

ASYMPTOTIC ERROR DISTRIBUTIONS FOR THE
EULER METHOD FOR STOCHASTIC
STOCHASTIC DIFFERENTIAL EQUATIONS

by

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Technical Report #96-36

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August 1996

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August 12, 1996

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1 Introduction

The classical Itô-type stochastic differential equation is of the form

$$X_t = x_0 + \int_0^t a(X_s) dW_s + \int_0^t b(X_s) ds \quad (1.1)$$

with a, b matrices of functions and W a multi-dimensional Brownian motion. By replacing dW_t and dt with a vector of semimartingales dY_t we consider the more general equation

$$X_t = x_0 + \int_0^t f(X_{s-}) dY_s \quad (1.2)$$

where f denotes a matrix $f = (f^{ij})$ of functions. In applications one often wants to solve equation (1.2) numerically, when possible. Because of simulation difficulties, and because one often combines a numerical solution of (1.2) with a (slow) Monte Carlo technique, it is usually advisable to solve (1.2) numerically with an Euler scheme, rather than a more complicated, faster one. (See the survey paper of Talay [14] for a discussion of this issue.)

Without loss we will take the time interval to be $[0, 1]$ rather than $[0, T]$ for some (non-random) $T > 0$. We will assume $[0, 1]$ is partitioned by $\Pi^n = \{0 = t_0 < t_1 < \dots < t_n = 1\}$ with $t_i = i/n$, $0 \leq i \leq n$. Rates of convergence will thus be given relative to this partition scheme. For equation (1.1) if $a \equiv 0$ then the rate of convergence of the Euler scheme is classically $1/n$; if a does not vanish then it is also classical that the rate is $1/\sqrt{n}$. The distribution of the (normalized) asymptotic error, however, is not at all classical and was established only recently for equation (1.1) (see [8]).

In this paper we mainly aim to give a class of equations of type (1.2) that converge at the rate $1/\sqrt{n}$ and determine their asymptotic error, although we also examine some equations providing the rate $1/n$. To give a flavor of our results in a very simple setting, consider the 1-dimensional case (for Y and X as well) when Y is continuous and is either (1) nondecreasing, or (2) a local martingale. Denote by X^n the “continuous” Euler approximation for (1.2) and by \bar{X}^n the “discretized” one (see Section 3 for the definitions), so the error processes are respectively $U^n = X^n - X$ and $\bar{U}^n = \bar{X}^n - X$.

The first situation corresponds to a purely deterministic problem:

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Theorem 1.1. *If Y is a non-decreasing continuous function, there is equivalence between*

- a) *For $x_0 = 1$ and $f(x) = x$ (i.e. $X = e^Y$ in (1.2)), the sequence of numbers $nU_1^n = n\bar{U}_1^n$ is bounded.*
- b) *For all starting points x_0 and all C^1 functions f with at most linear growth, the functions nU^n and $n\bar{U}^n$ converge uniformly to a limit U .*
- c) *The function Y has the form*

$$Y_t = \int_0^t y_s ds, \quad \text{with} \quad \int_0^1 y_s^2 ds < \infty.$$

In this case, the limiting process U is the solution of the following linear equation:

$$U_t = \int_0^t f'(X_s)U_s y_s ds - \frac{1}{2} \int_0^t f(X_s) f'(X_s) y_s^2 ds. \quad (1.3)$$

This covers in particular the case of an ordinary differential equation of the form

$$dX_t = f(X_t) y_t dt,$$

where the coefficient f is C^1 and $s \rightsquigarrow y_s$ is a given nonnegative function: then the Euler approximation converges at the rate $1/n$ on the interval $[0, 1]$ *if and only if* we have $\int_0^1 y_s^2 ds < \infty$, which seems to be a new result.

In the second situation, we denote by C the quadratic variation process of Y . We then have the following:

Theorem 1.2. *If Y is a continuous local martingale, there is equivalence between*

- a) *For all $x_0 = 1$ and $f(x) = x$ (i.e. $X = \mathcal{E}(Y)$, the Doléans exponential of X), the sequence of random variables $\sup_t |\sqrt{n}U_t^n|$ is tight.*
- b) *For all starting points x_0 and all C^1 functions f with at most linear growth, the processes $(Y, \sqrt{n}U^n)$ and $(Y, \sqrt{n}\bar{U}^n)$ converge in law to a limit (Y, U) .*
- c) *The quadratic variation has the form*

$$C_t = \int_0^t c_s ds, \quad \text{with} \quad \int_0^1 c_s^2 ds < \infty.$$

In this case, the limiting process U is the solution of the following linear equation:

$$U_t = \int_0^t f'(X_s)U_s dY_s - \frac{1}{\sqrt{2}} \int_0^t f(X_s) f'(X_s) c_s dW_s, \quad (1.4)$$

where W is a standard Brownian motion, independent of Y .

Note that in (1.4) we have some “additional” randomness provided by the extra Brownian motion W : this is a typical feature of the limiting error process, when the driving term Y itself is random.

Surprisingly, the situation is very different when the driving term Y is discontinuous. Consider for example the case where Y is a 1-dimensional discontinuous Lévy process. Then two situations

occur: first, if there is no Brownian part, then $\sqrt{n}U^n$ and $\sqrt{n}\bar{U}^n$ converge in law to 0, which means that the rate is faster than $\frac{1}{\sqrt{n}}$ (but we do not know the correct rate, or even if there is a rate at all). Second, if there is a Brownian part in Y , then $\sqrt{n}\bar{U}^n$ converges to a limit U , but $\sqrt{n}U^n$ does *not* converge in the usual sense (i.e. for the Skorohod topology on the set of càdlàg functions). It does converge to U , however, for weaker topologies: the one induced by convergence in (Lebesgue) measure, which is known as the Meyer-Zheng topology [10], and also the new S-topology introduced by Jakubowski [6].

The paper is organized as follows: In Section 2 some preliminaries are given, and this section may be skipped at first reading (except for the definitions of the so-called stable convergence, and of the UCV property). Section 3 is devoted to general results (extending [8]) on rates of convergence. We have given in Section 4 some results in the case Y is of finite variation, because this is simpler than the general case while it shows already all the pathologies of this problem; this section may also be skipped, although it contains the proof of Theorem 1.1. Section 5 is devoted to continuous semimartingales, and this is the most useful part of this paper as far as applications are concerned, and it contains the proof of Theorem 1.2. Finally, the case of Lévy processes is considered in Section 6.

