# PERCOLATION ON FUCHSIAN GROUPS

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Abstract. It is shown that, for site percolation on the Cayley graph of a co-compact Fuchsian group of genus $\geq 2$, infinitely many infinite connected clusters exist almost surely for certain values of the parameter $p = P\{\text{site is open}\}$. In such cases, the set of limit points at $\infty$ of an infinite cluster is shown to be a perfect, nowhere dense set of Lebesgue measure 0. These results are also shown to hold for a class of hyperbolic triangle groups.

## 1. Introduction

Percolation on a "Euclidean" graph, such as the standard integer lattice $\mathbb{Z}^d$, exhibits a single threshold probability $p_c$, above which infinite clusters exist with probability 1 and below which they exist with probability 0. In the percolation regime $p > p_c$ the infinite cluster is unique [2]. The purpose of this paper is to show that percolation on a "noneuclidean" graph may exhibit several threshold probabilities, and in particular that for some values of $p$ *infinitely many* infinite clusters may coexist, while for other values of $p$ there is only one infinite cluster. We shall consider only *site* percolation on the Cayley graph of a co-compact Fuchsian group, but it will be clear that most of our results have analogues for *bond* percolation.

**1.1. Fuchsian Groups and their Cayley Graphs.** A *Fuchsian group* is a discrete group $\Gamma$ of isometries of the hyperbolic plane $\mathbb{H}$ (the unit disk endowed with the Poincaré metric $d_H$) . See [5], chapters 2-4, or [7], chapters 1-2, for succinct expositions of the basic theory of Fuchsian groups. The group $\Gamma$ is *co-compact* if the quotient space $\Gamma \setminus \mathbb{H}$ is compact, equivalently, if it has a compact Dirichlet fundamental polygon. We shall assume throughout that the origin 0 is not a fixed point of any element of $\Gamma$ (this may always be arranged by a change of variable, so this assumption entails no loss of generality). The *Dirichlet fundamental polygon* $T$ centered at the origin is defined to be the set of all points $\xi \in \mathbb{H}$ such that for all $g \in \Gamma$,

$$d_H(\xi, 0) \leq d_H(\xi, g(0));$$

the polygon $T$ and its images $g(T), g \in \Gamma$, *tessellate* the hyperbolic plane (their union is all of $\mathbb{H}$, and distinct images $g(t), g'(T)$ intersect either in a single point, a geodesic segment, or not at all). We shall refer to the polygons $g(T)$ as *tiles*, and to $T$ as the *fundamental tile.*

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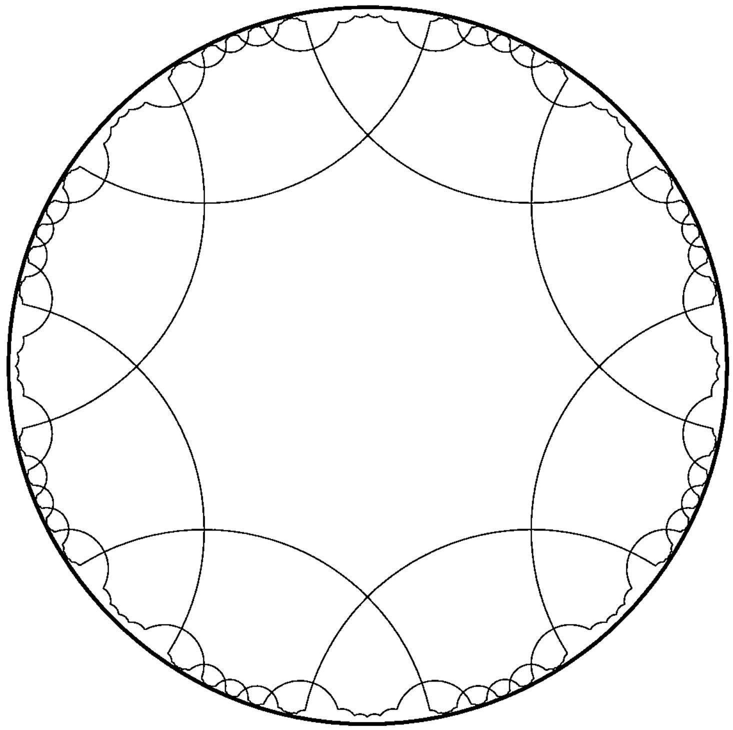


Since 0 is not a fixed point, the elements $g$ of $\Gamma$ are in one-to-one correspondence with the $\Gamma$-orbit of the origin 0, which in turn is in one-to-one correspondence with the set of tiles. Henceforth, we will (usually) identify the group element $g$ and the point $g(0) \in \mathbb{H}$, and (sometimes) the tile $g(T)$. Two graphs will play a central role in the percolation processes, both having vertex set $\Gamma$. The *Cayley graph*, designated $G_{\text{blue}}$, has an edge connecting $g$ and $g'$ iff the tiles $g(T)$ and $g'(T)$ intersect in a geodesic segment. The *extended Cayley graph*, designated $G_{\text{red}}$, has an edge connecting $g$ and $g'$ iff the tiles $g(T)$ and $g'(T)$ intersect (either in a geodesic segment or a point). Both graphs should be visualized as embedded in the hyperbolic plane $\mathbb{H}$, with geodesic segments representing the edges. For the Cayley graph $G_{\text{blue}}$ edges never cross. For the extended Cayley graph, edges $[g, g']$ and $[h, h']$ may cross, but only if the four tiles $g(T), g'(T), h(T), h'(T)$ have a point in common.

Elements of $\Gamma$ are either *elliptic* or *hyperbolic*. An *elliptic* isometry $g$ is conjugate to a rotation (i.e., for some isometry $h$ and some rotation $R$ about the origin, $g = hRh^{-1}$); every elliptic element has a unique fixed point in $\mathbb{H}$, and has finite order in $\Gamma$. A *hyperbolic* isometry has no fixed points in $\mathbb{H}$, but has two fixed points $\zeta_+, \zeta_-$ on $\partial\mathbb{H}$ =the unit circle, one ($\zeta_+$) attractive, the other repulsive; every hyperbolic element has infinite order in $\Gamma$. If $g \in \Gamma$ is hyperbolic, then for every $\xi \in (\mathbb{H} \cup \partial\mathbb{H}) - \{\zeta_-\}$,

$$\lim_{n\to\infty} g^n(\xi) = \zeta_+$$

in the usual (Euclidean) topology on the closed unit disk $\mathbb{H} \cup \partial\mathbb{H}$, and this convergence is uniform on compact subsets of $(\mathbb{H} \cup \partial\mathbb{H}) - \{\zeta_-\}$.

**1.2. Site Percolation on $\Gamma$.** Fix $p \in (0, 1)$. Color each tile $g(T)$ *blue* or *red*, blue with probability $p$ and red with probability $q = 1 - p$, with colors chosen independently for different tiles. If there is an infinite connected set of vertices in $G_{\text{blue}}$ all of which are colored blue, say that *blue percolation* (or *site percolation*) has occurred. If there is an infinite connected set of vertices in $G_{\text{red}}$ all of which are colored red, say that *red percolation* has occurred. In any case, define a *blue cluster* to be a maximal connected set of blue vertices in $G_{\text{blue}}$, and a *red cluster* to be a maximal connected set of red vertices in $G_{\text{red}}$. Thus, blue percolation occurs iff there is an infinite blue cluster, and red percolation occurs iff there is an infinite red cluster.

Similarly, define a *blue path* to be a (connected) path in the graph $G_{\text{blue}}$ all of whose vertices are colored blue, and define a *red path* to be a (connected) path in the graph $G_{\text{red}}$ all of whose vertices are colored red. Such paths will be identified with piecewise-geodesic paths in the hyperbolic plane. When topological properties of infinite blue paths or red paths are discussed, the implicit topology will always be the usual Euclidean topology on the closed unit disk $\mathbb{H} \cup \partial\mathbb{H}$. The following topological facts will be of crucial importance:

**Fact 1.** *No blue path can cross a red path.*

**Fact 2.** *If there are no infinite blue (red) clusters, then for every $n \geq 1$ there is a closed red (blue) path surrounding the (hyperbolic) circle of radius $n$ centered at the origin.*

**Fact 3.** *If $A, B, C, D$ are nonoverlapping arcs arranged in clockwise order on the unit circle $\partial\mathbb{H}$, then the existence of a doubly infinite red path connecting the arcs $A$ and $C$ precludes the existence of a doubly infinite blue path connecting the arcs $B$ and $D$ (and vice versa).*

**1.3. Principal Results.** The principal results of the paper concern the existence of a percolation phase in which infinitely many infinite blue clusters and infinitely many red clusters co-exist. In section 6, we will prove a stronger form of the following theorem:

**Theorem A.** *For any co-compact Fuchsian group of genus $g \geq 2$ there exist $0 < p_1 < p_2 < 1$ such that*

(1) *For $p < p_1$ there is a single infinite red cluster and no infinite blue cluster, with probability 1.*
(2) *For $p > p_2$ there is a single infinite blue cluster and no infinite red cluster, with probability 1.*
(3) *For $p_1 < p < p_2$ there are infinitely many infinite red clusters and infinitely many infinite blue clusters, with probability 1.*

We conjecture that this is true for *all* co-compact Fuchsian groups. In section 8, we will show that it is true also for a class of triangle groups (which have genus $g = 0$). Benjamini and Schramm [1] have made the more far-reaching conjecture that a similar statement holds for all *nonamenable* finitely generated discrete groups.

