n}^*(x) - d_{1G}(x)]\ W_1(x) = 0$ for all $x = 0, 1, \ldots$. By the definition of the events $E_{2n}E_{1n}^c(j), j = 1, 3$, and the decision rules $d_{1n}^*$ and $d_{1G}$, we obtain

$$
\begin{aligned}
I_1(n) &= \sum_x \tau_n^*(x)[d_{1n}^*(x) - d_{1G}(x)]W_1(x)J(E_{2n}E_{1n}^c(1)) \\
&\quad + \sum_x \tau_n^*(x)[d_{1n}^*(x) - d_{1G}(x)]W_1(x)J(E_{2n}E_{1n}^c(3)) \\
&= [\sum_{x=m_{1n}^*+1}^{m_1} \tau_n^*(x)W_1(x)]J(E_{2n}E_{1n}^c(1)) \\
&\quad + [\sum_{x=M_1}^{m_{1n}^*} \tau_n^*(x)[-W_1(x)]]J(E_{2n}E_{1n}^c(3)) \\
&\le [\sum_{x=0}^{m_1} W_1(x)]J(E_{2n}E_{1n}^c(1))
\end{aligned}
$$

$$+[\sum_{x=M_1}^{m_2}[-W_1(x)]]J(E_{2n}E_{1n}^c(3)).$$

Note that when $m_2 < M_1$, $E_{2n}E_{1n}^c(3) = \phi$, since on $E_{2n}E_{1n}^c(3)$, it implies that $m_2 = m_{2n}^* \geq m_{1n}^* \geq M_1$, which is a contradiction.

(I.2) On $E_{2n}^c(1)$, $m_{1n}^* \leq m_{2n}^* < m_2$. Let $a_n = \min(m_{1n}^*, m_1)$, $b_n = \max(m_{1n}^*, m_1) \leq m_2$.

Then

$$\begin{aligned} I_2(n) &= \sum_x \tau_n^*(x)[d_{1n}^*(x) - d_{1G}(x)]W_1(x)J(E_{2n}^c(1)E_{1n}^c) \\ &= [\sum_{x=a_n}^{b_n} \tau_n^*(x)|W_1(x)|]J(E_{2n}^c(1)E_{1n}^c) \\ &\leq [\sum_{x=0}^{m_2}|W_1(x)|]J(E_{2n}^c(1)). \end{aligned}$$

(I.3) On $E_{1n}^c(2)$, $[d_{1n}^*(x) - d_{1G}(x)]W_1(x) = 0$ for all $x = 0, 1, \ldots$. Therefore,

$$\begin{aligned} I_3(n) &= \sum_x \tau_n^*(x)[d_{1n}^*(x) - d_{1G}(x)]W_1(x)J(E_{2n}^c(2)E_{1n}^c(1)) \\ &\quad + \sum_x \tau_n^*(x)[d_{1n}^*(x) - d_{1G}(x)]W_1(x)J(E_{2n}^c(2)E_{1n}^c(3)) \\ &= [\sum_{x=m_{1n}^*+1}^{m_1} \tau_n^*(x)W_1(x)]J(E_{2n}^c(2)E_{1n}^c(1)) \\ &\quad + [\sum_{x=M_1}^{m_{1n}^*} \tau^*(x)[-W_1(x)]]J(E_{2n}^c(2)E_{1n}^c(3)) \\ &\leq [\sum_{x=0}^{m_1} W_1(x)]J(E_{2n}^c(2)E_{1n}^c(1)) + [\sum_{x=M_1}^{m_2+1}[-W_1(x)]]J(E_{2n}^c(2)E_{1n}^c(3)). \end{aligned}$$

(I.4)

$$\begin{aligned} I_4(n) &= \sum_x \tau_n^*(x)[d_{1n}^*(x) - d_{1G}(x)]W_1(x)J(E_{2n}^c(3)E_{1n}^c) \\ &\leq [\sum_{x=0}^{m_1} W_1(x) + \sum_{x=M_1}^{\infty}[-W_1(x))]]J(E_{2n}^c(3)E_{1n}^c). \end{aligned}$$

(I.5) On $E_{2n}^c(1)$, $m_{2n}^* < m_2$. Also, $\tau_n^*(x) = 1$ for $x > m_{2n}^*$. Let

$$T_1(m_2) = \{(x, y)|x + y \leq m_2,\ x, y = 0, 1, 2, \ldots\}.$$

Then,

$$
\begin{aligned}
I_5(n) &= \sum_{x=0}^{m_{2n}^*}[1-\tau_n^*(x)]\{\sum_{y=m_{2n}^*-x}^{m_2-x}[d_{2n}^*(x,y)-d_{2G}(x,y)]W_2(x,y)\}J(E_{2n}^c(1)) \\
&\le [\sum_{(x,y)\in T_1(m_2)} W_2(x,y)]J(E_{2n}^c(1)).
\end{aligned}
$$

(I.6) Let $T_2(m_2, M_2) = \{(x,y)|m_2 < x+y < M_2,\ x,y = 0,1,2,\ldots\}$. On $E_{2n}^c(2)$, $m_2 < m_{2n}^* < M_2$. Also, $1-\tau_n^*(x) = 0$ for $x > m_{2n}^*$. $d_{2n}^*(x,y) - d_{2G}(x,y) = 0$ for $(x,y) \in T_1(m_2)$, $W_2(x,y) = 0$ for $(x,y) \in T_2(m_2, M_2)$. Hence, $I_6(n) = 0$.

(I.7) Let $T_3(M_2) = \{(x,y)|x+y \ge M_2,\ x,y = 0,1,\ldots\}$. On $E_{2n}^c(3)$, $m_{2n}^* \ge M_2$. Hence,

$$
\begin{aligned}
I_7(n) &= \sum_x[1-\tau_n^*(x)]\{\sum_{y=M_2-x}^{m_{2n}^*-x}[d_{2n}^*(x,y)-d_{2G}(x,y)]W_2(x,y)\}J(E_{2n}^c(3)) \\
&\le [\sum_{(x,y)\in T_3(M_2)} [-W_2(x,y)]]J(E_{2n}^c(3)).
\end{aligned}
$$

### 4.1.2. Analysis of II(n)

First, we have

$$
\begin{aligned}
II(n) &= II(n)J(E_{2n}E_{1n}) + II(n)J(E_{2n}E_{1n}^c(1)) + II(n)J(E_{2n}E_{1n}^c(2)) \\
&\quad + II(n)J(E_{2n}E_{1n}^c(3)) + II(n)J(E_{2n}^c(1)) + II(n)J(E_{2n}^c(2)E_{1n}) \\
&\quad + II(n)J(E_{2n}^c(2)E_{1n}^c(1)) + II(n)J(E_{2n}^c(2)E_{1n}^c(2)) \\
&\quad + II(n)J(E_{2n}^c(2)E_{1n}^c(3)) + II(n)J(E_{2n}^c(3)) \\
&= \sum_{j=1}^{10} II_j(n).
\end{aligned} \tag{4.5}
$$

(II.1) Let $S_1 = \{0 \le x \le m_1|\Psi(x, d_{1G}, d_{2G}) \ne 0\}$ and $S_2 = \{m_1+1 \le x \le m_2|\Psi(x, d_{1G}, d_{2G}) \ne 0\}$. On $E_{2n}, m_{2n}^* = m_2$. By the definition of $\tau_n^*$ and $\tau_G$, $\tau_n^*(x) = \tau_G(x) = 1$ for $x > m_2$. Thus,

$$
\begin{aligned}
II_1(n) &= II(n)J(E_{2n}E_{1n}) \\
&= \sum_{x\in S_1}[\tau_n^*(x)-\tau_G(x)]\Psi(x, d_{1G}, d_{2G})J(E_{2n}E_{1n}) \\
&\quad + \sum_{x\in S_2}[\tau_n^*(x)-\tau_G(x)]\Psi(x, d_{1G}, d_{2G})J(E_{2n}E_{1n}),
\end{aligned}
$$

(II.2)

$$
\begin{aligned}
II_2(n) &= II(n)J(E_{2n}E_{1n}^c(1)) \\
&= \sum_{x \in S} [\tau_n^*(x) - \tau_G(x)]\Psi(x, d_{1G}, d_{2G})J(E_{2n}E_{1n}^c(1)) \\
&\leq [\sum_{x \in S} |\Psi(x, d_{1G}, d_{2G})|]J(E_{2n}E_{1n}^2(1)),
\end{aligned}
$$

where $S = S_1 \bigcup S_2$.