2 Preliminaries

In this paper we will mainly be dealing with weak convergence in the Skorohod topology: weak convergence for this topology is denoted by “ \Rightarrow ”. We need to give a review of and some complements to weak convergence.

First we recall some facts about *stable convergence*. Let X_n be a sequence of random variables with values in a Polish space E , all defined on the same probability space (Ω, \mathcal{F}, P) . We say that X_n *converges stably in law* to X , written “ $X_n \Rightarrow^{stably} X$ ”, if X is an E -valued random variable defined on an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ of the original space and if

$$\lim_n E(Uf(X_n)) = \tilde{E}(Uf(X)) \quad (2.1)$$

for every bounded continuous $f : E \rightarrow \mathbb{R}$ and all bounded measurable random

variables U . This convergence was introduced by Renyi [12] and studied by Aldous and Eagleson [1], see also [3]. It is obviously stronger than convergence in law.

If Y is another variable with values in another Polish space F we have the following equivalence:

Lemma 2.1. *If $X_n \Rightarrow^{stably} X$, then we have $(Y, X_n) \Rightarrow^{stably} (Y, X)$ for the product topology on $E \times F$.*

Conversely if (Y, X_n) weakly converges to a limit, we can realize this limit as (Y, X) with X defined on an extension of the space on which Y is defined, and $X_n \Rightarrow^{stably} X$ as soon as Y generates the σ -field \mathcal{F} .

Proof. The first claim is trivial. Conversely, assume that (Y, X_n) weakly converges to a limit (Y', X') . Call $Q(y, dx)$ a version of the regular conditional distribution of X' given Y' . Set $\tilde{\Omega} = \Omega \times E$, and $\tilde{\mathcal{F}} = \mathcal{F} \otimes \mathcal{E}$ where \mathcal{E} is the Borel σ -field of E , and $\tilde{P}(d\omega, dx) = P(d\omega)Q(Y(\omega), dx)$. We thus define an extension of the original space, with the “canonical” variable $X(\omega, x) = x$, and the pairs (Y, X) and (Y', X') clearly have the same law.

Observe that (2.1) holds for all $U = g(Y)$, where g is continuous and bounded on F , and what we need to prove is that it holds when g is measurable and bounded. But we then can find a sequence g_q of bounded continuous functions such that $g_q(Y) \rightarrow g(Y)$ in $\mathbb{L}^1(P)$, and the result readily follows. \square

Note that all this applies when X_n, X are \mathbb{R}^d -valued processes with càdlàg paths, as well as Y : we can then view them as random variables with values in the Skorohod space \mathcal{D} . However, in this situation we should be careful: the stable convergence of X_n implies the weak convergence of the pair (Y, X_n) for the product topology on $\mathcal{D}(\mathbb{R}^q) \times \mathcal{D}(\mathbb{R}^d)$, which is *not* the Skorohod topology on $\mathcal{D}(\mathbb{R}^{q+d})$, and we do not have in general weak convergence of (Y, X^n) in the usual sense.

Next, we prove a result on weak convergence and discretization which might be well known, but we could not find it in the literature. First, a standard result asserts that if x is a function belonging to \mathcal{D} and if η_n is a sequence of increasing piecewise constant functions from $[0, 1]$ into $[0, 1]$ which converges to the identity, then the sequence of “discretized” functions $x \circ \eta_n$ converges to x . More generally, we have:

Lemma 2.2. *If a sequence X^n of (possibly multidimensional) processes weakly converges to X , then the processes $X_{\eta_n(n)}^n$ weakly converge to the same limit.*

Proof. By the Skorohod representation Theorem, we can replace weak convergence by a.s. convergence, so that we only need to prove that if $x_n \rightarrow x$ in \mathcal{D} , then the sequence $y_n = x_n \circ \eta_n$ also converges to x . There are time-changes λ_n converging to the identity, and such that $x_n - x \circ \lambda_n$ goes uniformly to 0. Then $y_n - x \circ \lambda'_n$ also goes uniformly to 0, where $\lambda'_n = \lambda_n \circ \eta_n$. Now we have seen above that $x \circ \lambda'_n \rightarrow x$ in \mathcal{D} , and we are finished. \square

Next, we recall some facts about convergence of stochastic integrals, coming from the work of [5], [7]. See [9] for an expository account. First recall that for every $\delta > 0$, any semimartingale can be written as

$$X_t = X_0 + A(\delta)_t + M(\delta)_t + \sum_{s \leq t} \Delta X_s 1_{\{|\Delta X_s| > \delta\}}, \quad (2.2)$$

where $A(\delta)$ is a predictable process with finite variation, null at 0, $M(\delta)$ is a local martingale null at 0, and ΔX_s denotes the jump size of X at time s . As usual $\langle M, M \rangle$ denotes the predictable bracket of two local martingales M and N , if it exists. All these notions are relative to some filtration (\mathcal{F}_t) on our probability space. We also write, for any (possibly multidimensional) process V :

$$V^* = \sup_{t \in [0, 1]} \|V_t\|. \quad (2.3)$$

Definition. Let $X^n = (X^{n,i})_{1 \leq i \leq d}$ be a sequence of \mathbb{R}^d -valued semimartingales, with $A(\delta)^{n,i}$ and $M(\delta)^{n,i}$ associated with $X^{n,i}$ as in (2.2). We say that the sequence (X^n) has *uniformly controlled variation* (UCV) if for some $\delta > 0$ and for each i the sequence

$$\langle M(\delta)^{n,i}, M(\delta)^{n,i} \rangle_1 + \int_0^1 |dA(\delta Z)^{n,i}| + \sum_{0 < s \leq 1} |\Delta X_s^{n,i}| 1_{\{|\Delta X_s^{n,i}| > \delta\}}$$

is tight. This notion does not depend on the particular choice of $\delta > 0$ (recall that our time interval here is $[0, 1]$; it is important to emphasize that this notion does depend on the underlying filtrations (\mathcal{F}_t^n)). \square

It turns out that this property is equivalent to the notion of *uniform tightness* (UT) as introduced by Jakubowski, Mémin and Pagès [5], and its usefulness derives from the following fundamental set of results (see for example [9]). Below, we denote by $H \cdot X$ the stochastic integral process of H w.r.t. X , and it is understood that these two processes have matching dimensions.