In section 5, we shall consider topological and metric properties of the set of limit points in $\partial\mathbb{H}$ of an infinite cluster (red or blue). We will prove a series of propositions leading to the following theorem:

**Theorem B.** *If for some $p$ it is almost sure that there are infinite red paths and infinite blue paths that converge to points of $\partial\mathbb{H}$, then for any infinite cluster (red or blue) the set $\Lambda$ of its limit points in $\partial\mathbb{H}$ is closed, perfect, nowhere dense, and has Lebesgue measure 0.*

We conjecture that in these circumstances it is always the case that $\Lambda$ has Hausdorff dimension strictly less than 1.

Theorems A and B were largely inspired by results obtained in [6] concerning branching Brownian motion in the hyperbolic plane. *Branching Brownian motion* is the branching process in which individual particles follow (hyperbolic) Brownian paths and undergo binary fissions at rate $\lambda > 0$. As shown in [6], there is a threshold value $\lambda_c = 1/8$ (corresponding to the threshold $p_2$ in Theorem A above) such that (i) for $\lambda > \lambda_c$ the process is "recurrent" in the sense that, with probability 1, for every compact subset $K$ of the hyperbolic plane there are particles in $K$ at indefinitely large times; and (ii) for $\lambda < \lambda_c$ the process is "transient" in the sense that with probability 1 it dies out in every compact set eventually. For $\lambda > \lambda_c$, every point of $\partial\mathbb{H}$ is an accumulation point of the traces of particle trajectories, but for $\lambda < \lambda_c$ the set of such accumulation points is, with probability 1, a closed, perfect, nowhere dense set with Hausdorff dimension

$$\delta = \frac{1 - \sqrt{1 - 8\lambda}}{2}.$$

Thus, the recurrence/transience dichotomy is reflected in the topological and metric properties of the limit set.

**Acknowledgment.** The author is grateful to ITAI BENJAMINI and ODED SCHRAMM for sending him a preprint of their article [1], which contains several conjectures and general results for percolation on infinite graphs. As noted above, Benjamini and Schramm have conjectured that the existence of a percolation phase where infinitely many infinite clusters co-exist holds for all nonamenable groups. Also, Proposition 1 is borrowed from [1], and a result related to Proposition 13 is proved in [1].

## 2. 0-1 LAWS

A *configuration* is a function from the group $\Gamma$ to the two-element set $\{0, 1\}$. Configurations may be identified with two-colorings (with 0 = red, 1 = blue) of the vertex set of either the Cayley graph $G_{\text{blue}}(\Gamma)$ of $\Gamma$ or the extended Cayley graph $G_{\text{red}}(\Gamma)$. The probability measure $P_p$ is the product Bernoulli measure on configuration space $\Omega$, i.e., the probability measure on the Borel subsets of configuration space that makes the coordinate random variables $(\xi_g)_{g\in\Gamma}$ independent, identically distributed Bernoulli-$p$. A *tail event* is a Borel subset $B$ of $\Omega$ with the following property: for any two configurations $\xi, \xi'$ that differ in only finitely many entries, either $\xi \in B$ and $\xi' \in B$ or $\xi \in B^c$ and $\xi' \in B^c$.

**Lemma 1.** *Every tail event has $P_p-$probability 0 or 1.*

*Proof.* This is the Kolmogorov 0-1 Law. □

The group $\Gamma$ acts on the configuration space by left translation: for $g \in \Gamma$, the left translation $L_g$ is defined by $(L_g\xi)_{gh} = \xi_h$. For each $g \in \Gamma$ the left translation $L_g$ is $P_p-$measure-preserving. A random variable $X$ is said to be *$g-$invariant* if $X = X \circ L_g$ a.s. $P_p$. An event $B$ is called *$g-$invariant* if its indicator function is $g-$invariant.

**Lemma 2.** *If $g \in \Gamma$ is non-elliptic then the measure-preserving system $(\Omega, P_p, L_g)$ is ergodic and mixing.*

*Proof.* It suffices to prove that the system is mixing, as this implies ergodicity. For this it suffices, by a routine approximation argument, to prove that for any two cylinder events $A, B$ (events whose indicator functions depend only on finitely many coordinates),

$$\lim_{n\longrightarrow\infty} P_p(A \cap L_g^{-n}B) = P_p(A)P_p(B).$$

If $g$ is non-elliptic, then $g^{\pm n} \to \infty$, and consequently for every $h \in \Gamma$, $g^{\pm n}h \to \infty$. Since each of the indicators $1_A(\xi), 1_B(\xi)$ depends on only finitely many coordinates of $\xi$, it follows that for sufficiently large $n$ the indicators $1_A$ and $1_B \circ L_g$ depend on disjoint sets of coordinates, and therefore are independent under $P_p$. □

**Corollary 1.** *If $g \in \Gamma$ is non-elliptic then every $g$−invariant event has $P_p$−probability 0 or 1.*

The use of the 0-1 laws is facilitated by the following comparison lemma. For any configuration $\xi \in \Omega$ and any finite subset $K$ of $\Gamma$, define configurations $\xi^{0,K}$ and $\xi^{1,K}$ by

$$\begin{aligned} \xi_g^{i,K} &= \xi_g && \text{if } g \notin K; \\ \xi_g^{i,K} &= i && \text{if } g \in K. \end{aligned}$$

For any event $B$ and any finite subset $K$ of $\Gamma$, define events $B_K^0$ and $B_K^1$ by

$$B_K^i = \{\xi^{i,K} \mid \xi \in B\}.$$

**Lemma 3.** *For any event $B$ and any finite subset $K$ of $\Gamma$, if $P_p(B) > 0$ then $P_p(B_K^i) > 0$ for $i = 0$ and $i = 1$.*

*Proof.* Define $B_K^+$ to be the set of all configurations $\omega$ such that there exists a configuration $\omega' \in B$ that agrees with $\omega$ in all coordinates except possibly those in $K$. Clearly, $B$ is a subset of $B_K^+$, so if $P_p(B) > 0$ then $P_p(B_K^+) > 0$. Since the coordinate variables are independent under $P_p$,

$$P_p(B_K^0) = q^{|K|} P_p(B_K^+) > 0$$

and

$$P_p(B_K^1) = p^{|K|} P_p(B_K^+) > 0.$$

□

## 3. Consequences of the 0-1 Laws

The next result is taken from [1].

**Proposition 1.** *Let $N_R$ and $N_B$ be the number of* infinite *red clusters and blue clusters, respectively. Then with probability 1, $N_R$ and $N_B$ are constant, each taking one of the values 0, 1, or $\infty$.*

*Proof.* Since $\Gamma$ is nonelementary, it contains nonelliptic elements. Let $g \in \Gamma$ be nonelliptic. For either $i = R$ or $i = B$ and for any $k = 0, 1, 2, \ldots$ or $k = \infty$, the event $\{N_i = k\}$ is $g$−invariant. Consequently, by Corollary 1, it has probability 0 or 1. Thus, $N_i$ is almost surely constant.

Suppose that for some $k \in (1, \infty)$ the event $\{N_i = k\}$ had positive probability. Let $B_n$ be the event that all infinite $i$−clusters intersect the ball of radius $n$ centered at the origin of $\mathbb{H}$; then for sufficiently large $n$,

$$P_p(\{N_i = k\} \cap B_n) > 0.$$

Let $K$ be the set of all $g \in \Gamma$ such that $g(0)$ is contained in the ball of radius $n$ centered at 0. By Lemma 3,

$$P_p((\{N_i = k\} \cap B_n)^{i,K}) > 0.$$

But this is impossible, because on $B_n^{i,K}$ there is only one infinite $i$−cluster. □

For any $\zeta \in \partial\mathbb{H}$, say that $\zeta$ is an *$i-$cluster point* ($i =$Red or Blue) if there is an infinite $i-$path that has $\zeta$ as a cluster point. Similarly, say that $\zeta$ is an *$i-$limit point* ($i =$ Red or Blue) if there is an infinite $i-$path that converges to $\zeta$. It is not *a priori* necessary that an $i-$cluster point be an $i-$limit point, nor is it even *a priori* necessary that the existence of infinite $i-$clusters implies the existence of $i-$limit points. However, we shall see that at, least for a large class of Fuchsian groups $i-$cluster points must also be $i-$limit points.

**Proposition 2.** *If $P_p\{N_i > 0\} = 1$ then with probability 1 the set of $i-$cluster points is dense in $\partial\mathbb{H}$.*

*Proof.* Every infinite $i-$cluster has at least one cluster point in $\partial\mathbb{H}$. Since with probability one there is (by hypothesis) at least one infinite $i-$cluster, there is a nonempty open arc $A$ of $\partial\mathbb{H}$ such that with positive probability there is an $i-$cluster point in $A$.

Let $g \in \Gamma$ be hyperbolic. By Lemma 2 and the Birkhoff Ergodic Theorem,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} 1\{\exists\, i - \text{cluster point in } g^n A\} = P_p\{\exists\, i - \text{cluster point in } A\} > 0;$$

hence, with probability 1, there are $i-$cluster points in infinitely many of the arcs $g^n A$. In particular, $i-$cluster points accumulate at the attractive fixed point of $g$. Since the attractive fixed points of hyperbolic elements are dense in $\partial\mathbb{H}$ ([5], Theorem 3.4.4) it follows that with probability 1 the $i-$cluster points are dense in $\partial\mathbb{H}$. □

This result might lead one to suspect (however briefly) that in the $i-$percolation regime *all* points of $\partial\mathbb{H}$ are $i-$cluster points. Later we will show that this is not the case: When red and blue *sector* percolation occur simultaneously (see section 4 for the definition) and there are infinitely many $i-$clusters, the set of $i-$cluster points has (Lebesgue) measure zero, with probability 1. Thus, the *size* (as measured, for instance, by Hausdorff dimension) of the set of $i-$cluster points is an interesting quantity.