(II.3) Let $S_2(m_1, M_1) = \{x \in S_2 |\ m_1 < x < M_1$ and $\Psi(x, d_{1G}, d_{2G}) \neq 0\}$ and $S_2^c(m_1, M_1) = \{x \in S_2 \backslash S_2(m_1, M_1)\}$. Then,

$$
\begin{aligned}
II_3(n) &= II(n)J(E_{2n}E_{1n}^c(2)) \\
&= \sum_{x \in S_1} [\tau_n^*(x) - \tau_G(x)]\Psi(x, d_{1G}, d_{2G})J(E_{2n}E_{1n}^c(2)) \\
&\quad + \sum_{x \in S_2^c(m_1, M_1)} [\tau_n^*(x) - \tau_G(x)]\Psi(x, d_{1G}, d_{2G})J(E_{2n}E_{1n}^c(2)) \\
&\quad + \sum_{x \in S_2(m_1, M_1)} [\tau_n^*(x) - \tau_G(x)]\Psi(x, d_{1G}, d_{2G})J(E_{2n}E_{1n}^c(2)).
\end{aligned}
$$

(II.4)

$$
\begin{aligned}
II_4(n) &= II(n)J(E_{2n}E_{1n}^c(3)) \\
&= \sum_{x \in S} [\tau_n^*(x) - \tau_G(x)]\Psi(x, d_{1G}, d_{2G})J(E_{2n}E_{1n}^c(3)) \\
&\leq [\sum_{x \in S} |\Psi(x, d_{1G}, d_{2G})|]J(E_{2n}E_{1n}^c(3)).
\end{aligned}
$$

(II.5) On $E_{2n}^c(1)$, $m_{2n}^* < m_2$. By the definitions of $\tau_n^*$ and $\tau_G$, $\tau_n^*(x) = \tau_G(x) = 1$ for $x > m_2$. Therefore,

$$
\begin{aligned}
II_5(n) &= II(n)J(E_{2n}^c(1)) \\
&= \sum_{x \in S} [\tau_n^*(x) - \tau_G(x)]\Psi(x, d_{1G}, d_{2G})J(E_{2n}^c(1)) \\
&\leq [\sum_{x \in S} |\Psi(x, d_{1G}, d_{2G})|]J(E_{2n}^c(1)).
\end{aligned}
$$

(II.6) Let $S_3 = \{x | m_2 < x < M_2, \Psi(x, d_{1G}, d_{2G}) \neq 0\}$. On $E_{2n}^c(2)$, $m_2 < m_{2n}^* < M_2$. Thus,

$\tau_n^*(x) = \tau_G(x) = 1$ for $x \geq M_2$. Therefore,

$$\begin{aligned} II_6(n) &= II(n)J(E_{2n}^c(2)E_{1n}) \\ &= \sum_{x\in S_1}[\tau_n^*(x) - \tau_G(x)]\Psi(x, d_{1G}, d_{2G})J(E_{2n}^c(2)E_{1n}) \\ &\quad + \sum_{x\in S_2}[\tau_n^*(x) - \tau_G(x)]\Psi(x, d_{1G}, d_{2G})J(E_{2n}^c(2)E_{1n}) \\ &\quad + \sum_{x\in S_3}[\tau_n^*(x) - \tau_G(x)]\Psi(x, d_{1G}, d_{2G})J(E_{2n}^c(2)E_{1n}). \end{aligned}$$

(II.7) On $E_{2n}^c(2)E_{1n}^c(1)$, $m_2 < m_{2n}^* < M_2$ and $m_{1n}^* < m_1$. Thus,

$$\begin{aligned} II_7(n) &= II(n)J(E_{2n}^c(2)E_{1n}^c(1)) \\ &= \sum_{x\in S\bigcup S_3}[\tau_n^*(x) - \tau_G(x)]\Psi(x, d_{1G}, d_{2G})J(E_{2n}^c(2)E_{1n}^c(1)) \\ &\leq [\sum_{x\in S\bigcup S_3}|\Psi(x, d_{1G}, d_{2G})|]J(E_{2n}^c(2)E_{1n}^c(1)). \end{aligned}$$

(II.8) On $E_{2n}^c(2)E_{1n}^c(2)$, $m_2 < m_{2n}^* < M_2$, $m_1 < m_{1n}^* < M_1$, $\tau_n^*(x) = \tau_G(x) = 1$ for $x \geq M_2$. Thus,

$$\begin{aligned} II_8(n) &= II(n)J(E_{2n}^c(2)E_{1n}^c(2)) \\ &= \sum_{x\in S_1}[\tau_n^*(x) - \tau_G(x)]\Psi(x, d_{1G}, d_{2G})J(E_{2n}^c(2)E_{1n}^c(2)) \\ &\quad + \sum_{x\in S_2^c(m_1,M_1)}[\tau_n^*(x) - \tau_G(x)]\Psi(x, d_{1G}, d_{2G})J(E_{2n}^c(2)E_{1n}^c(2)) \\ &\quad + \sum_{x\in S_2(m_1,M_1)}[\tau_n^*(x) - \tau_G(x)]\Psi(x, d_{1G}, d_{2G})J(E_{2n}^c(2)E_{1n}^c(2)) \\ &\quad + \sum_{x\in S_3}[\tau_n^*(x) - \tau_G(x)]\Psi(x, d_{1G}, d_{2G})J(E_{2n}^c(2)E_{1n}^c(2)). \end{aligned}$$

(II.9)

$$\begin{aligned} II_9(n) &= II(n)J(E_{2n}^c(2)E_{1n}^c(3)) \\ &= \sum_{x\in S\bigcup S_3}[\tau_n^*(x) - \tau_G(x)]\Psi(x, d_{1G}, d_{2G})J(E_{2n}^c(2)E_{1n}^c(3)) \\ &\leq [\sum_{x\in S\bigcup S_3}|\Psi(x, d_{1G}, d_{2G})|]J(E_{2n}^c(2)E_{1n}^c(3)). \end{aligned}$$

(II.10)

$$\begin{aligned} II_{10}(n) &= II(n)J(E_{2n}^c(3)) \\ &= \sum_x [\tau_n^*(x) - \tau_G(x)]\Psi(x, d_{1G}, d_{2G})J(E_{2n}^c(3)) \\ &\le [\sum_x |\Psi(x, d_{1G}, d_{2G})|]J(E_{2n}^c(3)). \end{aligned}$$

### 4.2. An Upper Bound of Regret Bayes Risk

According to the preceding analysis, we can obtain an upper bound of the regret Bayes risk of $(d_{1n}^*, d_{2n}^*, \tau_n^*)$ which is given as follows.

$$\begin{aligned} &R(G, (d_{1n}^*, d_{2n}^*, \tau_n^*)) - R(G, (d_{1G}, d_{2G}, \tau_G)) \\ &= E_{\underset{\sim}{X}(n)}[I(n)] + E_{\underset{\sim}{X}(n)}[II(n)] \\ &\le \left[\sum_{x=0}^{m_1} W_1(x)\right] E_{\underset{\sim}{X}(n)}[J(E_{2n}E_{1n}^c(1))] + \left[\sum_{x=M_1}^{m_2} [-W_1(x)]\right] E_{\underset{\sim}{X}(n)}[J(E_{2n}E_{1n}^c(3))] \\ &\quad + \left[\sum_{x=0}^{m_2} |W_1(x)|\right] E_{\underset{\sim}{X}(n)}[J(E_{2n}^c(1))] \\ &\quad + \left[\sum_{x=0}^{m_1} W_1(x)\right] E_{\underset{\sim}{X}(n)}[J(E_{2n}^c(2)E_{1n}^c(1))] \\ &\quad + \left[\sum_{x=M_1}^{m_2+1} [-W_1(x)]\right] E_{\underset{\sim}{X}(n)}[J(E_{2n}^c(2)E_{1n}^c(3))] \\ &\quad + \left[\sum_{x=0}^{m_1} W_1(x) + \sum_{x=M_1}^{\infty} [-W_1(x)]\right] E_{\underset{\sim}{X}(n)}[J(E_{2n}^c(3)E_{1n}^c)] \\ &\quad + \left[\sum_{(x,y)\in T_1(m_2)} W_2(x,y)\right] E_{\underset{\sim}{X}(n)}[J(E_{2n}^c(1))] \\ &\quad + \left[\sum_{(x,y)\in T_3(M_2)} [-W_2(x,y)]\right] E_{\underset{\sim}{X}(n)}[J(E_{2n}^c(3))] \\ &\quad + \sum_{x\in S_1} |\Psi(x, d_{1G}, d_{2G})| E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)| J(E_{2n}E_{1n})] \\ &\quad + \sum_{x\in S_2} |\Psi(x, d_{1G}, d_{2G})| E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)| J(E_{2n}E_{1n})] \end{aligned}$$