Theorem 2.3. Let X^n and Y^n be two sequences of \mathbb{R}^d -valued semimartingales, relative to the filtrations (\mathcal{F}_t^n) .

- a) If both sequences X^n and Y^n have UCV, then so has the sequence $X^n + Y^n$.
- b) If each X^n is of finite variation and if the sequence $\int_0^1 |dX_s^n|$ is tight, then the sequence X^n has UCV.
- c) Let H^n be a sequence of (\mathcal{F}_t^n) -predictable processes such that the sequence H^{n*} is tight. If the sequence X^n has UCV, so has the sequence $H^n \cdot X^n$.
- d) Let H^n and H'^n be two sequences of (\mathcal{F}_t^n) -predictable processes such that the sequence H^{n*} is tight and that $(H^n - H'^n)^* \xrightarrow{P} 0$. If the sequence X^n has UCV, then $(H^n \cdot X^n - H'^n \cdot X^n)^* \xrightarrow{P} 0$.
- e) Suppose that X^n weakly converges. Then UCV is necessary and sufficient for the following property (called “goodness”):

For any sequence H^n of (\mathcal{F}_t^n) -adapted, right continuous and left-hand limited, processes such that the sequence (H^n, X^n) weakly converges to a limit (H, X) , then X is a semimartingale w.r.t. the filtration generated by the process (H, X) , and we have $(H^n, X^n, H_-^n \cdot X^n) \Rightarrow (H, X, H_- \cdot X)$.

We finally turn our attention to stochastic differential equations. General results are available (see e.g. [9] or [13]), but we confine ourselves to linear equations of the type:

$$X_t = J_t + \int_0^t X_{s-} H_s dY_s, \quad (2.4)$$

where Y is a given semimartingale, and J is an adapted càdlàg process, and H is a predictable process. All these terms can be multidimensional, with matching dimensions.

Let us begin with a comparison lemma, where X' is the solution of another equation (2.4) associated with J' , H' , and with *the same* semimartingale Y .

Lemma 2.4. For all $\varepsilon > 0$ $A > 0$ there is a constant K

depending on ε , A and on the semimartingale Y , such that for all $\eta > 0$, $u > 0$, $v \in (0, A]$, $w \in (0, u]$ we have

$$P((X - X')^* > \eta) \leq \varepsilon + P(H^* > A) + P(J^* > u) + P((H - H')^* > v) + P((J - J')^* > w) + \frac{uv + w}{\eta} K. \quad (2.5)$$

Proof. Let us introduce first a notation: if Z is a càdlàg process and T a stopping time, we write Z^{T-} for the process $Z_t^{T-} = Z_t 1_{[0, T)}(t) + Z_{T-} 1_{[T, 1]}(t)$.

We will use the “slicing technique” of Doléans-Dade (see [11]), which says three things: first, for any semimartingale Y and any $\alpha > 0$, $\varepsilon > 0$ there is a stopping time T such that the semimartingale Y^{T-} is α -sliceable and that $P(T \leq 1) \leq \varepsilon$. Second, if Y is α -sliceable for some α then $E(\sup_t |\int_0^t H_s dY_s|) \leq K_Y E(H^*)$ where K_Y only depends on Y . Third if Y is α -sliceable for some α and if we consider Equation (2.4) with $|H| \leq A$, then $E(X^*) \leq K_{A,Y} E(J^*)$ for a constant $K_{A,Y}$ depending on A, Y , provided $\alpha \leq C_A$ for some $C_A > 0$ depending on A only.

Now we fix $A > 0$ and $\varepsilon > 0$, and we take $\alpha = C_A$. Then we choose a stopping time T such that $P(T \leq 1) \leq \varepsilon$ and that $\bar{Y} = Y^{T-}$ is α -sliceable. Then we set $S = \inf(t : |H_t| > A \text{ or } |J_t| > u \text{ or } |H_t - H'_t| > v \text{ or } |J_t - J'_t| > w) \wedge T$ and $\bar{J} = J^{S-}$, $\bar{J}' = J'^{S-}$, and define the i^{th} component of \bar{H} as $\bar{H}^i = \bar{H}^i \wedge A \vee -A$, and similarly for \bar{H}' . These last two processes are predictable, and we can consider the solutions \bar{X} and \bar{X}' of (2.4), associated with $(\bar{J}, \bar{H}, \bar{Y})$ and

$(\bar{J}', \bar{H}', \bar{Y})$ respectively. Note that

$$X = \bar{X}, \quad X' = \bar{X}' \quad \text{on the set } \{S > 1\}. \quad (2.6)$$

Note also that $\bar{X}'' = \bar{X}' - \bar{X}$ is the solution of (2.4) associated with $(\bar{J}'', \bar{H}, \bar{Y})$, where $\bar{J}_t'' = \bar{J}_t' - \bar{J}_t + \int_0^t (\bar{H}_s' - \bar{H}_s) \bar{X}_{s-}' d\bar{Y}_s$. Using the properties of sliceable semimartingales recalled above, we get if $v \leq A$ and $w \leq u$:

$$E(\bar{X}''^*) \leq KE(\bar{J}''^*), \quad E(\bar{J}''^*) \leq w + KvE(\bar{X}'^*), \quad E(\bar{X}'^*) \leq (u+w)K \leq (u+A)K$$

where K only depends on A and \bar{Y} , so indeed on A and ε and Y . (2.5) readily follows from these estimates and from (2.6), once observed that $P(S \leq 1)$ is smaller than the sum of the first four terms on the right of (2.5). \square

Now we consider a sequence of SDE's like (2.4):

$$X_t^n = J_t^n + \int_0^t X_{s-}^n H_s^n dY_s, \quad (2.7)$$

all defined on the same filtered probability space and with the same dimensions. Let also ρ_n be an auxiliary sequence of random variables with values in some Polish space E , all defined on the same space again.