Essentially the same proof as in the previous proposition yields the following.

**Proposition 3.** *If an $i-$limit point exists with positive probability then with probability 1 the set of $i-$limit points is dense in $\partial\mathbb{H}$.*

For any nonempty arc $A$ (possibly a single point) of $\partial\mathbb{H}$, say that there is an $i-$path converging to $A$ if there is an infinite $i-$path all of whose cluster points are in $A$.

**Proposition 4.** *Suppose that there is a nonempty arc $A$ of $\partial\mathbb{H}$, whose complement in $\partial\mathbb{H}$ contains a nonempty open arc, such that, with positive probability, there is an $i-$path converging to $A$. Then for* every *nonempty open arc $A' \neq \mathbb{H}$ of $\partial\mathbb{H}$ there exists, almost surely, an $i-$path converging to $A'$.*

*Proof.* Choose a hyperbolic element $g \in \Gamma$ whose attractive fixed point is contained in $A'$. By Lemma 2 and the Ergodic Theorem, for infinitely many of the arcs $g^n A$ there are infinite $i-$paths that converge to $g^n A$, with probability 1. Since the attractive fixed point of $g$ is contained in $A'$, all but finitely many of the arcs $g^n A$ are contained in $A'$. □

## 4. Sector Percolation

Say that *$i-$sector percolation* occurs if there is an infinite *$i-$ path* that converges to a nonempty open arc $A$ of $\partial\mathbb{H}$ such that $A \neq \partial\mathbb{H}$. By the last proposition of the preceding section, if $i-$sector percolation occurs with positive probability then, with probability 1, for every nonempty open arc $A$ of $\partial\mathbb{H}$ there are infinite $i-$paths converging to $A$. Hence, $i-$sector percolation is a 0-1 event.

**Conjecture 1.** *Percolation implies sector percolation.*

We will show that this is true for a large class of co-compact Fuchsian groups (see Corollary 4 below), but a general proof has eluded us.

**Lemma 4.** *If $i-$sector percolation occurs then for every pair $A, A'$ of nonempty arcs in $\partial\mathbb{H}$ the probability that there is a doubly infinite $i-$path connecting $A$ with $A'$ is positive.*

*Proof.* With probability 1 there exist infinite paths converging to $A$ and $A'$. Consequently, for sufficiently large $n$ there exist, with positive probability, infinite paths converging to $A$ and $A'$, respectively, both originating in $B_n =$ the set of all vertices $g \in \Gamma$ at hyperbolic distance $\leq n$ from the origin. Clearly, on the event that *all* vertices in $B_n$ are colored $i$, any two such infinite paths could be connected to form a doubly infinite path connecting $A$ with $A'$. It follows from Lemma 3 that this happens with positive probability. □

**Corollary 2.** *If $i-$sector percolation occurs then with probability* 1 *there exist doubly infinite $i-$paths connecting nonoverlapping arcs of $\partial\mathbb{H}$.*

*Proof.* The event that there exist doubly infinite $i-$paths connecting nonoverlapping arcs of $\partial\mathbb{H}$ is $g-$invariant for every hyperbolic $g \in \Gamma$, so the result follows from Corollary 1. □

**Corollary 3.** *If $i-$sector percolation and $j-$percolation both occur with positive probability (where $i =$Red, $j =$Blue or $i =$Blue, $j =$Red) then $j-$sector percolation occurs, and with probability one there are infinitely many infinite red clusters and infinitely many infinite blue clusters.*

*Proof.* Suppose that $i-$sector percolation occurs. Then with probability 1 there exists a doubly infinite $i-$path $\gamma$ connecting disjoint open arcs $A, A'$of $\partial\mathbb{H}$. These arcs partition $\partial\mathbb{H}$ into four nonempty segments $A, A', B, B'$. If there exists an infinite $j-$cluster $C$, it must lie on one side or the other of $\gamma$, and consequently, any infinite self-avoiding $j-$path in $C$ must converge either to $B$ or to $B'$. Thus, there is $j-$sector percolation.

By Corollary 2, there exist doubly infinite red paths and doubly infinite blue paths connecting disjoint arcs of $\partial\mathbb{H}$. It follows from Proposition 4 that $N_{Red} \geq 2$ and $N_{Blue} \geq 2$. Hence, Proposition 1 implies that $N_{Red} = N_{Blue} = \infty$. □

**Corollary 4.** *Suppose that for some value of $p$ red sector percolation and blue sector percolation both occur with positive $P_p-$probability. Then for* all *values of $p$ and $i =$red or blue, if $i-$percolation occurs with positive $P_p-$probability then $i-$*sector *percolation occurs with $P_p-$probability* 1.

*Proof.* If for some $p_*$ red sector percolation and blue sector percolation both occur with positive $P_{p_*}-$probability, then they occur with $P_{p_*}-$probability 1. Hence, for any $p \geq p_*$, *blue* sector percolation occurs with $P_p-$probability 1, and for any $p \leq p_*$, *red* sector percolation occurs with $P_p-$probability 1. Thus, for every value of $p$, it is $P_p-$ almost sure that $i-$sector percolation occurs for either $i =$blue or $i =$red. Corollary 3 therefore implies that if $j-$percolation occurs with positive $P_p-$probability then $j-$*sector* percolation occurs with $P_p-$probability 1. □

Let $\zeta \in \partial\mathbb{H}$ and let $A$ be a closed arc of $\partial\mathbb{H}$ contained in $\partial\mathbb{H} - \{\zeta\}$. The complement of $A \cup \{\zeta\}$ in $\partial\mathbb{H}$ is the union of two nonoverlapping, nonempty open arcs $B$ and $B'$. Say that a doubly infinite $i-$path separates $\zeta$ from $A$ if it connects closed sub-arcs of $B$ and $B'$.

**Lemma 5.** *Assume that $i-$sector percolation occurs. Then for every hyperbolic fixed point $\zeta \in \partial\mathbb{H}$ and every closed arc $A \subset \partial\mathbb{H}-\{\zeta\}$ there exists a doubly infinite $i-$path that separates $\zeta$ from $A$.*

*Proof.* Since $\zeta$ is a hyperbolic fixed point there is a hyperbolic element $g \in \Gamma$ whose *attractive* fixed point is $\zeta$. Let $\zeta'$ be its repulsive fixed point, and let $B_1, B_2$ be the (disjoint) open arcs of $\partial\mathbb{H}$ with endpoints $\zeta$ and $\zeta'$. Choose nonempty open arcs $C_1, C_2$ whose closures are contained in $B_1, B_2$, respectively. As $n \to \infty$ the arcs $g^n C_1$ and $g^n C_2$ converge to $\zeta$ – in particular, for sufficiently large $n$ any doubly infinite $i-$path connecting $g^n C_1$ and $g^n C_2$ will separate $\zeta$ from $A$. But Lemma 4, Lemma 2, and the Ergodic Theorem imply that, with probability 1, for infinitely many $n$ there exist doubly infinite $i-$paths that connect $g^n C_1$ and $g^n C_2$. □

**Corollary 5.** *Suppose that red and blue sector percolation both occur with probability* 1. *Then with probability* 1, *no hyperbolic fixed point is a red cluster point or a blue cluster point.*

*Proof.* Lemma 5 implies that, for any particular hyperbolic fixed point $\zeta$, the probability that $\zeta$ is a red cluster point or a blue cluster point is 0. Since the set of hyperbolic fixed points is countable, the corollary follows. □

NOTE. A stronger result will be proved in Proposition 12 below.

**Proposition 5.** *Assume that $i-$sector percolation occurs but that $j-$percolation does* not *occur (where $i$ =Red, $j$ =Blue or $i$ =Blue, $j$ =Red). Then with probability* 1, *there is a single infinite $i-$cluster, and every $\zeta \in \partial\mathbb{H}$ is a limit point of this cluster.*

*Proof.* If $j-$percolation does not occur then all $j-$clusters are finite, and, consequently, surrounded by closed $i-$paths. It follows that for every $n \geq 1$ there is a closed $i-$path $\gamma_n$ surrounding the ball of radius $n$ centered at the origin 0. Any infinite $i-$cluster must intersect all but finitely many of the paths $\gamma_n$. But if two infinite $i-$clusters intersect the same $\gamma_n$ then they coincide, because they are connected by $\gamma_n$. It follows that there is only one infinite $i-$cluster.

For any nonempty open arc $A$ of $\partial\mathbb{H}$, define the *angular sector* $\hat{A}$ over $A$ to be the set of all $\xi \in \mathbb{H}$ such that the geodesic emanating from the origin and passing through $\xi$ converges to a point of $A$. The *edges* of this angular sector are the two geodesics emanating from the origin and converging to the endpoints of $A$. If $\gamma_n$ is any closed path in $\mathbb{H}$ that surrounds the ball of radius $n$ centered at the origin, then for any angular sector $\hat{A}_n$ there is a segment $\beta_n$ of $\gamma_n$ that connects the edges of $\hat{A}_n$ and lies entirely in the closure of $\hat{A}_n$.