$$
\begin{aligned}
&+ \left[\sum_{x \in S} |\Psi(x, d_{1G}, d_{2G})|\right] E_{\underset{\sim}{X}(n)}[J(E_{2n}E_{1n}^c(1))] \\
&+ \sum_{x \in S_1} |\Psi(x, d_{1G}, d_{2G})| E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)| J(E_{2n}E_{1n}^c(2))] \\
&+ \sum_{x \in S_2^c(m_1, M_1)} |\Psi(x, d_{1G}, d_{2G})| E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)| J(E_{2n}E_{1n}^c(2))] \\
&+ \sum_{x \in S_2(m_1, M_1)} |\Psi(x, d_{1G}, d_{2G})| E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)| J(E_{2n}E_{1n}^c(2))] \\
&+ [\sum_{x \in S} |\Psi(x, d_{1G}, d_{2G})|] E_{\underset{\sim}{X}(n)}[J(E_{2n}E_{1n}^c(3))] \\
&+ \left[\sum_{x \in S} |\Psi(x, d_{1G}, d_{2G})|\right] E_{\underset{\sim}{X}(n)}[J(E_{2n}^c(1))] \\
&+ \sum_{x \in S_1} |\Psi(x, d_{1G}, d_{2G})| E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)| J(E_{2n}^c(2)E_{1n})] \\
&+ \sum_{x \in S_2} |\Psi(x, d_{1G}, d_{2G})| E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)| J(E_{2n}^c(2)E_{1n})] \\
&+ \sum_{x \in S_3} |\Psi(x, d_{1G}, d_{2G})| E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)| J(E_{2n}^c(2)E_{1n})] \\
&+ \left[\sum_{x \in S \bigcup S_3} |\Psi(x, d_{1G}, d_{2G})|\right] E_{\underset{\sim}{X}(n)}[J(E_{2n}^c(2)E_{1n}^c(1))] \qquad (4.6) \\
&+ \sum_{x \in S_1} |\Psi(x, d_{1G}, d_{2G})| E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)| J(E_{2n}^c(2)E_{1n}^c(2))] \\
&+ \sum_{x \in S_2^c(m_1, M_1)} |\Psi(x, d_{1G}, d_{2G})| E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)| J(E_{2n}^c(2)E_{1n}^c(2))] \\
&+ \sum_{x \in S_2(m_1, M_1)} |\Psi(x, d_{1G}, d_{2G})| E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)| J(E_{2n}^c(2)E_{1n}^c(2))] \\
&+ \sum_{x \in S_3} |\Psi(x, d_{1G}, d_{2G})| E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)| J(E_{2n}^c(2)E_{1n}^c(2))] \\
&+ \left[\sum_{x \in S \bigcup S_3} |\Psi(x, d_{1G}, d_{2G})|\right] E_{\underset{\sim}{X}(n)}[J(E_{2n}^c(2)E_{1n}^c(3))] \\
&+ \left[\sum_{x} |\Psi(x, d_{1G}, d_{2G})|\right] E_{\underset{\sim}{X}(n)}[J(E_{2n}^c(3))].
\end{aligned}
$$

We claim the finiteness of the following three summations.

**Lemma 4.1.** (a) $\sum_x |\Psi(x, d_{1G}, d_{2G})| < \infty$;

(b) Under the assumption that $\int \frac{\theta}{\beta(\theta)} dG(\theta) < \infty$, we have

$$\text{(b.1)} \quad \sum_{x=M_1}^{\infty} [-W_1(x)] < \infty \qquad \text{and (b.2)} \qquad \sum_{(x,y)\in T_3(M_2)} [-W_2(x,y)] < \infty.$$

**Proof:** (a) By (2.14),

$$\begin{aligned} \sum_x |\Psi(x, d_{1G}, d_{2G})| &= \sum_{x=0}^{m_2} |\Psi(x, d_{1G}, d_{2G})| + \sum_{x=m_2+1}^{\infty} cf(x) \\ &\le \sum_{x=0}^{m_2} |\Psi(x, d_{1G}, d_{2G})| + c < \infty \end{aligned}$$

since $|\Psi(x, d_{1G}, d_{2G})|$ is finite for each $x$ and $m_2$ is a finite number.

(b.1) By the definition of $W_1(x)$, $-W_1(x) > 0$ for $x \ge M_1$ and

$$\begin{aligned} \sum_{x=M_1}^{\infty} [-W_1(x)] &= \sum_{x=M_1}^{\infty} [h(x+1) - \theta_0 h(x)]a(x) \\ &< \sum_{x=M_1}^{\infty} a(x)h(x+1) \\ &= \sum_{x=M_1}^{\infty} a(x) \int \theta^{x+1} dG(\theta) \\ &= \int \theta \sum_{x=M_1}^{\infty} a(x)\theta^x dG(\theta) \\ &\le \int \theta \sum_{x=0}^{\infty} a(x)\theta^x dG(\theta) \\ &= \int \frac{\theta}{\beta(\theta)} dG(\theta) < \infty. \end{aligned}$$

(b.2) For $(x,y) \in T_3(M_2)$, $-W_2(x,y) = a(x)a(y)[k(x+y+1) - \theta_0 k(x+y)] > 0$, and

$$\begin{aligned} &\sum_{(x,y)\in T_3(M_2)} [-W_2(x,y)] \\ \le &\sum_{(x,y)\in T_3(M_2)} a(x)a(y)k(x+y+1) \end{aligned}$$

$$= \sum_x \sum_{y=M_2-x}^{\infty} \int a(x)a(y)\theta^{x+y+1}\beta(\theta)dG(\theta)$$

$$= \int \sum_x a(x)\theta^{x+1} \left[ \sum_{y=M_2-x}^{\infty} a(y)\theta^y\beta(\theta) \right] dG(\theta)$$

$$\leq \int \sum_x a(x)\theta^{x+1} dG(\theta) \quad \text{since} \quad 0 \leq \sum_{y=M_2-x}^{\infty} a(y)\theta^y\beta(\theta) \leq 1$$

$$= \int \frac{\theta}{\beta(\theta)} dG(\theta) < \infty. \qquad \square$$

For each $x$ and $y$, $|W_1(x)|$, $|W_2(x,y)|$ and $|\Psi(x, d_{1G}, d_{2G})|$ are finite. Also, $T_1(m_2)$, $S_1$, $S_2$ and $S_3$ are finite sets. In view of (4.6) and Lemma 4.1, to investigate the asymptotic optimality of the regret Bayes risk of $(d_{1n}^*, d_{2n}^*, \tau_n^*)$, it suffices to evaluate the following expectations.

(1) $E_{\underset{\sim}{X}(n)}[J(E_{2n}E_{1n}^c(1))]$,

(2) $E_{\underset{\sim}{X}(n)}[J(E_{2n}E_{1n}^c(3))]$,

(3) $E_{\underset{\sim}{X}(n)}[J(E_{2n}^c(2)E_{1n}^c(1))]$,

(4) $E_{\underset{\sim}{X}(n)}[J(E_{2n}^c(2)E_{1n}^c(3))]$,

(5) $E_{\underset{\sim}{X}(n)}[J(E_{2n}^c(1))]$,

(6) $E_{\underset{\sim}{X}(n)}[J(E_{2n}^c(3))]$,

(7) (a) for $x \in S_1$, $E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)|J(E_{2n}E_{1n})]$,

(b) for $x \in S_2$, $E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)|J(E_{2n}E_{1n})]$,

(8) (a) For $x \in S_1$, $E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)|J(E_{2n}E_{1n}^c(2))]$,

(b) for $x \in S_2^c(m_1, M_1)$, $E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)|J(E_{2n}E_{1n}^c(2))]$,

(c) for $x \in S_2(m_1, M_1)$, $E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)|J(E_{2n}E_{1n}^c(2))]$,

(9) (a) For $x \in S_1$, $E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)|J(E_{2n}^c(2)E_{1n})]$,

(b) for $x \in S_2$, $E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)|J(E_{2n}^c(2)E_{1n})]$,

(c) for $x \in S_3$, $E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)|J(E_{2n}^c(2)E_{1n})]$,

(10) (a) For $x \in S_1$, $E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)|J(E_{2n}^c(2)E_{1n}^c(2))]$,

(b) for $x \in S_2^c(m_1, M_1)$, $E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)|J(E_{2n}^c(2)E_{1n}^c(2))]$,

(c) for $x \in S_2(m_1, M_1)$, $E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)|J(E_{2n}^c(2)E_{1n}^c(2))]$,

(d) for $x \in S_3$, $E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)|J(E_{2n}^c(2)E_{1n}^c(2))]$.

For evaluating these expectations, certain useful lemmas are introduced in Section 5.

### 4.3. Evaluation of Expectations

**Proposition 4.1.** $E_{\underset{\sim}{X}(n)}[J(E_{2n}^c(1))] \le (m_2+1)\exp\{-nc_1\}$,

where $c_1 = 2 \min\limits_{0 \le z \le m_2} [(\theta_0 k(z) - k(z+1))/(\frac{\theta_0}{a(z)} + \frac{1}{a(z+1)})]^2 > 0$.