Theorem 2.5. a) *Tightness of both sequences J^{n*} and H^{n*} implies tightness of the sequence X^{n*} .*

b) *Suppose that we have another equation (2.7) with solution X'^n and coefficients J'^n and H'^n . If the sequences J^{n*} and H^{n*} are tight and if $(J^n - J'^n)^* \rightarrow^P 0$ and $(H^n - H'^n)^* \rightarrow^P 0$, then*

c) *Let $V_t^n = \int_0^t H_s^n dY_s$. Suppose that the sequence H^{n*} is tight and that the sequence (J^n, V^n, ρ^n) stably converges to a limit (J, V, ρ) defined on some extension of the space. Then V is a semimartingale on the extension (w.r.t. the filtration generated by the pair (J, V)), and $(J^n, V^n, X^n, \rho^n) \Rightarrow^{\text{stably}} (J, V, X, \rho)$, where X is the unique solution of*

$$X_t = J_t + \int_0^t X_{s-} dV_s. \quad (2.8)$$

The statement (a) above seems to be new, while (b,c) are variations on the so-called ‘‘stability’’ results for SDE's ((c) is recalled from [13], while (b) has a slightly new formulation). We have stated this theorem in a simple form, which is enough for our purposes, but instead of linear equations we could have more general (Lipschitz-continuous) coefficients. Also Y might be replaced by a sequence Y^n : in this case it is necessary to add the assumption that the sequence Y^n has UCV, which implies that in fact it is ‘‘uniformly’’ sliceable in some sense.

Proof. a) (2.5) applied with $J' = 0$ and $H' = 0$ yields

$$P(X^{n*} > \eta) \leq \varepsilon + 2P(H^{n*} \geq A) + 2P(J^{n*} \geq u) + \frac{u}{\eta} K_{\varepsilon, A}, \quad (2.9)$$

where $K_{\varepsilon, A}$ is a constant depending on ε , A and Y . If we choose first ε arbitrarily, then A, u big, then η big, we obtain that the left side of (2.8) is smaller than 2ε , hence (a) holds.

b) Similarly,

$$\begin{aligned} P((X^n - X'^n)^* > \eta) &\leq \varepsilon + P(H^{n*} > A) + P(J^{n*} > u) \\ &+ P((H^n - H'^n)^* > v) + P((J^n - J'^n)^* > w) + \frac{uv+w}{\eta} K_{\varepsilon, A}. \end{aligned}$$

So we obtain the result by choosing first ε, η arbitrarily, then A, u big, then v, w small, then n big.

c) The assumptions ensure that the sequence V^n has UCV. Thus if we do not introduce the variables ρ^n and if we replace stable convergence by ordinary (weak) convergence, this result is well known (see e.g. [11]). Since stable convergence is just weak convergence of (U, J^n, V^n, ρ^n) to (J, V, ρ) for any random variable U on the original probability space, our statement is proved. \square

3 The Fundamental Result on the Error Distribution

We let $Y = (Y^i)_{1 \leq i \leq d}$ be a semimartingale on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. We always assume that $Y_0 = 0$ (this is of course not a restriction here). The time interval is $[0, 1]$. We consider the q -dimensional *SDE*:

$$dX_t = f(X_{t-})dY_t, \quad X_0 = x_0. \quad (3.1)$$

Here $x_0 \in \mathbb{R}^q$ and f is a continuously differentiable function from \mathbb{R}^q into $\mathbb{R}^q \otimes \mathbb{R}^d$ with linear growth (i.e. $\|f(x)\| \leq K(1 + \|x\|)$ some constant K).

One knows that (3.1) has a unique (strong) solution. We consider the *Euler continuous approximation* X^n given by

$$dX_t^n = f(X_{\varphi_n(t)}^n)dY_t, \quad X_0^n = x_0, \quad (3.2)$$

where $\varphi_n(t) = [nt]/n$ if $nt \notin \mathbb{N}$ and $\varphi_n(t) = t - 1/n$ if $nt \in \mathbb{N}$, and the *Euler discontinuous approximation* \bar{X}^n given by

$$\bar{X}_t^n = X_{[nt]/n}^n. \quad (3.3)$$

The corresponding error processes are denoted by

$$U_t^n = X_t^n - X_t, \quad \bar{U}_t^n = \bar{X}_t^n - X_{[nt]/n} = U_{[nt]/n}^n. \quad (3.4)$$

Theorem 3.1. *If f is locally Lipschitz continuous and with linear growth, then U^{n*} and \bar{U}^{n*} tend to 0 in probability.*

Proof. The second statement follows clearly from the first one. The first claim is proved in [8] (Theorem 3.3) if f is bounded. In the general case we consider functions $h_m \in C_b^1(\mathbb{R}^q)$ with $1_{\{\|x\| \leq m\}} \leq h_m(x) \leq 1_{\{\|x\| \leq m+1\}}$, and set $f_m(x) = f(x)h_m(x)$. Let $X(m)$ be the solution of Equation (3.1) with the coefficient f_m and $X^n(m)$ be the corresponding Euler approximations.

Observe that $X = X(m)$ is $X^* < m$ and that $X^n = X^n(m)$ if $X^n(m)^* < m$. Hence $U^n = X^n(m) - X(m)$ on the set $\{X^* < m - 1, (X^n(m) - X(m))^* \leq 1\}$, and thus for $\varepsilon \in (0, 1]$ we get

$$P(U^{n*} > \varepsilon) \leq P(X^* \geq m - 1) + P((X^n(m) - X(m))^* > \varepsilon).$$

Since $X^n(m) \rightarrow X(m)$ in probability uniformly on $[0, 1]$ as $n \rightarrow \infty$ for each $m \geq 1$ and since $\lim_n P(X^* \geq m - 1) = 0$, the result follows. \square

Next, let us examine rates of convergence. By this, we mean a sequence α_n of constants going to $+\infty$, such that the processes $\alpha_n U^n$ or $\alpha_n \bar{U}^n$ are tight, with non-trivial limiting processes. If this is the case, we also are interested in the “error processes” which are the limits of either one of these two sequences. Indeed, as far as applications are concerned, the usual Euler scheme gives us \bar{X}^n and thus we would prefer to have results on \bar{U}^n , but mathematically speaking the processes U^n are easier to handle.