Fix $\zeta \in \partial\mathbb{H}$; we will construct an infinite $i-$path that converges to $\xi$. Let $\{A_n\}_{n\geq 1}$ be a nested sequence (i.e., the closure of each $A_n$ is contained in $A_{n-1}$) of nonempty open arcs such that

$$\bigcap_{n=1}^{\infty} A_n = \{\xi\}.$$

By Proposition 4, for each $n$ there is an infinite $i-$path $\alpha_n$ that converges to $A_n$. The path $\alpha_n$ may be chosen so that it lies entirely in the angular sector $\hat{A}_{n-1}$ over $A_{n-1}$. For each $n$ the path $\alpha_n$ must cross all but finitely many of the closed $i-$paths $\gamma_m$; in particular, for each $n$ there exists $m_n > m_{n-1}$ so large that $\alpha_n$ crosses $\gamma_m$ for all $m \geq m_n$. Build an infinite $i-$path as follows: Proceed along $\alpha_1$ until it first reaches a point of $\beta_2$; then follow $\beta_2$ until it first reaches a point of $\alpha_2$; then follow $\alpha_2$ until it first reaches a point of $\beta_3$; etc. The resulting path $\alpha$ will converge to $\zeta$. □

## 5. Simultaneous Red and Blue Sector Percolation

If $i$−sector percolation and $j$−percolation occur with positive probability, then by Corollary 3 red and blue sector percolation both occur almost surely. In this section we will investigate the consequences of simultaneous red and blue sector percolation. Throughout the section, the following standing hypothesis will be in force:

**Hypothesis 1.** *Red and blue sector percolation both occur almost surely.*

### 5.1. Limit Points and Cluster Points.

**Proposition 6.** *Every infinite self-avoiding $i$−path ($i$ =Red or Blue) converges to a point of $\partial\mathbb{H}$.*

*Proof.* For definiteness, let $i$ =blue. Let $\zeta, \zeta'$ be distinct points of $\partial\mathbb{H}$, and let $B, B'$ be the disjoint open arcs of $\partial\mathbb{H}$ with endpoints $\zeta$ and $\zeta'$. Since red sector percolation occurs with probability 1, there exist, by Proposition 4, infinite self-avoiding red paths $\gamma$ and $\gamma'$ converging to $B$ and $B'$, respectively. Let $g$ and $g'$ be the initial points of $\gamma$ and $\gamma'$, and let $\beta$ be a finite path in $G_{\text{red}}$ connecting $g$ and $g'$. The doubly infinite path comprised of the paths $\gamma, \beta$, and $\gamma'$ separates $\zeta$ from $\zeta'$; consequently, any infinite blue path that has both $\zeta$ and $\zeta'$ as cluster points must cross $\beta$ infinitely often. Since $\beta$ is finite, such a blue path could not possibly be self-avoiding. □

**Proposition 7.** *Every $i$−cluster point is an $i$−limit point ($i$ =Red or Blue).*

*Proof.* For definiteness, let $i$ =blue. Let $\zeta \in \partial\mathbb{H}$ be a blue cluster point; then by definition there exists an infinite blue path $\gamma$ that has $\zeta$ as a cluster point. We will use $\gamma$ to construct a self-avoiding blue path $\gamma'$ that converges to $\zeta$.

Since $\gamma$ has $\zeta$ as a cluster point, there exists a sequence of vertices $g_n$ on $\gamma$ such that $g_n \to \zeta$. Consequently, for each $n \geq 1$ there exists a (finite) self-avoiding blue path $\gamma_n$ that connects $g_0$ to $g_n$, obtained by following $\gamma$ from $g_0$ to $g_n$, excising any "loops" that occur along the way. Let

$$B_m = \{g \in \Gamma \mid d_H(g, 1) \leq m\}.$$

Since $g_n \to \zeta$, for each fixed $m$ the path $\gamma_n$ will exit $B_m$ for all sufficiently large $n$. Define $h_m^n$ to be the last vertex of $B_m$ visited by $\gamma_n$. Since $B_m$ is finite, some $h_m \in B_m$ must occur infinitely often in the sequence $(h_m^n)_{n\geq 1}$; by a routine diagonal argument, the terms $h_m$ may be chosen so that for some subsequence $n_k$,

$$h_m^{n_k} = h_m$$

for all $k \geq k_m$, with $k_m < \infty$ for each $m$. Consequently, there exist finite self-avoiding blue paths $\gamma'_m$ from $g_0$ to $h_m$ such that each $\gamma'_{m+1}$ is an extension of $\gamma'_m$. Define $\gamma'$ to be the direct limit of the sequence $\gamma'_m$, i.e., $\gamma'$ is the unique infinite path that is an extension of every $\gamma'_m$. Clearly, $\gamma'$ is a self-avoiding infinite blue path, so by the last proposition it converges to a point $\zeta'$ of $\partial\mathbb{H}$. Thus, to complete the proof, it suffices to show that $\zeta' = \zeta$, i.e., that $\zeta$ is the only possible cluster point of $\gamma'$. Note that the vertices $h_m$ converge to infinity; since they lie on $\gamma'$ it follows that $h_m \to \zeta'$.

Suppose that $\zeta' \neq \zeta$. Then by Proposition 4, there exist infinite *red* paths $\beta$ and $\beta'$ that converge to the open arcs of $\partial\mathbb{H}$ with endpoints $\zeta$ and $\zeta'$. Clearly, there is an integer $m \geq 1$

such that each of $\beta$ and $\beta'$ intersects $B_m$. Recall that $h_m \to \zeta'$, and that for all sufficiently large $j$ there exists a self-avoiding blue path from $h_m$ to $g_{n_j}$ that does not re-enter $B_m$. If $m$ is sufficiently large (so that $h_m$ is close to $\zeta'$), any path from $h_m$ to $g_{n_j}$ that does not re-enter $B_m$ must cross either $\beta$ or $\beta'$. Since $\beta$ and $\beta'$ are *red* paths, this is a contradiction. □

**5.2. Limit Set of an Infinite Cluster.** For $g \in \Gamma$ and $i$ =red or blue, define $\Lambda_g^i$ to be the set of limit points in $\partial\mathbb{H}$ of the $i-$cluster containing $g$. Observe that unless $g$ is contained in an *infinite* $i-$cluster, $\Lambda_g^i = \emptyset$; consequently, either $\Lambda_g^{\text{red}} = \emptyset$ or $\Lambda_g^{\text{blue}} = \emptyset$.

**Proposition 8.** *The set $\Lambda_g^i$ is closed.*

*Proof.* By Proposition 7, $\Lambda_g^i$ is also the set of *cluster* points of the $i-$cluster containing $g$. Suppose that $\zeta_n \in \Lambda_g^i$ is a sequence converging in $\partial\mathbb{H}$ to some point $\zeta$. We must show that there is an infinite path $\gamma$ in the (infinite) $i-$ cluster $\mathcal{C}_g$ containing $g$ that has $\zeta$ as a cluster point.

For each $n$ there is a self-avoiding path $\gamma_n$ in $\mathcal{C}_g$ that converges to $\zeta_n$. Since $\zeta_n \to \zeta$, there exist vertices $g_n \in \gamma_n$ such that $g_n \to \zeta$. Each $g_n$ is an element of $\mathcal{C}_g$; consequently, for each $n$ there is a finite path $\beta_n$ starting at $g$ and ending at $g_n$. Let $\gamma$ be the infinite path in $\mathcal{C}_g$ that first follows $\beta_1$ from $g$ to $g_1$ and then $\beta_1$ in reverse from $g_1$ back to $g$, then follows $\beta_2$ from $g$ to $g_2$ and then $\beta_2$ in reverse from $g_2$ back to $g$, etc. For each $n$ the path $\gamma$ visits $g_n$, so $\zeta$ is a cluster point of the path $\gamma$. □

**Proposition 9.** *The set $\Lambda_g^i$ is nowhere dense.*

*Proof.* For definiteness let $i$ =blue. Suppose that $\Lambda_g^{\text{blue}}$ contains two distinct points $\zeta$ and $\zeta'$. Let $A$ be one of the two closed arcs of $\partial\mathbb{H}$ with endpoints $\zeta$ and $\zeta'$, and let $B = \partial\mathbb{H} - A$. Since $\zeta \neq \zeta'$, the arc $B$ is a nonempty open arc, containing at least one hyperbolic fixed point $\zeta''$. By Hypothesis 1 both red and blue sector percolation occur with probability 1; consequently, by Lemma 5, there exists a doubly infinite *red* path separating $\zeta''$ from $A$. Thus, there is an open arc containing $\zeta''$ that cannot contain any points of $\Lambda_g^{\text{blue}}$. □

**Proposition 10.** *If $\Lambda_g^i \neq \emptyset$ then almost surely $|\Lambda_g^i| = \infty$.*

It obviously suffices to prove the proposition for $g = 1$, and for definiteness we shall consider only the case $i$ =blue. Write $\Lambda_1^{\text{blue}} = \Lambda$. The proof requires viewing the percolation process in "layers". Let $C_n$ be the hyperbolic circle of radius $n$ centered at the origin 0, and let $B_n$ be the set of vertices $g \in \Gamma$ such that the tile $g(T)$ does not lie entirely outside $C_n$. For each $n \geq 1$ define $A_n$ to be the (random) set of all $g \in \Gamma$ such that (i) there exists a blue path $\gamma$ from the vertex 1 to the vertex $g$ such that all vertices on $\gamma$ are in $B_n$; and (ii) the tile $g(T)$ intersects the circle $C_n$.