**Proof:** By Lemmas 5.5 and 5.2,

$$\begin{aligned} E_{\underset{\sim}{X}(n)}[J(E_{2n}^c(1))] &\le \sum_{z=0}^{m_2} P\{\theta_0 k_n(z) - k_n(z+1) \le 0\} \\ &\le (m_2+1)\exp\{-nc_1\}. \end{aligned}$$

□

**Proposition 4.2.** Suppose that both $\{\frac{w_2(z)}{a(z)}\}_{z=0}^{\infty}$ and $\{\frac{w_2(z)}{a(z+1)}\}_{z=0}^{\infty}$ are nonincreasing in $z$ and bounded above by 1. Then,

$$E_{\underset{\sim}{X}(n)}[J(E_{2n}^c(3))] \le d\exp\{-nc_3\}.$$

where $c_3 = 2[L(M_2)\min(\theta_0^{-1}, 1)/4]^2$, $L(M_2) = [k(M_2+1) - \theta_0 k(M_2)]w_2(M_2)$ and $d$ is a positive constant independent of the distribution function $F$, the marginal distribution function of the random variable $X$.

**Proof:** By Lemma 5.4 and an inequality of Dvoretzky, Kiefer and Wolfowitz (1956), we obtain

$$\begin{aligned} E_{\underset{\sim}{X}(n)}[J(E_{2n}^c(3))] &\le P\{\sup_{t \ge 0}|F_n(t) - F(t)| > \frac{L(M_2)\min(\theta_0^{-1}, 1)}{4}\} \\ &\le d\exp\{-2n[L(M_2)\min(\theta_0^{-1}, 1)/4]^2\} \\ &= d\exp\{-nc_3\}. \end{aligned}$$

□

**Proposition 4.3.** $E_{\underset{\sim}{X}(n)}[J(E_{2n}E_{1n}^c(1))] \le (m_1+1)\exp\{-nc_2\}$, where

$$c_2 = \min_{z\in A}\{\ell n\ [m_z(\theta_0 h(z) - h(z+1))]^{-1}\} > 0, A = \{z|0 \le z \le m_1 \text{ or } M_1 \le z \le M_2\}$$

and the function $m_z(\cdot)$ is defined in(5.1).

**Proof:** This proposition directly follows from Lemmas 5.3 and 5.6 and the definition of $c_2$. □

From Lemmas 5.7 and 5.3, we can obtain the following result.

**Proposition 4.4.** If $M_1 > m_2$, then $E_{\underset{\sim}{X}(n)}[J(E_{2n}E_{1n}^c(3))] = 0$. If $M_1 \le m_2$, then $E_{\underset{\sim}{X}(n)}[J(E_{2n}E_{1n}^c(3))] \le (m_2 - M_1 + 1)\exp\{-nc_2\}$.

From Lemmas 5.8 and 5.3, we can obtain

**Proposition 4.5.** When $M_2 = m_2+1$, $E_{\underset{\sim}{X}(n)}[J(E_{2n}^c(2)E_{1n}^c(1))] = 0$. When $M_2 = m_2+2$, $E_{\underset{\sim}{X}(n)}[J(E_{2n}^c(2)E_{1n}^c(1))] \le (m_1+1)\exp\{-nc_2\}$.

From Lemmas 5.9 and 5.3, we can obtain

**Proposition 4.6.** When either $M_2 = m_2+1$ or $M_1 = M_2$, $E_{\underset{\sim}{X}(n)}[J(E_{2n}^c(2)E_{1n}^c(3))] = 0$. When $M_2 = m_2+2$ and $M_2 > M_1$, $E_{\underset{\sim}{X}(n)}[J(E_{2n}^c(2)E_{1n}^c(3))] \le (M_2 - M_1)\exp\{-nc_2\}$.

**Proposition 4.7.** (a) For each $x \in S_1$,

$$E_{\underset{\sim}{X}(n)}[\tau_n^*(x) - \tau_G(x)|J(E_{2n}E_{1n})] \le \exp\{-n\ \ell n\ [m_{u,x,m_2}(\Psi(x,d_{1G},d_{2G}))]^{-1}\} + \exp\{-n\ \ell n\ [m_{u,x,m_2}(-\Psi(x,d_{1G},d_{2G}))]^{-1}\}.$$

(b) For each $x \in S_2$

$$E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)|J(E_{2n}E_{1n})] \le \exp\{-n\ \ell n\ [m_{v,x,m_2}(\Psi(x,d_{1G},d_{2G}))]^{-1}\} + \exp\{-n\ \ell n\ [m_{v,x,m_2}(-\Psi(x,d_{1G},d_{2G}))]^{-1}\}.$$

**Proof:** (a) On $E_{2n}E_{1n} = \{m_{2n}^* = m_2,\ m_{1n}^* = m_1\}$, for $x \in S_1$, $\Psi(x,d_{1G},d_{2G}) \ne 0$ and $\Psi_n^*(x) = \frac{1}{n}\sum_{j=1}^{n} u(x,X_j,m_2)$, where $u(x,X_j,m_2)$ is defined in Section 5. Here, $u(x,X_j,m_2)$,

$j = 1, \ldots, n$, are iid and $E_{\underset{\sim}{X}(n)}\, u(x, X_j, m_2) = \Psi(x, d_{1G}, d_{2G})$. Therefore, by Lemma 5.10,

$$\begin{aligned}
&E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)|J(E_{2n}E_{1n})] \\
&\leq P\{|\Psi_n^*(x) - \Psi(x, d_{1G}, d_{2G})| \geq |\Psi(x, d_{1G}, d_{2G})| \text{ and } E_{2n}E_{1n}\} \\
&\leq P\{|\frac{1}{n}\sum_{j=1}^{n} u(x, X_j, m_2) - \Psi(x, d_{1G}, d_{2G})| \geq |\Psi(x, d_{1G}, d_{2G})|\} \\
&\leq \exp\{-n\ \ell n\ [m_{u,x,m_2}(\Psi(x, d_{1G}, d_{2G}))]^{-1}\} \\
&\qquad + \exp\{-n\ \ell n\ [m_{u,x,m_2}(-\Psi(x, d_{1G}, d_{2G}))]^{-1}\}
\end{aligned}$$

(b) On $E_{2n}E_{1n}$, for $x \in S_2$, $\Psi(x, d_{1G}, d_{2G}) \neq 0$, $\Psi_n^*(x) = \frac{1}{n}\sum_{j=1}^{n} v(x, X_j, m_2)$, where $v(x, X_j, m_2)$ is defined in Section 5. Since $v(x, X_j, m_2)$, $j = 1, \ldots, n$ are iid and $E_{\underset{\sim}{X}(n)}\, v(x, X_j, m_2) = \Psi(x, d_{1G}, d_{2G})$, by Lemma 5.10,

$$\begin{aligned}
&E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)|J(E_{2n}E_{1n})] \\
&\leq P\{|\frac{1}{n}\sum_{j=1}^{n} v(x, X_j, m_2) - \Psi(x, d_{1G}, d_{2G})| \geq |\Psi(x, d_{1G}, d_{2G})|\} \\
&\leq \exp\{-n\ \ell n\ [m_{v,x,m_2}(\Psi(x, d_{1G}, d_{2G}))]^{-1}\} \\
&\qquad + \exp\{-n\ \ell n\ [m_{v,x,m_2}(-\Psi(x, d_{1G}, d_{2G}))]^{-1}\}. \qquad \square
\end{aligned}$$

**Proposition 4.8.** (a) For $x \in S_1$

$$\begin{aligned}
&E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)|J(E_{2n}E_{1n}^c(2))] \\
&\leq \exp\{-n\ \ell n\ [m_{u,x,m_2}(\Psi(x, d_{1G}, d_{2G}))]^{-1}\} \\
&\qquad + \exp\{-n\ \ell n\ [m_{u,x,m_2}(-\Psi(x, d_{1G}, d_{2G}))]^{-1}\}.
\end{aligned}$$

(b) For $x \in S_2^c(m_1, M_1)$,

$$\begin{aligned}
&E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)|J(E_{2n}E_{1n}^c(2))] \\
&\leq \exp\{-n\ \ell n\ [m_{v,x,m_2}(\Psi(x, d_{1G}, d_{2G}))]^{-1}\} \\
&\qquad + \exp\{-n\ \ell n\ [m_{v,x,m_2}(-\Psi(x, d_{1G}, d_{2G}))]^{-1}\}.
\end{aligned}$$

(c) For $x \in S_2(m_1, M_1)$,

$$\begin{aligned}
&E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)|J(E_{2n}E_{1n}^c(2))] \\
&\leq \exp\{-n\ \ell n\ [m_{u,x,m_2}(\Psi(x, d_{1G}, d_{2G}))]^{-1}\} \\
&\qquad + \exp\{-n\ \ell n\ [m_{u,x,m_2}(-\Psi(x, d_{1G}, d_{2G}))]^{-1}\}.
\end{aligned}$$

**Proof:** $E_{2n}E_{1n}^c(2) = \{m_{2n}^* = m_2 \text{ and } m_1 < m_{1n}^* < M_1\}$, which is empty if $M_1 = m_1 + 1$. In such a situation, the inequalities hold. Thus, in the following, it is assumed that $M_1 = m_1 + 2$, and therefore, $E_{2n}E_{1n}^c(2) = \{m_{2n}^* = m_2 \text{ and } m_{1n}^* = m_1 + 1\}$.