Here we give an improvement on a result by Kurtz-Protter, who essentially proved the implication (a) \Rightarrow (b) below. For this, we need to introduce some notation. For any process V we

write

$$\Delta_t^n V = V_{i/n} - V_{(i-1)/n}, \quad V_t^{(n)} = V_t - V_{[nt]/n}. \quad (3.5)$$

For any two semimartingales U, V , we write

$$Z_t^n(U, V) = \int_0^t U_{s-}^{(n)} dV_s. \quad (3.6)$$

Theorem 3.2. *Let $Z^n = (Z^{n,ij} := Z^n(Y^i, Y^j))_{1 \leq i, j \leq d}$, and let (α_n) be a deterministic sequence of positive numbers. There is equivalence between:*

- a) *The sequence $\alpha_n Z^n$ has UCV and $(Y, \alpha_n Z^n) \Rightarrow (Y, Z)$.*
- b) *For any starting point x_0 and any C^1 function f with linear growth, the sequence $\alpha_n U^n$ has UCV and $(Y, \alpha_n U^n) \Rightarrow (Y, U)$.*

In this case, we can realize the limits Z and U above on the same extension of the space on which Y is defined, and they are connected by

$$dU_t^i = \sum_{j=1}^d \sum_{k=1}^q f_k^{i,j}(X_{t-}) \left[U_{t-}^k dY_t^j - \sum_{\ell=1}^d f^{k\ell}(X_{t-}) dZ_t^{\ell j} \right], \quad U_0^i = 0, \quad (3.7)$$

($f_k^{i,j}$ is the k^{th} partial derivative of f^{ij}), and $(Y, Z^n, U^n) \Rightarrow (Y, Z, U)$ and also the sequence \bar{U}^n stably converges in law to U .

Remark 3.1. In view of Lemma 2.1, in (a) and (b) above we also have stable convergence in law of $\alpha_n Z^n$ and $\alpha_n U^n$, but these stable convergences are not enough to imply the convergences in (a) and (b).

Note also that we do *not* have $(Y, \bar{U}^n) \Rightarrow (Y, U)$ (except when Y or Z are continuous) in the last claim. \square

Remark 3.2. We will see later than when Y is continuous, then the assumption (a) is satisfied under mild hypotheses on Y . It is also satisfied when Y has jumps *and* each jump time is contained in an interval of constancy of Y , provided the “continuous part” of Y satisfies again the mild assumptions referred to above. In all other cases, we conjecture that indeed either the limits in this theorem are all 0 (i.e., the rate α_n is not the correct rate), or the sequence $\alpha_n Z^n$ is not even tight for the Skorohod topology: This conjecture is supported by the results of Section 4-2 and 6 below. \square

Remark 3.3. As we shall see in the proof, (a) is in fact equivalent to the property (b) stated for a single (judiciously chosen) equation: Namely, let $\lambda \in \mathbb{R} \setminus \{0\}$ be such that

$$P(\Delta Y_t^i = -\lambda \text{ for some } i \leq d \text{ and } t \in (0, 1]) = 0. \quad (3.8)$$

Then it is enough to have (b) for the d^2 -dimensional equation

$$X_t^{ij} = \delta_{ij} + (1 - \delta_{ij})Y_t^j + \lambda \int_0^t X_{s-}^{ij} dY_s^i, \quad 1 \leq i, j \leq d. \quad \square \quad (3.9)$$

Proof. 1) The implication (a) \Rightarrow (b), as well as (3.7) is proved in [8] in the case when ∇f is bounded. The fact that $\alpha_n U^n$ satisfies UCV immediately follows from Equation (3.21) in [8] and from the fact that $\alpha_n Z^n$ satisfies UCV.

When f is C^1 with linear growth, let f_m and $X(m)$ be defined as in Theorem 1.1, and $U^n(m)$ be associated with $X(m)$ by (3.4). Then ∇f_m is bounded. We know that $(Y, \alpha_n Z^n, \alpha_n U^n(m))$ weakly converges to $(Y, Z, U(m))$ for all m where $U(m)$ is the solution of (3.7) written for f_m and $X(m)$.

Denote by U the solution of (3.7) for f and X . As in Theorem 3.1 we have $U = U(m)$ and $U^n(m) = U^n$ on the set $\{X^* < m - 1, U^n(m)^* \leq 1\}$, while $U^n(m)^* \xrightarrow{P} 0$ as $n \rightarrow \infty$, hence

$$\lim_m \limsup_n P(U \neq U^m \text{ or } U^n \neq U^n(m)) = 0.$$

Then $(Y, \alpha_n Z^n, \alpha_n U^n(m)) \Rightarrow (Y, Z, U(m))$ for all m readily implies that $(Y, \alpha_n Z^n, \alpha_n U^n) \Rightarrow (Y, Z, U)$, and we have (b) and (3.7). The last claim is obvious by Lemma 2.1, since from what precedes we have stable convergence of U^n to U .

2) Suppose now that we have (b) for Equation (3.9). We have $U^n = (U^{n,ij})_{1 \leq i, j \leq d}$ and $X^n = (X^{n,ij})_{1 \leq i, j \leq d}$. A simple calculation shows that

$$U_t^{n,ij} = \begin{cases} \int_0^t [\lambda U_{s-}^{n,ii} dY_s^i - \lambda^2 X_{\varphi_n(s)}^{n,ii} dZ_s^{n,ii}], & \text{if } j = i, \\ \int_0^t [\lambda U_{s-}^{n,ij} dY_s^i - \lambda^2 X_{\varphi_n(s)}^{n,ij} dZ_s^{n,ii}] - \lambda Z_t^{n,ji}, & \text{if } j \neq i. \end{cases} \quad (3.10)$$