**Lemma 6.** *On the event $\{\Lambda \neq \emptyset\}$, $|A_n| \longrightarrow \infty$ almost surely.*

*Proof.* First note that if $\Lambda \neq \emptyset$ then each $A_n$ is nonempty, because if $A_n = \emptyset$ then the blue cluster $\mathcal{B}_1$ containing vertex 1 must lie entirely inside $C_n$. Let $\nu$ be the cardinality of the set of generators of $\Gamma$ and $q = 1 - p$ be the probability that a vertex is colored red. Suppose that $|A_n| \leq k$; then there is (conditional) probability at least $q^{\nu k}$ that $\mathcal{B}_1$ is "cut off" at $C_n$, i.e., that all the tiles outside $C_n$ bordering tiles $g(T)$ where $g \in A_n$ are colored red. If $\mathcal{B}_1$ is "cut off" at $C_n$ then clearly $A_{n+d} = \emptyset$, where $d$ is the smallest integer smaller than the diameter of the tile $T$, and so $\Lambda = \emptyset$. By Lévy's version of the Borel-Cantelli Lemma, if $|A_n| \leq k$ for infinitely many $n$, then, with probability 1, for some $n$ it will happen that

$\mathcal{B}_1$ is "cut off" at $C_n$. Consequently, on the event $\{\Lambda \neq \emptyset\}$, it cannot happen that $|A_n| \leq k$ for infinitely many $n$; thus, $|A_n| \to \infty$ a.s. □

Let $\gamma$ be an oriented doubly infinite geodesic $\gamma$ that intersects the tile $T$, and define $\mathcal{R}(\gamma, 1)$ to be the event that there is an infinite self-avoiding blue path starting at 1 and passing only through tiles $g(T)$ that intersect the half-plane to the right of $\gamma$.

**Lemma 7.** *There exists $\rho > 0$ such that for* every *oriented doubly infinite geodesic $\gamma$ that intersects the tile $T$,*

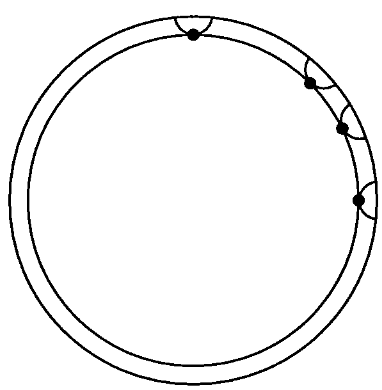
$$P_p(\mathcal{R}(\gamma, 1)) \geq \rho.$$

*Proof.* Choose finitely many half-planes $H_1, \ldots, H_l$ such that for every oriented doubly infinite geodesic $\gamma$ intersecting the tile $T$, one of the half-spaces $H_i$ lies entirely to the right of $\gamma$. (If $l$ is suitably large then the half-spaces bounded by the geodesics joining successive points $e^{2\pi j/l}$ on $\partial\mathbb{H}$ will work, because the tile $T$ is compact in $\mathbb{H}$.) For each $H_i$ there is a vertex $g_i \in \Gamma$ such that, with positive probability, there is an infinite self-avoiding blue path starting at $g_i$ and lying entirely in $H_i$ – this follows from Proposition 4. Now let $\beta_i$ be the geodesic segment from the vertex 1 (at the origin) to the vertex $g_i$, and let $K_i$ be the set of all $g \in \Gamma$ such that $\beta_i$ intersects the tile $g(T)$. If there is an infinite self-avoiding blue path starting at $g_i$ and lying entirely in $H_i$, and if all of the vertices in $K_i$ are colored blue, then, clearly, for every oriented doubly infinite geodesic $\gamma$ that intersects the tile $T$ and contains $H_i$ entirely to its right, there is an infinite self-avoiding blue path starting at 1 and passing only through tiles $g(T)$ on the right of $\gamma$. But by Lemma 3 the probability that all of the vertices in $K_i$ are colored blue and that there is an infinite self-avoiding blue path starting at $g_i$ and lying entirely in $H_i$ is positive. The lemma follows. □

*Proof of Proposition 10.* By Lemma 6, $|A_n| \to \infty$ a.s. on the event $\{\Lambda \neq \emptyset\}$, so there exist subsets $A_n^* \subset A_n$ such that $|A_n^*| \to \infty$ and such that

$$\min_{\substack{g,h \in A_n^* \\ g \neq h}} d_H(g, h) \longrightarrow \infty$$

as $n \to \infty$, where $d_H$ denotes the hyperbolic distance. For each $g_i \in A_n^*$, choose a geodesic $\gamma_i$ tangent to the circle $C_n$ that passes through the tile $g_i(T)$, and let $H_i$ be the half-plane bounded by $\gamma_i$ exterior to $C_n$. Because the hyperbolic distance between any two distinct $g_i \in A_n^*$ is large, the half-planes $H_i$ do not overlap (in fact the minimum distance between distinct $H_i$ converges to $\infty$ as $n \to \infty$).



Let $F_i$ be the event that there is an infinite self-avoiding blue path starting at $g_i$ and passing only through tiles that intersect $H_i$. Since the half-planes do not overlap, the events $F_i$ are conditionally independent (given the assignment of colors to vertices inside $C_n$), and by Lemma 7, each $F_i$ has conditional probability at least $\rho$. For each event $F_i$ that occurs there is a distinct limit point in $\Lambda$. Hence, since $|A_n^*| \to \infty$, the Weak Law of Large Numbers implies that, for every $m < \infty$, the probability that $|\Lambda| \leq m$ is 0. □

**Corollary 6.** *For every half-plane $H \subset \mathbb{H}$ there exists, with probability 1, an infinite blue cluster contained entirely in $H$ with infinitely many distinct limit points in $\partial\mathbb{H}$.*

*Proof.* Let $A \subset \partial\mathbb{H}$ be the boundary arc of the half-plane $H$. Since hyperbolic fixed points are dense in $A$, Lemma 5 implies that, with probability 1, there is a doubly infinite red path connecting non-overlapping closed arcs $B, B'$ contained in $A$. By Proposition 2, there is an infinite blue cluster that has a cluster point on the arc of $A$ between $B$ and $B'$. By Proposition 10, this infinite blue cluster has infinitely many distinct limit points in $\partial\mathbb{H}$. But all limit points of the cluster must be contained in the arc of $A$ between $B$ and $B'$, because the blue cluster cannot cross the doubly infinite red path connecting $B$ and $B'$. □

**Proposition 11.** *If $\Lambda_g^i \neq \emptyset$ then almost surely $\Lambda_g^i$ is a perfect set.*

*Proof.* Without loss of generality we may take $g = 1$. We will consider only the case $i$ =blue.

Suppose the statement were false. Then there would exist a nonempty open arc $A$ of the circle $\partial\mathbb{H}$ such that, with positive probability, $\Lambda \cap A$ is a nonempty *finite* set. Moreover, since infinite *red* clusters accumulate at a dense set of points in $A$, for some $n < \infty$ there would exist, with positive probability, infinite red clusters $\mathcal{C}_1, \mathcal{C}_2$ such that (i) each of $\mathcal{C}_1, \mathcal{C}_2$ contains a vertex at hyperbolic distance $< n$ from the origin; and (ii) the limit sets of $\mathcal{C}_1, \mathcal{C}_2$ contain points $\zeta_1, \zeta_2 \in A$, respectively, such that all points of $\Lambda \cap A$ lie between $\zeta_1$ and $\zeta_2$. Let $B$ be the event that all of these things occur, i.e., that $\Lambda \cap A$ is a nonempty *finite* set and there exist infinite red clusters $\mathcal{C}_1, \mathcal{C}_2$ with the properties detailed above. By hypothesis,

$$P_p(B) > 0.$$

Consider the event $B^0$ consisting of all configurations $\omega$ such that, for some configuration $\omega' \in B$, $\omega$ is obtained from $\omega'$ by changing to red the colors of all vertices $g$ at hyperbolic distance $< n$ from the origin. Observe that changing these vertices to red has the effect of disconnecting the infinite blue cluster that (before the color changes) contained the vertex 1, leaving at least one infinite blue cluster all of whose limit points are in $A$, and therefore with only finitely many limit points. Since $P_p(B) > 0$, Lemma 3 implies that

$$P_p(B^0) > 0.$$

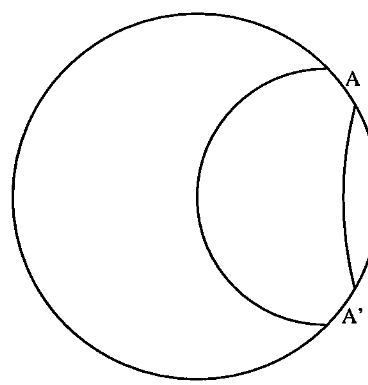
But on the event $B^0$ there is, by construction, an infinite blue cluster whose limit set is *finite*. This contradicts Proposition 10. □

**Proposition 12.** *For every $\zeta \in \partial\mathbb{H}$,*

$$P_p\{\zeta \in \Lambda_g^i\} = 0.$$

*Proof.* It suffices to prove the statement for $g = 1$. For definiteness, let $i$ =blue.

Let $A, B, A', B'$ be nonoverlapping closed arcs of $\partial\mathbb{H}$, each of length $l < \pi/2$, such that $B, A'$, and $B'$ are obtained by rotating $A$ by $\pi/2, \pi$, and $3\pi/2$ radians, respectively. (Thus, the geodesics in $\mathbb{H}$ from the centers of $A$ and $B$ to the centers of $A'$ and $B'$, respectively, meet at the central point 0 of $\mathbb{H}$ at right angles.) By Lemma 4, the probability that there is a doubly infinite *red* path connecting $A$ and $A'$ is positive. Such a path cannot approach $\partial\mathbb{H} - (A \cup A')$, so it must lie entirely inside a region of $\mathbb{H}$ bounded by two hypercycles joining the endpoints of $A$ to those of $A'$ (a *hypercycle* is just the segment of a Euclidean circle that intersects the disc $\mathbb{H}$). Call such a region a *hypercyclic strip* bounded by $A$ and $A'$ at infinity. It now follows that for some hypercyclic strip $S_a$ bounded by $A$ and $A'$ at infinity, the probability that there is a doubly infinite red path connecting $A$ and $A'$ and lying entirely in $S_a$ is positive.