(a) On $E_{2n}E_{1n}^c(2)$, for $x \in S_1$, by definition, $\Psi_n^*(x) = \frac{1}{n} \sum_{j=1}^{n} u(x, X_j, m_2)$. Thus the proof is similar to that of Proposition 4.7 (a). The detail is omitted here.

(b) On $E_{2n}E_{1n}^c(2)$, for $x \in S_2^c(m_1, M_1)$, $\Psi_n^*(x) = \frac{1}{n} \sum_{j=1}^{n} v(x, X_j, m_2)$. The proof is similar to that of Proposition 4.7 (b). The detail is omitted here.

(c) $S_2(m_1, M_1) = \{x | x = m_1 + 1 \text{ and } \Psi(x, d_{1G}, d_{2G}) \neq 0\}$. If $\Psi(m_1 + 1, d_{1G}, d_{2G}) = 0$, $S_2(m_1, M_1)$ is an empty set. Therefore, the proof is complete. So, assume that $\Psi(m_1+1, d_{1G}, d_{2G}) \neq 0$. On $E_{2n}E_{1n}^c(2)$, $\Psi_n^*(m_1+1) = \frac{1}{n} \sum_{j=1}^{n} u(m_1+1, X_j, m_2)$. Since $\theta_0 h(m_1 + 1) - h(m_1 + 2) = 0$, $E_{\underset{\sim}{X}(n)} u(m_1 + 1, X_j, m_2) = \Psi(m_1 + 1, d_{1G}, d_{2G})$. Then, by Lemma 5.10, we can conclude the result. □

**Proposition 4.9.** (a) For $x \in S_1$,

$$\begin{aligned}
&E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)|J(E_{2n}^c(2)E_{1n})] \\
&\leq \exp\{-n\ \ell n\ [m_{u,x,m_2+1}(\Psi(x, d_{1G}, d_{2G}))]^{-1}\}. \\
&\qquad + \exp\{-n\ \ell n\ [m_{u,x,m_2+1}(-\Psi(x, d_{1G}, d_{2G}))]^{-1}\}.
\end{aligned}$$

(b) For $x \in S_2$,

$$\begin{aligned}
&E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)|J(E_{2n}^c(2)E_{1n})] \\
&\leq \exp\{-n\ \ell n\ [m_{v,x,m_2+1}(\Psi(x, d_{1G}, d_{2G}))]^{-1}\} \\
&\qquad + \exp\{-n\ \ell n\ [m_{v,x,m_2+1}(-\Psi(x, d_{1G}, d_{2G}))]^{-1}\}.
\end{aligned}$$

(c) For $x \in S_3$,

$$E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)| J(E_{2n}^c(2)E_{1n})]$$
$$\leq \exp\{-n\ \ell n\ [m_{v,x,m_2+1}(\Psi(x, d_{1G}, d_{2G}))]^{-1}\}$$
$$+ \exp\{-n\ \ell n\ [m_{v,x,m_2+1}(-\Psi(x, d_{1G}, d_{2G}))]^{-1}\}$$

**Proof:** $E_{2n}^c(2)E_{1n} = \{m_2 < m_{2n}^* < M_2 \text{ and } m_{1n}^* = m_1\}$, which is empty if $M_2 = m_2 + 1$. In such a situation, the inequalities hold. So, in the following, assume that $M_2 = m_2 + 2$, which occurs only when $\theta_0 k(m_2 + 1) - k(m_2 + 2) = 0$.

(a) On $E_{2n}^c(2)E_{1n} = \{m_{2n}^* = m_2 + 1 \text{ and } m_{1n}^* = m_1\}$, for $x \in S_1$, $\Psi_n^*(x) = \frac{1}{n}\sum_{j=1}^{n} u(x, X_j, m_2 + 1)$. Since $\theta_0 k(m_2 + 1) - k(m_2 + 2) = 0$, $E_{\underset{\sim}{X}(n)} u(x, X_j, m_2 + 1) = \Psi(x, d_{1G}, d_{2G})$. The remaining part of the proof is similar to that of Proposition 4.7 (a). The detail is omitted here.

(b) On $E_{2n}^c(2)E_{1n}$, for $x \in S_2$ , $\Psi_n^*(x) = \frac{1}{n}\sum_{j=1}^{n} v(x, X_j, m_2+1)$, where $E_{\underset{\sim}{X}(n)}\ v(x, X_j, m_2+1) = \Psi(x, d_{1G}, d_{2G})$. The remaining part of the proof is similar to that of Proposition 4.7 (b). The detail is omitted here.

(c) $S_3 = \{x | x = m_2 + 1 \text{ and } \Psi(x, d_{1G}, d_{2G}) \neq 0\}$. From (2.14), $\Psi(m_2 + 1, d_{1G}, d_{2G}) = -cf(m_2 + 1) < 0$. On $E_{2n}^c(2)E_{1n}$, $\Psi_n^*(m_2 + 1) = \frac{1}{n}\sum_{j=1}^{n} v(m_2 + 1, X_j, m_2 + 1)$.

Since $\theta_0 k(m_2+1) - k(m_2+2) = 0$, $E_{\underset{\sim}{X}(n)}\ v(m_2+1, X_j, m_2+1) = \Psi(m_2+1, d_{1G}, d_{2G})$. Then the result follows from Lemma 5.10. □

**Proposition 4.10.** (a) For $x \in S_1$,

$$E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)| J(E_{2n}^c(2)E_{1n}^c(2))]$$
$$\leq \exp\{-n\ \ell n\ [m_{u,x,m_2+1}(\Psi(x, d_{1G}, d_{2G}))]^{-1}\}$$
$$+ \exp\{-n\ \ell n\ [m_{u,x,m_2+1}(-\Psi(x, d_{1G}, d_{2G}))^{-1}\}.$$

(b) For $x \in S_2^c(m_1, M_1)$,

$$E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)| J(E_{2n}^c(2)E_{1n}^c(2))]$$
$$\leq \exp\{-n\ \ell n\ [m_{v,x,m_2+1}(\Psi(x, d_{1G}, d_{2G}))]^{-1}\}$$
$$+ \exp\{-n\ \ell n\ [m_{v,x,m_2+1}(-\Psi(x, d_{1G}, d_{2G}))^{-1}\}.$$

(c) For $x \in S_2(m_1, M_1)$,

$$E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)|J(E_{2n}^c(2)E_{1n}^c(2))]$$
$$\le \exp\{-n\ \ell n\ [m_{u,x,m_2+1}(\Psi(x, d_{1G}, d_{2G}))]^{-1}\}$$
$$+ \exp\{-n\ \ell n\ [m_{u,x,m_2+1}(-\Psi(x, d_{1G}, d_{2G}))]^{-1}\}.$$

(d) For $x \in S_3$,

$$E_{\underset{\sim}{X}(n)}[|\tau_n^*(x) - \tau_G(x)|J(E_{2n}^c(2)E_{1n}^c(2))]$$
$$\le \exp\{-n\ \ell n\ [m_{v,x,m_2+1}(\Psi(x, d_{1G}, d_{2G}))]^{-1}\}$$
$$+ \exp\{-n\ \ell n\ [m_{v,x,m_2+1}(-\Psi(x, d_{1G}, d_{2G}))]^{-1}\}.$$

**Proof:** $E_{2n}^c(2)E_{1n}^c(2) = \{m_2 < m_{2n}^* < M_2$ and $m_1 < m_{1n}^* < M_1\}$ which is empty if either $M_2 = m_2 + 1$ or $M_1 = m_1 + 1$. $E_{2n}^c(2)E_{1n}^c(2) \neq \phi$ when $M_2 = m_2 + 2$ and $M_1 = m_1 + 2$ which occur iff $\theta_0 h(m_1 + 1) - h(m_1 + 2) = 0$ and $\theta_0 k(m_2 + 1) - k(m_2 + 2) = 0$.