Observe that for each i , $X^{ii} = \mathcal{E}(\lambda Y^i)$ (the Doléans-Dade, or stochastic exponential). Thus (3.8) implies that a.s. X^{ii} does not vanish, and $X^{n,ii}$ does not vanish either on $[0, 1]$ for n large enough (because of Theorem 3.1). Hence (3.10) can be “inverted” to yield

$$Z_t^{n,ji} = \begin{cases} \int_0^t [(U_{s-}^{n,ii} / \lambda X_{\varphi_n(s)}^{n,ii}) dY_s^i - (1/\lambda^2 X_{\varphi_n(s)}^{n,ii}) dU_s^{n,ii}], & \text{if } i = j, \\ \int_0^t [U_{s-}^{n,ij} dY_s^i - \lambda X_{\varphi_n(s)}^{n,ij} dZ_s^{n,ii}] - U_t^{n,ij} / \lambda, & \text{if } i \neq j. \end{cases} \quad (3.11)$$

One deduces from the hypothesis that $(Y, \alpha_n U^n, X_{[n \cdot]/n}^n) \Rightarrow (Y, U, X)$, and the pair $(Y, \alpha_n U^n)$ has UCV, so (a) readily follows from Theorems 2.3 and 3.1. \square

In view of Remark 3.2 above, the following result has some interest: although not providing the limit of the error process, it actually gives the convergence rate (recall that if a sequence V^n has UCV, then a fortiori the sequence V^{n*} is tight).

Theorem 3.3. *With the notation of Theorem 3.2, there is equivalence between:*

- a) *The sequence $\alpha_n Z^n$ has UCV.*
- b) *For any starting point x_0 and any C^1 function f with linear growth, the sequence $\alpha_n U^n$ has UCV.*

Proof. Assume first (b) and consider Equation (3.9) for a λ having (3.8), so (3.11) holds. We have seen that $1/X_{\varphi_n(s)}^{n,ii}$ goes to $1/X_s^{ii}$ uniformly in s , in probability, while X^{ii} does not vanish, hence the sequence $(1/X_{\varphi_n(\cdot)}^{n,ii})^*$ is tight. Then UCV for (U^n) yields that $(U^{n,ii}/X_{\varphi_n(\cdot)}^{n,ii})^*$ also is tight, and Theorem 2.3(a) applied to the first part of (3.11) yields that $(Z^{n,ii})$ has UCV. Then apply the same result to the second part of (3.11) to get that $(Z^{n,ij})$ has UCV for $i \neq j$, and (a) holds.

For the converse, we need to introduce the equation satisfied by U^n , which is (with matrix notation):

$$dU_t^n = (f(X_t^n) - f(X_t))dY_t - (f(X_{t-}^n) - f(X_{\varphi_n(t)}^n))dY_t.$$

Now, with any càdlàg process V we set

$$k(V)_t^n = f(V_{\varphi_n(t)}) \int_0^1 \nabla f(V_{\varphi_n(t)} + u(V_{t-} - V_{\varphi_n(t)})) du, \quad (3.12)$$

which is left-continuous. Apply Taylor expansion and the fact that $X_{t-}^n - X_{\varphi_n(t)}^n = f(X_{\varphi_n(t)}^n)(Y_{t-} - Y_{\varphi_n(t)})$ to get the following, where \tilde{X}_{t-}^n is in between X_{t-} and X_{t-}^n :

$$d(\alpha_n U^n)_t = (\alpha_n U^n)_{t-} \nabla f(\tilde{X}_{t-}^n) dY_t - k(X^n)_t^n d(\alpha_n Z^n)_t. \quad (3.13)$$

The sequence $k(X^n)^{n*}$ is tight by Theorem 3.1, hence the sequence $\int_0^\cdot k(X^n)_t^n d(\alpha_n Z^n)_t$ has UCV as soon as (a) holds by Theorem 2.3. Since the sequence $\nabla f(\tilde{X}^n)^*$ is also tight, Theorem 2.5 gives the tightness of the sequence $(\alpha_n U^{n,*})$, and another application of Theorem 2.3 yields (b). \square

4 Processes with finite variation

We treat here the case where the driving process Y in (3.1) is of finite variation, with the rate $1/n$ in view. In this case (3.1) is truly an “ ω -wise” (or deterministic) equation, and the reason for looking at this case is not practical importance but rather methodological implications. When Y is continuous, we find a necessary and sufficient condition for getting this rate $1/n$, and this seems to be new even although it concerns only “ordinary differential equations”.

However, when Y has jumps together with a non-trivial continuous part, the picture changes radically: the rate is still $1/n$, in the sense that nU^{n*} and $n\bar{U}^{n*}$ remain tight, but in the deterministic case these processes have no limit. In the random case, rather mild conditions imply the convergence of these processes to a limit involving “additional randomness”.

For simplicity we only consider the 1-dimensional case. Extensions to several dimensions are straightforward and left to the reader.

1) The continuous case. Here we assume that Y is of finite variation and continuous. Remember that $Y_0 = 0$. We write $Z^n = Z^n(Y, Y)$ (see (3.6)). An integration by parts shows that

$$Z_t^n = \frac{1}{2} \left(\sum_{i=1}^{[nt]} (\Delta_i^n Y)^2 + (A_t^n)^2 \right) \geq 0. \quad (4.1)$$

Theorem 4.1. *Assume that Y is continuous with finite variation. There is equivalence between:*

a) *We have*

$$Y_t = \int_0^t y_s ds, \quad \int_0^1 y_s^2 ds < \infty \quad \text{a.s.} \quad (4.2)$$

b) *The sequence of random variables $(2^n Z_1^{2^n})_{n \geq 1}$ is tight.*

c) *The sequence of random variables $(n Z_1^n)_{n \geq 1}$ is tight.*

d) *$\sup_n Z_1^n < \infty$ a.s.*

e) *The processes nZ^n converge a.s. uniformly in time to a process Z .*

Moreover in this case we have $Z_t = \frac{1}{2} \int_0^t y_s^2 ds$ and $\sup_n \int_0^1 |dZ_s^n| < \infty$ a.s.

Note that (e) and the last claim imply Condition (a) of Theorem 3.2, with $\alpha_n = n$, while the latter implies (c): hence the above result completely solves the question of whether the rate for U_n is $1/n$ or not, for processes Y as above. Note also that all statements above are “ ω -wise” (a.s.), that is this result is deterministic in nature, and indeed we begin with two lemmas which are concerned with the deterministic case.