Similarly, there is a hypercyclic strip $S_b$ bounded by $B$ and $B'$ at infinity such that, with positive probability, there is a doubly infinite red path connecting $B$ and $B'$ and lying entirely in $S_b$.

Fix $\zeta \in \partial\mathbb{H}$, and let $\zeta'$ be the antipodal point of $\partial\mathbb{H}$. Let $J$ and $J'$ be the *open* arcs of $\partial\mathbb{H}$ with endpoints $\zeta$ and $\zeta'$, and let $\gamma$ be the geodesic ray emanating from the origin 0 that converges to $\zeta$. The geodesic $\gamma$ passes through a sequence $g_n(T)$ of tiles, beginning with $g_0 = 1$ (since by convention the origin is an interior point of $T$). For each $g_n$ at least one of the hypercyclic strips $g_n(S_a), g_n(S_b)$ "cuts" the geodesic $\gamma$, i.e., the boundary arcs $g_n(A)$ and $g_n(A')$ (or the boundary arcs $g_n(B)$ and $g_n(B')$) are contained in opposite arcs $J, J'$. Hence, for either $i = a$ or $i = b$ (or both), the hypercyclic strip $g_n(S_i)$ cuts $\gamma$ for infinitely many $g_n$. It follows that there is a subsequence $h_k$ of the sequence $g_n$ such that for $i = a$ or $i = b$

(1) Each hypercyclic strip $h_k(S_i)$ cuts $\gamma$; and
(2) Distinct hypercyclic strips $h_k(S_i), h_l(S_i)$ are *strongly nonoverlapping*, in the sense that every tile $g(T)$ of the tessellation intersects the closure of at most one of the strips.

Let $F_k$ be the event that there is a doubly infinite *red* path lying entirely inside the hypercyclic strip $h_k(S_i)$ that connects its opposite boundary arcs ($h_k(A)$ and $h_k(A')$ if $i = a$, and $h_k(B)$ and $h_k(B')$ if $i = b$). Since distinct strips $h_k(S_i)$ are strongly nonoverlapping, the events $F_1, F_2, \ldots$ are independent; and since the strips $h_k(S_i)$ are all congruent by an element of $\Gamma$, the events $F_k$ all have the same probability. By construction, this probability is *positive*. Consequently, by the Strong Law of Large Numbers, infinitely many of the strips $h_k(S_i)$ contain doubly infinite red paths connecting opposite arcs $J, J'$. But the existence of *any* such red path precludes the possibility of an infinite blue path that starts at the vertex $i$ and has $\zeta$ as a cluster point. □

**Corollary 7.** *With probability* 1, *the Lebesgue measure of* $\Lambda_g^i$ *is* 0.

## 6. Groups of Genus $\geq 2$

Every finitely generated Fuchsian group has a *signature* that determines a presentation in terms of the generating set $\mathcal{G}$. (Recall that $\mathcal{G}$ is the set of side-pairing transformations for the fundamental tile $T$.) The signature $\Gamma(e_1, e_2, \dots, e_n; g)$ consists of $n \geq 0$ *exponents* $e_i$, which are integers $\geq 2$, and the *genus* $g$, which is a nonnegative integer. The generating set $\mathcal{G}$ has $2n + 4g$ elements

$$c_i^{\pm 1}, a_j^{\pm 1}, b_j^{\pm 1},$$

with $1 \leq i \leq n$ and $1 \leq j \leq g$. These generators satisfy the relations

$$c_i^{e_i} = 1, \; 1 \leq i \leq n, \text{ and}$$

$$c_1 c_2 \cdots c_n \prod_{j=1}^{g} (a_j b_j a_j^{-1} b_j^{-1}) = 1,$$

and all other relations can be derived from these. See [8], page 98, for further details.

The *surface group* $\Gamma_g$ is the Fuchsian group with 0 exponents and genus $g$; it is the fundamental group of a compact, orientable surface of genus $g$. For genus 0, $\Gamma_g$ is the trivial group, and for genus 1, $\Gamma_g = \mathbb{Z}^2$. These are both elementary groups, so we exclude them from further consideration. For the surface group $\Gamma_g$ there is one fundamental relation, $\prod_{j=1}^{g}(a_j b_j a_j^{-1} b_j^{-1}) = 1$; the expression $R = \prod_{j=1}^{g}(a_j b_j a_j^{-1} b_j^{-1})$ is called the *fundamental relator.*

**Dehn's Theorem.** [3, 4] *Let $\Gamma_g$ be the surface group of genus $g$ with $g \geq 2$. Any nonempty word $W$ in the generators $a_i^{\pm 1}, b_i^{\pm 1}$ that represents the identity can be shortened in at least one of the following ways:*

1. *Delete a subword $xx^{-1}$, where $x$ is one of the generators.*
2. *Replace a subword $A$ by the shorter word $B$, where $A, B$ are such that $AB^{-1}$ is a cyclic permutation of the fundamental relator $R$ or its inverse $R^{-1}$.*

This theorem is often referred to as "Dehn's algorithm" because it provides an automatic way to determine whether a word $w$ in the generators represents the identity, and therefore an automatic way to determine whether two words $W_1, W_2$ represent the same group element of the surface group.

Say that a word $W$ is *reduced* if it contains no subwords $xx^{-1}$ with $x$ a generator, and say that it is *Dehn reduced* if it cannot be shortened by either of the methods specified in the theorem.

**Corollary 8.** *Let $\Gamma_g$ be the surface group of genus $g$ with $g \geq 2$. Let $\mathcal{F}$ be a subset of the set $\mathcal{G}$ of generators such that for each index $i$, $\mathcal{F}$ contains elements of only one of the pairs $\{a_i, a_i^{-1}\}, \{b_i, b_i^{-1}\}$. Then two* reduced *words $U$ and $V$ containing letters only from $\mathcal{F}$ represent the same element of the surface group $\Gamma_g$ if and only if they are identical as words, i.e., if and only if $U$ and $V$ have the same length and $U_j = V_j$ for every index $j$.*

*Proof.* If $U$ and $V$ represent the same element of $\Gamma_g$, then $UV^{-1}$ represents the identity. Because the letters of $U$ and $V$ come from $\mathcal{F}$, it is impossible to shorten $UV^{-1}$ by the second of the two methods specified in Dehn's theorem, so it must be possible to remove a spur $xx^{-1}$, $x \in \mathcal{G}$. But $U$ and $V$ (and hence also $V^{-1}$) are reduced, so if a spur occurs in $UV^{-1}$ it must be at the point of conjunction, i.e., $U$ and $V$ have the same last letter. A routine induction argument now shows that $U$ and $V$ have the same length and $U_j = V_j$ for every index $j$. $\square$

**Corollary 9.** *Let $\mathcal{F} = \mathcal{F}_g$ be the subgroup of the surface group $\Gamma_g$ generated by the elements $\{a_j^{\pm 1}\}_{1\leq j\leq g}$. Then $\mathcal{F}$ is a free group on $2g$ generators.*

*Proof.* By the preceding corollary, if two words $U, V$ in these generators represent the same element, then they are identical as words. Consequently, there are no relations in the generators $a_i^{\pm 1}$, and so $\mathcal{F}$ is a free group. □

**Theorem 1.** *Let $\Gamma_g$ be the surface group of genus $g \geq 2$. For all $p \in (1/(2g-1), 1 - 1/(2g-1))$, red and blue sector percolation occur with $P_p$−probability 1 on $\Gamma_g$.*

*Proof.* Let $\mathcal{F}$ be the subgroup of $\Gamma_g$ consisting of all words in the generators $a_i^{\pm 1}$. By the preceding corollary, $\mathcal{F}$ is a free group on $2g$ generators, so its Cayley graph $C(\mathcal{F})$, which is contained in that of $\Gamma_g$, is a homogeneous tree of degree $2g$. If $p > 1/(2g-1)$, blue percolation occurs with $P_p$−probability 1 on $C(\mathcal{F})$, because if $Z_n$ is the number of elements of $\mathcal{F}$ of word length $n$ connected to 1 by a blue path in $C(\mathcal{F})$ then $Z_n$ is a Galton-Watson process with mean offspring number $(2g-1)p$. Furthermore, blue percolation in $C(\mathcal{F})$ is necessarily blue *sector* percolation, because $C(\mathcal{F})$ is embedded as a tree in the hyperbolic plane $\mathbb{H}$. Thus, for $p > 1/(2g-1)$ blue sector percolation occurs on $\Gamma_g$ with $P_p$− probability 1. The same argument shows that for all $p < 1-(1/(2g-1))$, *red* sector percolation occurs with $P_p$−probability 1. □

**Theorem 2.** *Let $\Gamma = \Gamma(e_1, e_2, \ldots, e_n; g)$ be a co-compact Fuchsian group with genus $g \geq 2$. Then for all $p \in (1/(2g-1), 1 - 1/(2g-1)$, red and blue sector percolation occur with $P_p$−probability 1 on $\Gamma$.*

*Proof.* Let $c_i^{\pm 1}, a_j^{\pm 1}, b_j^{\pm 1}$ be the generators of $\Gamma$. By a basic result of combinatorial group theory ([9], Corollary 1.1.3) there is a natural homomorphism $\varphi : \Gamma \to \Gamma_g$ to the surface group $\Gamma_g$ of genus $g$ such that

$$\begin{aligned} \varphi(c_i) &= 1; \\ \varphi(a_j) &= a_j; \\ \varphi(b_j) &= b_j. \end{aligned}$$

Consider the subgroup $\mathcal{F}^*$ of $\Gamma$ generated by $\{a_j, b_j\}_{1\leq j\leq g}$; the homomorphism $\varphi$ maps $\mathcal{F}^*$ isomorphically onto $\mathcal{F}$, so $\mathcal{F}^*$ is free on the generators $a_j^{\pm 1}$ and $b_j^{\pm 1}$. The same argument as used in the proof of the preceding theorem now applies. □

**Corollary 10.** *Let $\Gamma$ be a co-compact Fuchsian group with genus $g \geq 2$. There exist constants $0 < p_1 \leq 1/(2g-1)$ and $1 - 1/(2g-1) \leq p_2 < 1$ such that*

(1) *For $p < p_1$ there is a single infinite red cluster and no infinite blue cluster, with probability* 1.
(2) *For $p > p_2$ there is a single infinite blue cluster and no infinite red cluster, with probability* 1.
(3) *For $p_1 < p < p_2$ there are infinitely many infinite red clusters and infinitely many infinite blue clusters, with probability* 1.