(a) On $E_{2n}^c(2)E_{1n}^c(2) = \{m_{2n}^* = m_2 + 1$ and $m_{1n}^* = m_1 + 1\}$, for $x \in S_1$, $\Psi_n^*(x) = \frac{1}{n}\sum_{j=1}^{n} u(x, X_j, m_2 + 1)$. Since $\theta_0 k(m_2 + 1) - k(m_2 + 2) = 0$, $E_{\underset{\sim}{X}(n)} u(x, X_j, m_2 + 1) = \Psi(x, d_{1G}, d_{2G})$. The remaining part of the proof is similar to that of Proposition 4.7 (a) and hence is omitted.

(b) On $E_{2n}^c(2)E_{1n}^c(2)$, for $x \in S_2^c(m_1, M_1)$, $\Psi_n^*(x) = \frac{1}{n}\sum_{j=1}^{n} v(x, X_j, m_2 + 1)$ and $E_{\underset{\sim}{X}(n)}$ $v(x, X_j, m_2 + 1) = \Psi(x, d_{1G}, d_{2G})$. Then, analogous to that of Proposition 4.7 (b), we can conclude the result.

(c) $S_2(m_1, M_1) = \{x | x = m_1 + 1$, and $\Psi(x, d_{1G}, d_{2G}) \neq 0\}$. On $E_{2n}^c(2)E_{1n}^c(2)$, $\Psi_n^*(m_1 + 1) = \frac{1}{n}\sum_{j=1}^{n} u(m_1 + 1, X_j, m_2 + 1)$. Thus, the remaining part of the proof is similar to that of Proposition 4.8 (c). The detail is omitted.

(d) $S_3 = \{x | x = m_2 + 1\}$. On $E_{2n}^c(2)E_{1n}^c(2)$, $\Psi_n^*(m_2 + 1) = \frac{1}{n}\sum_{j=1}^{n} v(m_2 + 1, X_j, m_2 + 1)$. The remaining part of the proof is similar to that of Proposition 4.9 (c). The detail is omitted here.

## 4.4. Rate of Convergence

Define

$$c_4 = \min_{S_2} \min\{\ell n\ [m_{v,x,m_2}(\Psi(x, d_{1G}, d_{2G}))]^{-1},\ \ell n\ [m_{v,x,m_2}(-\Psi(x, d_{1G}, d_{2G}))]^{-1}\},$$

$$c_5 = \min_{S_2 \bigcup S_3} \min\{\ell n\ [m_{v,x,m_2+1}(\Psi(x, d_{1G}, d_{2G}))]^{-1},\ \ell n\ [m_{v,x,m_{2+1}}(-\Psi(x, d_{1G}, d_{2G}))]^{-1}\},$$

$$c_6 = \min_{S_1 \bigcup S_2(m_1, M_1)} \min\{\ell n\ [m_{u,x,m_2}(\Psi(x, d_{1G}, d_{2G}))]^{-1},\ \ell n\ [m_{u,x,m_2}(-\Psi(x, d_{1G}, d_{2G}))]^{-1}\},$$

$$c_7 = \min_{S_1 \bigcup S_2(m_1, M_1)} \min\{\ell n\ [m_{u,x,m_2+1}(\Psi(x, d_{1G}, d_{2G}))]^{-1},\ \ell n\ [m_{u,x,m_2+1}(-\Psi(x, d_{1G}, d_{2G}))]^{-1}\}.$$

Since $S_1$, $S_2$ and $S_3$ are finite sets, $c_j > 0$ for each $j = 4, 5, 6, 7$. Let $c = \min_{1 \le j \le 7} c_j$. So $c > 0$.

We now give the main result of the paper, the rate of convergence of the empirical Bayes two-stage test $(d^*_{1n},\ d^*_{2n},\ c^*_n)$, as a theorem as follows.

**Theorem 4.1.** Let $(d^*_{1n}, d^*_{2n}, \tau^*_n)$ be the empirical Bayes two-stage test constructed previously. Suppose that

(a) $\int \frac{\theta}{\beta(\theta} dG(\theta) < \infty$;

(b) Both $\{\frac{w_2(z)}{a(z)}\}_{z=0}^{\infty}$ and $\{\frac{w_2(z)}{a(z+1)}\}_{z=0}^{\infty}$ are nonincreasing and bounded above by 1, and,

(c) $M_2 < \infty$.

Then, $R(G, (d^*_{1n}, d^*_{2n}, \tau^*_n)) - R(G, (d_{1G}, d_{2G}, \tau_G)) = O(\exp(-nc))$.

## 5. Lemmas

The following lemmas are helpful for evaluating the expectations listed in (1)–(10) of Section 4.2.

Lemma 5.1 is from Liang (1991).

**Lemma 5.1.** Let $\{a_m\}$ be a sequence of real numbers and $\{b_m\}$ be a sequence of nonincreasing, positive numbers with $b_1 \le 1$. Then, for a fixed constant $d > 0$, $\sup_{n \ge 1} |\sum_{m=1}^{n} a_m b_m| \ge (>) d$ implies that $\sup_{n \ge 1} |\sum_{m=1}^{n} a_m| \ge (>) d$.

Let $c_1 = 2 \min_{0 \le z \le m_2} \left[(\theta_0 k(z) - k(z+1)) / \left[\frac{\theta_0}{a(z)} + \frac{1}{a(z+1)}\right]\right]^2$. Since $\theta_0 k(z) - k(z+1) > 0$ for all $z = 0, 1, \ldots, m_2$, $c_1 > 0$.

**Lemma 5.2.** For each $z = 0, 1, \ldots, m_2$,

$$P\{\theta_0 k_n(z) - k_n(z+1) \leq 0\} \leq \exp\{-nc_1\}.$$

**Proof:** Let $c(z, X_j) = \frac{\theta_0}{a(z)} I_{\{z\}}(X_j) - \frac{1}{a(z+1)} I_{\{z+1\}}(X_j)$, $j = 1, \ldots, n$. Then $c(z, X_j)$, $j = 1, \ldots, n$, are iid, $-\frac{1}{a(z+1)} \leq c(z, X_j) \leq \frac{\theta_0}{a(z)}$, $E_{\underset{\sim}{X}(n)} c(z, X_j) = \theta_0 k(z) - k(z+1)$, and $\theta_0 k_n(z) - k_n(z+1) = \frac{1}{n} \sum_{j=1}^{n} c(z, X_j)$. Thus, by Hoeffding's inequality,

$$\begin{aligned}
&P\{\theta_0 k_n(z) - k_n(z+1) \leq 0\} \\
&= P\{\frac{1}{n} \sum_{j=1}^{n} [c(z, X_j) - \theta_0 k(z) + k(z+1)] \leq -[\theta_0 k(z) - k(z+1)]\} \\
&\leq \exp\{-2n[\theta_0 k(z) - k(z+1)]^2 / [\frac{\theta_0}{a(z)} + \frac{1}{a(z+1)}]^2\} \\
&\leq \exp\{-nc_1\}.
\end{aligned}$$

□

Let $b(z, X_j) = \frac{\theta_0 a(X_j - z) - a(X_j - z - 1)}{a(X_j)}$, $z = 0, 1, \ldots,$ and $j = 1, 2, \ldots, n$. Then, for each $z$, $b(z, X_j)$, $j = 1, \ldots, n$, are iid, $E_{\underset{\sim}{X}(n)} b(z, X_j) = \theta_0 h(z) - h(z+1)$, and $\theta_0 h_n(z) - h_n(z+1) = \frac{1}{n} \sum_{j=1}^{n} b(z, X_j)$.

Let $\mathcal{M}_z(t)$ denote the moment-generating function of $b(z, X_j) - [\theta_0 h(z) - h(z+1)]$. For each real value $v$, define

$$m_z(v) = \inf_t e^{-vt} \mathcal{M}_z(t). \tag{5.1}$$

Following an argument analogous to that of Lemma 3.6 of Gupta and Liang (1991), we have: for each positive constant $c$,

$$0 < m_z(c) < 1 \text{ and } 0 < m_z(-c) < 1. \tag{5.2}$$

**Lemma 5.3.** (a) For each $z = 0, 1, \ldots, m_1$,

$$P\{\theta_0 h_n(z) - h_n(z+1) \leq 0\} \leq \exp\{-n\, \ell n\, [m_z(\theta_0 h(z) - h(z+1))]^{-1}\};$$

(b) For each $z$, $M_1 \leq z \leq M_2$,

$$P\{\theta_0 h_n(z) - h_n(z+1) > 0\} \leq \exp\{-n\, \ell n\, [m_z(\theta_0 h(z) - h(z+1))]^{-1}\}.$$

**Proof:** By the definitions of $m_1$ and $M_1$, $\theta_0 h(z) - h(z+1) > (<)\ 0$ iff $z \le m_1 (z \ge M_1)$. Then, by a direct application of a theorem of Chernoff (1952), we have: for each $z = 0, 1, \ldots, m_1$,

$$\begin{aligned}
&P\{\theta_0 h_n(z) - h_n(z+1) \le 0\} \\
&= P\{\frac{1}{n}\sum_{j=1}^{n}[b(z, X_j) - \theta_0 h(z) + h(z+1)] \le -[\theta_0 h(z) - h(z+1)]\} \\
&\le [m_z(\theta_0 h(z) - h(z+1))]^n \\
&= \exp\{-n\ \ell n\ [m_z(\theta_0 h(z) - h(z+1))]^{-1}\},
\end{aligned}$$

and for each $z$, $M_1 \le z \le M_2$,

$$\begin{aligned}
&P\{\theta_0 h_n(z) - h_n(z+1) > 0\} \\
&= P\{\frac{1}{n}\sum_{j=1}^{n}[b(z, X_j) - \theta_0 h(z) + h(z+1)] \ge -[\theta_0 h(z) - h(z+1)]\} \\
&\le [m_z(\theta_0 h(z) - h(z+1))]^n \\
&= \exp\{-n\ \ell n\ [m_z(\theta_0 h(z) - h(z+1))]^{-1}\}. \qquad \square
\end{aligned}$$

We let $c_2 = \min_{z \in A}\{\ell n\ [m_z(\theta_0 h(z) - h(z+1))]^{-1}\}$, where $A = \{z | 0 \le z \le m_1$ or $M_1 \le z \le M_2\}$. Note that $c_2 > 0$ by (5.2).