Lemma 4.2. *Assume that Y is continuous with finite variation and deterministic. Only two cases are possible:*

- i) (4.2) holds and $\sup_n nZ_1^n < \infty$.
- ii) (4.2) does not hold and $2^n Z_1^{2^n} \rightarrow \infty$.

Proof. Consider the following measures on $(0, 1]$: $\lambda(dt) = dt$, $\mu(dt) = dY_t$, $\mu'(dt) = |dY_t|$, and set $\rho = 1 + \mu'((0, 1])$. Then $\nu = \frac{1}{\rho}(\mu' + \lambda)$ is a probability measure, and we introduce the Radon-Nikodym derivatives:

$$V = \frac{d\lambda}{d\nu}, \quad U = \frac{d\mu}{d\nu},$$

which satisfy $\{V = 0\} \subset \{|U| = \rho\}$. With the convention $a/0 = +\infty$ (resp. $-\infty$) if $a > 0$ (resp. $a < 0$), we also set $L = U/V$.

Let \mathcal{G}_n be the σ -field of $(0, 1]$ generated by the intervals $(\frac{i}{n}, \frac{i+1}{n}]$, and set

$$V_n = \frac{d\lambda}{d\nu|_{\mathcal{G}_n}}, \quad U_n = \frac{d\mu}{d\nu|_{\mathcal{G}_n}}, \quad L_n = \frac{U_n}{V_n}.$$

These are finite-valued functions, with $V_n = \nu(V|\mathcal{G}_n)$ and $U_n = \nu(U|\mathcal{G}_n)$. A simple computation yields $L_n(s) = n\Delta_i^n Y$ for $s \in (\frac{i-1}{n}, \frac{i}{n}]$, so by (4.1):

$$nZ_1^n = \int L_n^2 d\lambda = \int L_n^2 V_n d\mu'. \quad (4.3)$$

The sequences U_{2^n} and V_{2^n} are uniformly integrable ν -martingales w.r.t. the filtration (\mathcal{G}_{2^n}) , converging ν -a.s. to U and V , hence L_{2^n} ν -a.s. converges to L (because $\{V = 0\} \subset \{|U| = \rho > 0\}$). On the set $\{V > 0\}$ we have $L_{2^n}^2 V_{2^n} \rightarrow L^2 V$ ν -a.s., while on the complement we have $L_{2^n}^2 V_{2^n} = L_{2^n}^2 U_{2^n} \rightarrow +\infty$ ν -a.s. Then (4.3) and Fatou's lemma yield

$$\liminf_n 2^n Z_1^{2^n} \geq \int L^2 d\lambda + \infty \mu'(V = 0). \quad (4.4)$$

If $\liminf_n 2^n Z_1^{2^n} < \infty$, we deduce $\mu'(V = 0) = 0$ from (4.4), hence the first half of (4.2) holds true with $y = L$; another application of (4.4) yields the second half of (4.2). Conversely, assume (4.2): we have $L = y$, which belongs to $L^2(d\lambda)$, and $L_n = \lambda(L|\mathcal{G}_n)$, hence $\sup_n nZ_1^n = \sup_n \int L_n^2 d\lambda < \infty$. This finishes to prove the lemma. \square

Lemma 4.3. *Assume that Y is deterministic and satisfies (4.2) with y being piecewise constant. Then nZ_t^n converges uniformly in t to $Z_t = \frac{1}{2} \int_0^t y_s^2 ds$.*

Proof. We have $y_s = u_i$ for $t_{i-1} < s < t_i$, where $0 = t_0 < \dots < t_k = 1$. Let $C = \sup_i |u_i|$, and $\tau_n(s) = s - [ns]/n$. We have $|Y_s^n| \leq C/n$, and also $Y_s^n = u_i \tau_n(s)$ if $t_{i-1} < [ns]/n \leq s < t_i$. Hence

$$nZ_t^n = n \sum_{i=1}^k u_i^2 \int_{t_{i-1} \wedge t}^{t_i \wedge t} \tau_n(s) ds + R_n(t),$$

where $|R_n(t)| \leq 2knC^2/n^2 \rightarrow 0$. A simple calculation shows that $\int_u^v n\tau_n(s)ds \rightarrow \frac{v-u}{2}$, hence

$$Z_t^{1n} := n \sum_{i=1}^k u_i^2 \int_{t_{i-1} \wedge t}^{t_i \wedge t} \tau_n(s)ds \rightarrow \frac{1}{2} \sum_{i=1}^k u_i^2 (t_i \wedge t - t_{i-1} \wedge t) = Z_t.$$

Since Z^{1n} and Z are continuous non-decreasing, this convergence is uniform in t over $[0, 1]$. \square

Proof of Theorem 4.1. That (e) \Rightarrow (d) \Rightarrow (c) \Rightarrow (b) is obvious. Let B be the set of all ω such that the function $t \rightarrow Y_t(\omega)$ does not satisfy (4.2). By Lemma 4.2, $2^n Z_1^{2^n}(\omega) \rightarrow \infty$ if $\omega \in B$, then (b) implies $P(B) = 0$ and (a).

It remains to prove that (a) implies (e) and the last claim, so below we assume (4.2). A density argument shows that there is a sequence $y(p)$ of bounded processes, piecewise constant in time, such that

$$\int_0^1 y(p)_s^2 ds \leq \int_0^1 y_s^2 ds, \quad \eta_p := \int_0^1 |y_s - y(p)_s|^2 ds \rightarrow 0 \quad \text{a.s.} \quad (4.5)$$

Let $Y(p)_t = \int_0^t y(p)_s ds$, with the associated processes $Z^n(p) = Z^n(Y(p), Y(p))$, and set $Z_t = \frac{1}{2} \int_0^t y_s^2 ds$ and $Z(p)_t = \frac{1}{2} \int_0^t y(p)_s^2 ds$. By Lemma 4.3, $Z^n(p)$ converges for all ω and p to $Z(p)$, uniformly in time, as $n \rightarrow \infty$. By (4.5) it is obvious that $Z(p) \rightarrow Z$ uniformly in time as $p \rightarrow \infty$, a.s. So in order to prove (e) it is enough to show that $u_p^n = \sup_t |nZ_t^n(p) - nZ_t^n|$ satisfies $\sup_n u_p^n \rightarrow 0$ a.s. as $p \rightarrow \infty$.