*Proof.* By Theorem 2, there exist values of $p$ such that under $P_p$ both red and blue sector percolation occur with probability 1. Hence, by Corollary 4, for *all* values of $p$, if $i$−percolation ($i$ =red or blue) occurs with positive $P_p$−probability then $i$−*sector* percolation occurs with

$P_p$−probability 1. Define

$$p_1 = \inf\{p \mid P_p(\text{blue percolation}) > 0\};$$
$$p_2 = \sup\{p \mid P_p(\text{red percolation}) > 0\}.$$

By Theorem 2, $p_1 < 1/(2g-1)$ and $p_2 > (2g-2)/(2g-1)$. By Proposition 5, for $p < p_1$ there is, with $P_p$−probability 1, a single infinite red cluster, and for $p > p_2$ there is with $P_p$−probability 1, a single infinite blue cluster. By Corollary 3, for all $p \in (p_1, p_2)$, there are, with probability 1, infinitely many infinite red clusters and infinitely many infinite blue clusters. Thus, to complete the proof it suffices to show that $p_1 > 0$ and $p_2 < 1$.

Let $m$ be the cardinality of the set of generators of $\Gamma$, i.e., the number of sides of the fundamental tile $T$, and define $Z_n$ to be the number of tiles $g(T)$ at *word* distance $\leq n$ from $T$ in the blue cluster containing $T$. Observe that for every tile $g(T)$ in this blue cluster that is at word distance $n$ from $T$, the number of *neighboring* blue tiles $g'(T)$ at word distance $n+1$ from $T$ is dominated by a Binomial $(m,p)$ random variable, since $g(T)$ has only $m$ neighbors. Consequently, by an easy construction, there exists (possibly on an enlarged probability space) a Galton-Watson process $Y_n$ with offspring distribution Binomial $(m,p)$ such that

$$Z_n \leq \sum_{j=0}^{n} Y_j.$$

If $mp < 1$, the Galton-Watson process is subcritical, and $EZ_n < (1-mp)^{-1}$ for all $n \geq 1$. This implies that the blue cluster containing $T$ is finite with $P_p$probability 1. A similar argument shows that if $m'(1-p) < 1$, where $m'$ is the degree of each vertex in the graph $G_{\text{red}}$ (i.e., the number of tiles $g(T)$ that intersect $T$ in at least one point) then the *red* cluster containing $T$ is finite with $P_p$−probability 1. □

## 7. Estimates on the Critical Probabilities

**7.1. Critical Probabilities for Sector Percolation.** Let $\Gamma$ be an arbitrary co-compact Fuchsian group. That blue percolation occurs in $\Gamma$ with positive $P_p$−probability for *some* value of $p$ follows from a general theorem of Benjamini and Schramm [1], who show in particular that it occurs whenever

$$p > \frac{1}{1 + \text{Cheeger}(\Gamma)},$$

where Cheeger($\Gamma$) is the *Cheeger constant* of $\Gamma$. For a co-compact Fuchsian group the Cheeger constant is always positive, but not easily computed. We shall give another estimate that is easily computed, albeit rather crude.

By a more general theorem of Selberg [10], the group $\Gamma$ contains a torsion-free subgroup $H$ of finite index. Such a subgroup $H$ must be a surface group $\Gamma_g$ for some $g \geq 2$, because it is itself co-compact and has neither elliptic nor parabolic elements.

**Proposition 13.** *If $\Gamma$ contains a surface group $\Gamma_g$ as a subgroup of finite index $\nu$, then blue sector percolation occurs in $\Gamma$ with positive $P_p$−probability for all*

$$p > \frac{1}{(2g-1)^{\frac{1}{\nu}}}. \tag{1}$$

*Proof.* Let $T^*$ be the Dirichlet fundamental polygon for $\Gamma_g$ with center at the origin (i.e., $T^*$ is the set of all points in the hyperbolic plane that are closer to the vertex 1 than to any other vertex $h \in \Gamma_g$). Then $T^*$ is the union of $\nu$ tiles $h_i(T)$, where $T$ is the Dirichlet polygon for $\Gamma$ centered at the origin and $h_1\Gamma_g, \dots, h_\nu\Gamma_g$ is an enumeration of the cosets of $\Gamma_g$ in $\Gamma$. Furthermore, *every* tile $h(T^*)$, where $h \in \Gamma_g$, is the union of exactly $\nu$ tiles $g_i(T)$, where $g_i \in \Gamma$. Under $P_p$, the probability that for a given tile $h(T^*)$ in the $\Gamma_g-$tessellation all $\nu$ of the constituent tiles $g_i(T)$ in the $\Gamma-$tessellation are colored blue is $p^\nu$. Consequently, by Theorem 2, if (1) holds then blue percolation will occur with positive probability. □

REMARK. The same argument shows that if $(1-p)^\nu > (2g-1)^{-1}$ then *red* sector percolation occurs with $P_p-$probability 1.

Clearly, an infinite blue cluster may intersect a tile $h(T^*)$ in the $\Gamma_g-$tessellation without all of its constituent subtiles $g_i(T)$ being colored blue, so the bound is crude. Better estimates can in some cases be obtained by a more careful consideration of the possible red-blue configurations inside a tile $h(T^*)$ – see Section 8 below for an example.

**7.2. Critical Probability for Percolation.** Let $\Gamma$ be a co-compact Fuchsian group, and consider the Cayley graph $G_{\text{blue}} = G_{\text{blue}}(\Gamma)$. For $n \geq 2$ let $N_n$ be the number of self-avoiding paths in $G_{\text{blue}}$ that start at 1 and return to 1 after $n$ steps. Define

$$r_* = \limsup_{n\to\infty} N_n^{\frac{1}{n}}.$$

**Proposition 14.** *If $pr_* < 1$ then with $P_p-$probability 1 red percolation occurs.*

NOTE. A similar result (Theorem 5.1.1) is obtained by a somewhat different method in [1]. The result stated here for Fuchsian groups may be sharper than that in [1], because $r_* \leq |\mathcal{G}|\rho(\Gamma)$, where $\mathcal{G}$ is the set of (side-pairing) generators of $\Gamma$ and $\rho(\Gamma)$ is the (inverse) spectral radius of the simple nearest-neighbor random walk on $(\Gamma, \mathcal{G})$.

*Proof of Proposition 14.* If red percolation occurs with $P_p-$probability 0 then, with probability 1, for every $n \geq 1$ there is a closed blue path $\beta_n$ surrounding the circle of (hyperbolic) radius $n$ centered at the origin. This path may be chosen to be self-avoiding (up to the last step). It has a point (vertex) $g_n$ of closest approach to the origin, and has (word) length $L_n$ satisfying

$$L_n \geq Ce^{\|g_n\|},$$

where $\|g\|$ is the hyperbolic distance from vertex $g$ to the vertex 1. (This is because the hyperbolic circle of radius $t$ has hyperbolic circumference $\sim e^t$.)