**Lemma 5.4.** Suppose that both $\{\frac{w_2(z)}{a(z)}\}_{z=0}^{\infty}$ and $\{\frac{w_2(z)}{a(z+1)}\}_{z=0}^{\infty}$ are nonincreasing in $z$ and bounded above by 1. Then,

$$E_{2n}^c(3) \subset \{\sup_{t \ge 0} |F_n(t) - F(t)| > L(M_2) \min(\frac{1}{\theta_0}, 1)/4\},$$

where $L(M_2) = [k(M_2+1) - \theta_0 k(M_2)]w_2(M_2)$, and $F_n(t)$ is the empirical distribution based on $\underset{\sim}{X}(n)$, and $F(t)$ is the marginal distribution function of the random variable $X$.

**Proof:** For $z \ge M_2$, $k(z+1) - \theta_0 k(z) > 0$. Therefore, for $t \ge M_2$, $\sum_{z=M_2}^{t} w_2(z)[k(z+1) - \theta_0 k(z)] \ge L(M_2)$.

By the definition of $E_{2n}^c(3)$ and $m_{2n}^*$ and by Lemma 5.1,

$E_{2n}^c(3) = \{m_{2n}^* \ge M_2\}$

$$\subset \{B_{2n}(t) - B_{2n}(M_2 - 1) = \sum_{z=M_2}^{t} w_2(z)[\theta_0 k_n(z) - k_n(z+1)] > 0 \text{ for some } t \geq M_2\}$$

$$\subset \bigcup_{t=M_2}^{\infty} \{ \sum_{z=M_2}^{t} \frac{\theta_0 w_2(z)}{a(z)}[f_n(z) - f(z)] - \sum_{z=M_2}^{t} \frac{w_2(z)}{a(z+1)}[f_n(z+1) - f(z+1)] \geq L(M_2)\}$$

$$\subset \bigcup_{t=M_2}^{\infty} \left\{ \sum_{z=M_2}^{t} \frac{w_2(z)}{a(z)}[f_n(z) - f(z)] \geq \frac{L(M_2)}{2\theta_0} \right.$$

$$\left. \text{or } \sum_{z=M_2}^{t} \frac{w_2(z)}{a(z+1)}[f_n(z+1) - f(z+1)] \leq -\frac{L(M_2)}{2} \right\}$$

$$\subset \{| \sum_{z=M_2}^{t} \frac{w_2(z)}{a(z)}[f_n(z) - f(z)]| \geq \frac{L(M_2)}{2\theta_0} \text{ for some } t \geq M_2\}$$

$$\bigcup \{| \sum_{z=M_2}^{t} \frac{w_2(z)}{a(z+1)}[f_n(z+1) - f(z+1)]| \geq \frac{L(M_2)}{2} \text{ for some } t \geq M_2\}$$

$$\subset \{| \sum_{z=M_2}^{t} [f_n(z) - f(z)]| \geq \frac{L(M_2)}{2\theta_0} \text{ for some } t \geq M_2\}$$

$$\bigcup \{ \sum_{z=M_2}^{t} [f_n(z+1) - f(z+1)]| \geq \frac{L(M_2)}{2} \text{ for some } t \geq M_2\}$$

$$= \{|[F_n(t) - F(t)] - [F_n(M_2 - 1) - F(M_2 - 1)]| \geq \frac{L(M_2)}{2\theta_0} \text{ for some } t \geq M_2\}$$

$$\bigcup \{|[F_n(t+1) - F(t+1)] - [F_n(M_2) - F(M_2)]| \geq \frac{L(M_2)}{2} \text{ for some } t \geq M_2\}$$

$$\subset \{ \sup_{t \geq M_2 - 1} |F_n(t) - F(t)| \geq \frac{L(M_2)}{4\theta_0}\} \bigcup \{ \sup_{t \geq M_2} |F_n(t) - F(t)| \geq \frac{L(M_2)}{4}\}$$

$$\subset \{\sup_{t \geq 0} |F_n(t) - F(t)| \geq \frac{L(M_2)}{4} \min(1, \frac{1}{\theta_0})\}. \qquad \square$$

**Lemma 5.5.** $E_{2n}^c(1) \subset \bigcup_{z=0}^{m_2} \{\theta_0 k_n(z) - k_n(z+1) \leq 0\}$.

**Proof:** By definitions of $E_{2n}^c(1)$ and $m_{2n}^*$,

$$E_{2n}^c(1) = \{m_{2n}^* < m_2\}$$

$$\subset \{B_{2n}(m_2) - B_{2n}(t) = \sum_{z=t+1}^{m_2} w_2(z)[\theta_0 k_n(z) - k_n(z+1)] \leq 0 \text{ for some } t < m_2\}$$

$$\subset \bigcup_{z=0}^{m_2} \{\theta_0 k_n(z) - k_n(z+1) \leq 0\},$$

since $w_2(z)$ is positive. □

**Lemma 5.6.** $E_{2n}E_{1n}^c(1) \subset \bigcup_{z=0}^{m_1} \{\theta_0 h_n(z) - h_n(z+1) \leq 0\}$.

**Proof:**

$$E_{2n}E_{1n}^c(1) = \{m_{2n}^* = m_2 \text{ and } m_{1n}^* < m_1\}$$

$$\subset \{B_{1n}(m_1) - B_{1n}(t) = \sum_{z=t+1}^{m_1} w_1(z)[\theta_0 h_n(z) - h_n(z+1)] \leq 0 \text{ for some } t < m_1\}$$

$$\subset \bigcup_{z=0}^{m_1} \{\theta_0 h_n(z) - h_n(z+1) \leq 0\}.$$

since $w_1(z) > 0$ for all $z = 0, 1, 2, \ldots$ □

**Lemma 5.7.** (a) $E_{2n}E_{1n}^c(3) = \phi$ when $m_2 < M_1$.

(b) When $m_2 \geq M_1$,

$$E_{2n}E_{1n}^c(3) \subset \bigcup_{z=M_1}^{m_2} \{\theta_0 h_n(z) - h_n(z+1) > 0\}.$$

**Proof:** $E_{2n}E_{1n}^c(3) = \{m_2 = m_{2n}^* \geq m_{1n}^* \geq M_1\}$. Thus, $E_{2n}E_{1n}^c(3) = \phi$ when $M_1 > m_2$. When $m_2 \geq M_1$,

$$E_{2n}E_{1n}^c(3) \subset \{B_{1n}(t) - B_{1n}(M_1 - 1)$$

$$= \sum_{z=M_1}^{t} w_1(z)[\theta_0 h_n(z) - h_n(z+1)] > 0 \text{ for some } M_1 \leq t \leq m_2\}$$

$$\subset \bigcup_{z=M_1}^{m_2} \{\theta_0 h_n(z) - h_n(z+1) > 0\}.$$ □

**Lemma 5.8.** (a) When $M_2 = m_2 + 1$, $E_{2n}^c(2)E_{1n}^c(1) = \phi$.

(b) When $M_2 = m_2 + 2$,

$$E_{2n}^c(2)E_{1n}^c(1) \subset \bigcup_{z=0}^{m_1} \{\theta_0 h_n(z) - h_n(z+1) \le 0\}.$$

**Proof:** $E_{2n}^c(2)E_{1n}^c(1) = \{m_2 < m_{2n}^* < M_2 \text{ and } m_{1n}^* < m_1\}$. When $M_2 = m_2 + 1$, it is clear that $E_{2n}^c(2)E_{1n}^c(1) = \phi$. When $M_2 = m_2 + 2$,

$$\begin{aligned} E_{2n}^c(2)E_{1n}^c(1) &= \{m_{2n}^* = m_2 + 1 \text{ and } m_{1n}^* < m_1\} \\ &\subset \{B_{1n}(m_1) - B_{1n}(t) = \sum_{z=t+1}^{m_1} w_1(z)[\theta_0 h_n(z) - h_n(z+1)] \le 0 \\ &\qquad \text{for some } 0 \le t < m_1\} \\ &\subset \bigcup_{z=0}^{m_1} \{\theta_0 h_n(z) - h_n(z+1) \le 0\}. \end{aligned}$$

□

**Lemma 5.9.** (a) When either $M_2 = m_2 + 1$ or $M_1 = M_2$, $E_{2n}^c(2)E_{1n}^c(3) = \phi$.

(b) When $M_2 = m_2 + 2$ and $M_2 > M_1$, then,

$$E_{2n}^c(2)E_{1n}^c(3) \subset \bigcup_{z=M_1}^{M_2-1} \{\theta_0 h_n(z) - h_n(z+1) > 0\}.$$

**Proof:** $E_{2n}^c(2)E_{1n}^c(3) = \{m_2 < m_{2n}^* < M_2 \text{ and } m_{2n}^* \ge m_{1n}^* \ge M_1\}$. Therefore, if either $M_2 = m_2 + 1$ or $M_1 = M_2$, $E_{2n}^c(2)E_{1n}^c(3) = \phi$. When $M_2 = m_2 + 2$, and $M_2 > M_1$,

$$\begin{aligned} E_{2n}^c(2)E_{1n}^c(3) &= \{m_{2n}^* = m_2 + 1 \text{ and } m_{1n}^* \ge M_1\} \\ &\subset \{B_{1n}(t) - B_{1n}(M_1 - 1) = \sum_{z=M_1}^{t} w_1(z)[\theta_0 h_n(z) - h_n(z+1)] > 0 \\ &\qquad \text{for some } M_1 \le t \le m_{2n}^* = M_2 - 1\} \\ &\subset \bigcup_{z=M_1}^{M_2-1} \{\theta_0 h_n(z) - h_n(z+1) > 0\}. \end{aligned}$$

□

For each positive integer $\ell$ and each $x = 0, 1, \ldots, \ell$, define

$$v(x, X_j, \ell) = \sum_{y=0}^{\ell-x} a(x)a(y)[\frac{\theta_0 I_{\{x+y\}}(X_j)}{a(x+y)} - \frac{I_{\{x+y+1\}}(X_j)}{a(x+y+1)}] - cI_{\{x\}}(X_j),$$

and

$$u(x, X_j, \ell) = -\frac{a(x)[\theta_0 a(X_j - x) - a(X_j - x - 1)]}{a(X_j)} + v(x, X_j, \ell).$$

Note that $v(x, X_j, \ell)$, $j = 1, \ldots, n$, are iid, and $u(x, X_j, \ell)$, $j = 1, \ldots, n$ are iid. Let

$$\mu_u(x, \ell) = E_{\underset{\sim}{X}(n)} u(x, X_j, \ell), \quad \mu_v(x, \ell) = E_{\underset{\sim}{X}(n)} v(x, X_j, \ell),$$

and let $\mathcal{M}_{u,x,\ell}(t)$ and $\mathcal{M}_{v,x,\ell}(t)$ denote the moment-generating function of $u(x, X_j, \ell) - \mu_u(x, \ell)$ and $v(x, X_j, \ell) - \mu_v(x, \ell)$, respectively. For a real value $b$, define

$$m_{u,x,\ell}(b) = \inf_t e^{-bt} \mathcal{M}_{u,x,\ell}(t),$$
$$m_{v,x,\ell}(b) = \inf_t e^{-bt} \mathcal{M}_{v,x,\ell}(t).$$

Then, analogous to (5.2), we have: for a positive constant $b$,

$$0 < m_{u,x,\ell}(b) < 1, \qquad 0 < m_{u,x,\ell}(-b) < 1,$$
$$0 < m_{v,x,\ell}(b) < 1, \qquad 0 < m_{v,x,\ell}(-b) < 1.$$

By a theorem of Chernoff (1952), we have

**Lemma 5.10.** For a positive constant $b$,

(a) $$P\{\frac{1}{n}\sum_{j=1}^{n}[v(x, X_j, \ell) - \mu_v(x, \ell)] > b\} \leq [m_{v,x,\ell}(b)]^n = \exp\{-n \,\ell n\, [m_{v,x,\ell}(b)]^{-1}\},$$

(b) $$P\{\frac{1}{n}\sum_{j=1}^{n}[v(x, X_j, \ell) - \mu_v(x, \ell)] < -b\} \leq [m_{v,x,\ell}(-b)]^n = \exp\{-n \,\ell n\, [m_{v,x,\ell}(-b)]^{-1}\},$$

(c) $$P\{\frac{1}{n}\sum_{j=1}^{n}[u(x, X_j, \ell) - \mu_u(x, \ell)] > b\} \leq [m_{u,x,\ell}(b)]^n = \exp\{-n \,\ell n\, [m_{u,x,\ell}(b)]^{-1}\},$$

(d) $$P\{\frac{1}{n}\sum_{j=1}^{n}[u(x, X_j, \ell) - \mu_u(x, \ell)] < -b\} \leq [m_{u,x,\ell}(-b)]^n = \exp\{-n \,\ell n\, [m_{u,x,\ell}(-b)]^{-1}\}.$$

## 6. Examples

We use the following examples to demonstrate the asymptotic optimality of the empirical Bayes two-stage test $(d_{1n}^*, d_{2n}^*, \tau_n^*)$.

**Example 1.** *(The negative binomial distribution.)* Suppose that

$$f(x|\theta) = \binom{x+r-1}{r-1}\theta^x(1-\theta)^r, \quad x = 0, 1, 2, \ldots; \quad 0 < \theta < 1;$$

where $r$ is a fixed, positive value. Then, $\beta(\theta) = (1-\theta)^r$ and $a(x) = \binom{x+r-1}{r-1}$.

If we let $w_2(z) = 1$ for all $z = 0, 1, 2, \ldots$, then both $\frac{w_2(z)}{a(z)}$ and $\frac{w_2(z)}{a_2(z+1)}$ are decreasing in $z$ for $z = 0, 1, \ldots$, and bounded above by 1.

Suppose the prior distribution $G$ is a member of the family of beta distributions with parameter $(\alpha, \beta)$ for which $\beta > r+1$. Then, $\int_0^1 \frac{\theta}{\beta(\theta)} d\theta = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \theta^\alpha (1-\theta)^{\beta-r-1} d\theta < \infty$. Also, $\varphi_2(z) = \frac{k(z+1)}{k(z)} = \frac{z+\alpha}{z+\alpha+\beta+r}$, which tends to 1 as $z \to \infty$. Therefore, for a fixed $0 < \theta_0 < 1$, $M_2 \equiv \inf\{z|\varphi_2(z) \geq \theta_0\} < \infty$. Thus, by Theorem 4.1, the empirical Bayes two-stage test $(d_{1n}^*, d_{2n}^*, \tau_n^*)$ is asymptotically optimal, having a rate of convergence of order $O(\exp(-n\tau))$ for some $\tau > 0$.

**Example 2.** *(The Poisson distribution.)* Suppose that

$$f(x|\theta) = e^{-\theta}\theta^x/x! \qquad x = 0, 1, 2, \ldots, \quad 0 < \theta < \infty.$$

Then, $\beta(\theta) = e^{-\theta}$ and $a(x) = (x!)^{-1}$.

If we choose $w_2(z) = a(z+1)$, then $\frac{w_2(z)}{a(z)} = \frac{1}{z+1}$, $\frac{w_2(z)}{a(z+1)} = 1$. Both are nonincreasing in $z$ and bounded above by 1.

Suppose the prior distribution $G$ is a member of the family of gamma distributions with parameter $(\alpha, \beta)$ for which $\beta > 1$. Then, $\int_0^\infty \frac{\theta}{\beta(\theta)} dG(\theta) = \int_0^\infty \theta e^\theta \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} d\theta = \frac{\alpha\beta^\alpha}{(\beta-1)^{\alpha+1}} < \infty$. Also, $\varphi_2(z) = \frac{k(z+1)}{k(z)} = \frac{z+\alpha}{1+\beta}$, which tends to infinity as $z \to \infty$. Therefore, for a fixed $\theta_0 > 0$, $M_2 = \inf\{z|\varphi_2(z) \geq \theta_0\} < \infty$. Thus by Theorem 4.1, the empirical Bayes two-stage test $(d_{1n}^*, d_{2n}^*, \tau_n^*)$ is asymptotically optimal of order $O(\exp(-n\tau))$ for some $\tau > 0$.

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