The number of self-avoiding closed paths of length $l$ whose hyperbolic distance to the origin is less than $c \log l$ is no larger than $O(l^c N_l)$, since the number of vertices at distance less than $c \log l$ to the origin is $O(l^c)$. For any such path, the $P_p-$probability that all its vertices are colored blue is $p^l$. Hence, the *expected* number of such blue paths of length $l \geq l_*$ is

$$O(\sum_{l=l_*}^{\infty} p^l N_l l^c).$$

If $pr_* < 1$ this sum is finite, and so the number of such closed blue paths is $P_p-$almost surely finite. Consequently, if $pr_* < \infty$ then red percolation occurs almost surely. □

There is an analogous result for blue percolation. Let $G_{\text{red}} = G_{\text{red}}(\Gamma)$ be the extended Cayley graph of $\Gamma$ (see Section 1), and for $n \geq 2$ let $N_n^*$ to be the number of self-avoiding paths in $G_{\text{red}}$ that start at vertex 1 and return to vertex 1 after $n$ steps. Define

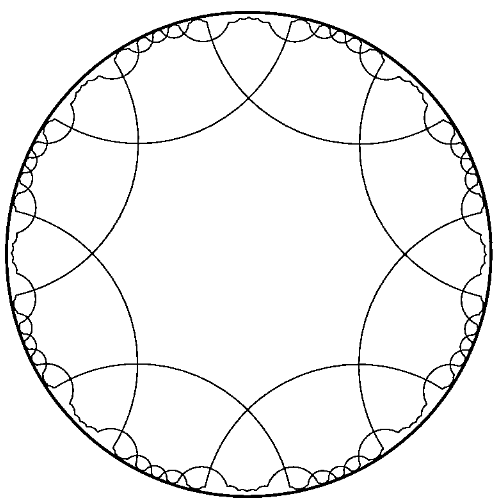
$$r_{**} = \limsup_{n \to \infty} (N_n^*)^{\frac{1}{n}}.$$

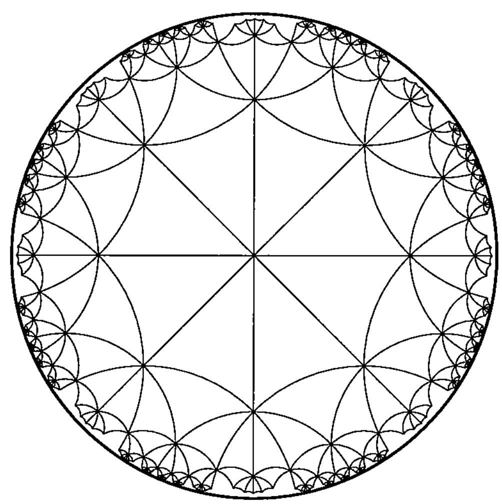
**Proposition 15.** *If $(1-p)r_{**} < 1$ then with $P_p$–probability 1, blue sector percolation occurs.*

## 8. The Triangle Groups $\Gamma(2, 4m, 4m; 0)$

The theorems of Dehn and Selberg can in some cases be used more efficiently than in Proposition 13. In this section we will use Dehn's theorem and a coset argument to show that theorems A-B extend to the triangle groups $\Gamma(2, 4m, 4m; 0)$ for $m \geq 5$.

The triangle group $\Gamma(2, 4m, 4m; 0)$ contains the surface group $\Gamma_m$ as a subgoup of index $4m$. Its action on the hyperbolic plane may be described as follows. Begin with a regular hyperbolic $4m$–gon $R$ with angles $\pi/2m$ at the corners and with center at the origin; this is a fundamental polygon for $\Gamma_m$. Partition $R$ into $4m$ congruent isosceles hyperbolic triangles $T_i, 1 \leq i \leq 4m$, by drawing geodesic segments from the origin 0 to the corners of $R$. Then $T_1$ is a fundamental polygon for $\Gamma(2, 4m, 4m; 0)$, and $\Gamma(2, 4m, 4m; 0)$ is generated by the hyperbolic rotations $\rho_1, \rho_2, \rho_3$ through angle $\pi/2m$ about the vertices $0, w_2, w_3$ of $T_1$, respectively. The tessellation $g(T_1)$, where $g \in \Gamma(2, 4m, 4m; 0)$, coincides with the tessellation $g(T_i)$, where $g \in \Gamma_m$ and $i = 1, 2, \cdots, 4m$; thus, the tiles $g(T_1)$, $g \in \Gamma(2, 4m, 4m; 0)$ are the triangles obtained by drawing the geodesic segments from the centers to the corners in all the $4m$–gons $g(R), g \in \Gamma_m$. The figure below shows the tessellation for the group $\Gamma(2, 8, 8; 0)$.





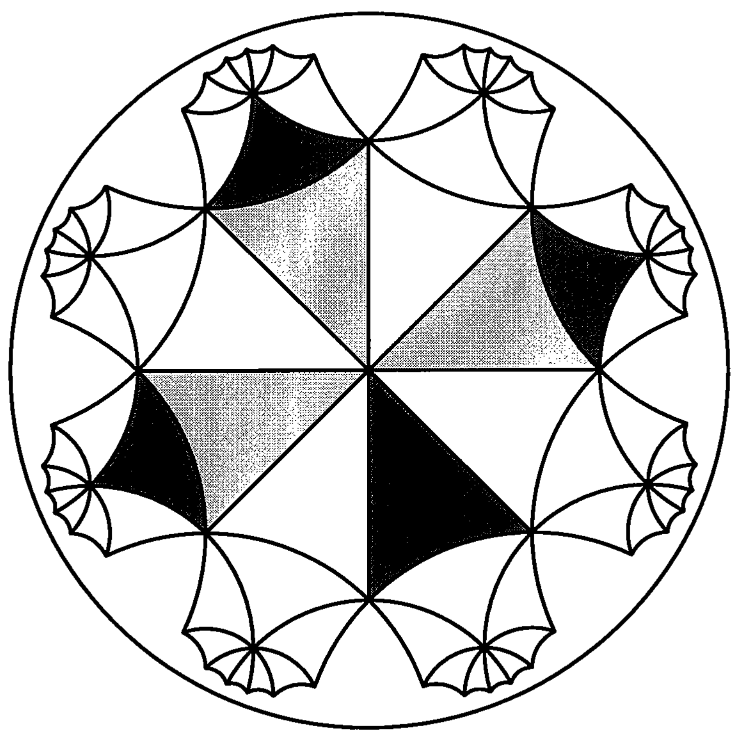
**Theorem 3.** *Let $\Gamma = \Gamma(2, 4m, 4m; 0)$. If $m \geq 5$ then there exist $0 < p_1 < p_2 < 1$ such that for all $p \in (p_1, p_2)$, red and blue sector percolation occur with $P_p$–probability 1 on $\Gamma$.*

*Proof.* Let $a_i^{\pm 1}, b_i^{\pm 1}$, $1 \leq i \leq m$, be the generators of the surface group $\Gamma_m$ contained as a subgroup in $\Gamma$, and let $\mathcal{F}_m$ be the subgroup generated by $a_i^{\pm 1}$, $1 \leq i \leq m$. By Corollary

9, $\mathcal{F}_m$ is a free group on the generators $a_i^{\pm 1}$, so its Cayley graph is a homogeneous tree of degree $2m$.

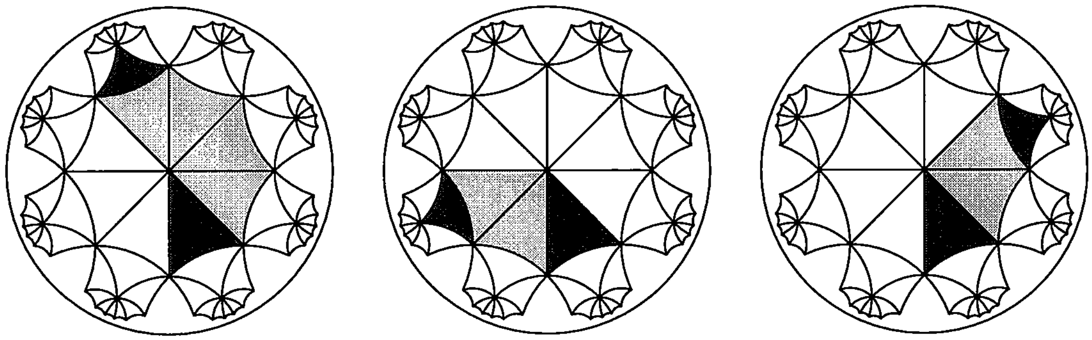
Consider tiles $f(T_1), f'(T_1)$ in the $\Gamma(2,4m,4m;0)$–tessellation such that $f'f^{-1}$ is one of the generators $a_i^{\pm 1}$. There is a path in $G_{\text{red}}$ of length 2 connecting $f(T_1)$ and $f'(T_1)$, with the connecting tile $g(T_1)$ lying in the $4m$–gon $f(R)$ ($g(T_1)$ intersects $f(T_1)$ in a point and $f'(T_1)$ in a geodesic segment). See the figure below for the case $m = 2$. Moreover, any two such paths connecting *distinct* pairs $f(T_1), f'(T_1)$ and $f''(T_1), f'''(T_1)$ overlap in at most one tile. Thus, if $Z_n$ is the number of tiles $f(T_1)$, $f \in \mathcal{F}_m$ connected to $T_1$ by a red path of length $2n$, then there is a Galton-Watson process $Y_n \leq Z_n$ with mean offspring number

$$\alpha_m(p) = (2m-1)p^2.$$



Similarly, there are paths in $G_{\text{blue}}$ of lengths $3, 5, 7, \ldots, 2m+1, 2m-1, \ldots, 5, 3$ connecting $f(T_1)$ to the $2m-1$ tiles $f'(T_1)$ such that each $f'f^{-1}$ is one of the generators $a_i^{\pm 1}$. See the figure below for the case $m = 2$. The intermediate tiles in these paths are all contained in the $4m$–gon $f(R)$. Thus, if $Z'_n$ is the number of tiles $f(T_1)$, with $f$ a reduced word of length $n$ in the generators $a_i^{\pm 1}$, such that $f(T_1)$ is connected to $T_1$ by a blue path of length $2n$, then there is a Galton-Watson process $Y'_n \leq Z'_n$ with mean offspring number

$$\beta_m(p) = 2(p^3 + p^5 + \cdots + p^{2m-1}) + p^{2m+1}.$$



A simple numerical calculation shows that for $m = 5$, $\alpha_m(p) > 1$ and $\beta_m(p) > 1$ for all $.6615 \leq p \leq .6665$. Since $\alpha_m(p)$ and $\beta_m(p)$ are clearly monotone in $m$, the same is true for all $m \geq 5$. Thus, both Galton-Watson processes $Y_n$ and $Y_n'$ are supercritical when $.6615 \leq p \leq .6665$. It follows that with positive $P_p$−probability there is red sector percolation and blue sector percolation. □

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