By Cauchy-Schwarz inequality, and with $\eta = \int_0^1 y_s^2 ds$, we get

$$\begin{aligned} u_p^n &\leq n \int_0^1 |Y(p)_s^n y(p)_s - Y_s^n y_s| ds \\ &\leq n \int_0^1 |Y(p)_s^n| |y(p)_s - y_s| ds + n \int_0^1 |Y(p)_s^n - Y_s^n| |y_s| ds \\ &\leq n\sqrt{\eta_p} \left(\int_0^1 |Y(p)_s^n|^2 ds \right)^{1/2} + m\sqrt{\eta} \left(\int_0^1 |Y(p)_s^n - Y_s^n|^2 ds \right)^{1/2} \end{aligned}$$

Now $n|Y(p)_s^n| \leq (n \int_{[ns]/n}^s y(p)_u^2 du)^{1/2}$, hence

$$\int_0^1 |Y(p)_s^n|^2 ds \leq \frac{1}{n} \int_0^1 ds \int_{[ns]/n}^s y(p)_u^2 du \leq \frac{1}{n^2} \int_0^1 y(p)_u^2 du \leq \frac{\eta}{n^2},$$

and similarly $\int_0^1 |Y(p)_s^n - Y_s^n|^2 ds \leq \eta_p/n^2$. Therefore $u_p^n \leq 2\sqrt{\eta\eta_p} \rightarrow 0$ as $p \rightarrow \infty$: hence we have (e) with Z as above.

Finally $\int_0^1 |dZ_s^n| \leq Z^n(Y', Y')_1$, where $Y'_t = \int_0^t |y_s| ds$. Since Y' also satisfies (4.2) we deduce the last claim. \square

Proof of Theorem 1.1. That (b) \Rightarrow (a) is obvious, and (c) implies (b) and the last claim by Theorems 3.2 and 4.1.

Suppose now (a). A simple computation shows that $X_1^n = X_1 \prod_{i=1}^n ((1 + \Delta_i^n Y) \exp -\Delta_i^n Y)$. By hypothesis $0 \leq -nU_1^n \leq K$ for some constant K (remember that Y is increasing); then if $A^n = \sum_{i=1}^n (\Delta_i^n Y - \log(1 + \Delta_i^n Y))$ it follows that $nA^n \leq K'$ for another constant K' , hence $n \sum_{i=1}^{[nt]} (\Delta_i^n Y)^2 \leq K''$ for yet another constant K'' . So (4.1) and Lemma 4.2 give (c). \square

2) The discontinuous case. In the remainder of this section, we study the case where $Y = A+B$,

with A a continuous process with finite variation, and with B as follows:

$$B_t = \sum_{j \geq 1} b_j 1_{\{T_j \leq t\}}, \quad (4.6)$$

where $T_0 = 0$, T_j is $[0, 1] \cup \{\infty\}$ -valued, non-decreasing in j , and with $T_j < T_{j+1}$ if $T_j < 1$, and also $K = \inf\{j : T_{j+1} = \infty\}$ is a.s. finite, and $b_j \neq 0$ if $j \leq K$.

When $A = 0$, the situation is particularly simple:

Theorem 4.4. *If $A = 0$, then $Z^n(\omega) = 0$ for all n large enough (depending on ω).*

Proof. Let ω be fixed. For n large enough, each interval $(\frac{i-1}{n}, \frac{i}{n}]$ contains at most one jump time of B , in which case the property $Z_i^n(\omega) = 0$ is readily verified. \square

In the above situation, Theorem 3.2 readily applies, but we have even more, namely that $U^n = \bar{U}^n = 0$ as soon as n is large enough.

The situation when $A \neq 0$ is more surprising, and we will find out that the sequences nZ^n and nU^n do not converge in law for the Skorohod topology, unless the process A is constant on a neighbourhood of each T_j . However, under mild assumptions the processes $n\bar{U}^n$ and $n\bar{Z}^n$ indeed converge, where

$$\bar{Z}_t^n = Z_{[nt]/n}^n. \quad (4.7)$$

We need some additional notation below:

$$T_+(n, j) = \inf\left(\frac{i}{n} : i \geq 1, \frac{i}{n} \geq T_j\right), \quad T_-(n, j) = T_+(n, j) - \frac{1}{n}, \quad (4.8)$$

$$\alpha_j^n = n(A_{T_j} - A_{T_-(n, j)}), \quad \beta_j^n = n(A_{T_+(n, j)} - A_{T_j}), \quad \gamma_j^n = \alpha_j^n + \beta_j^n, \quad (4.9)$$

$$\bar{\beta}_j^n = \int_{T_j}^{T_+(n, j)} |dA_s|, \quad (4.10)$$

with $\alpha_j^n = \beta_j^n = \bar{\beta}_j^n = 0$ if $j > K$. We are not especially interested in the processes $n\bar{Z}^n$, but they are simpler than $n\bar{U}^n$ and so we start with them.

Theorem 4.5. *We have equivalence between*

- a) $n\bar{Z}^n \Rightarrow^{stably} Z$.
- b) *The process A has (4.2) (with a density a , say), and $(\gamma_j^n)_{j \geq 1} \Rightarrow^{stably} (\gamma_j)_{j \geq 1}$ for the product topology on $\mathbb{R}^{\mathbb{N}}$.*

In this case, the limits in (a) and (b) above are connected by

$$Z_t = \frac{1}{2} \int_0^t a_s^2 ds + \sum_{j \geq 1} b_j \gamma_j 1_{\{T_j \leq t\}}, \quad (4.11)$$

and furthermore the sequence nZ^n converges stably in finite-dimensional laws along the (dense) set $J = \{t : P(\Delta Y_t \neq 0) = 0\}$.

Proof. Let us write $C^n = nZ^n(A, A)$ and $D^n = nZ^n(A, B) + nZ^n(B, A)$, and associate with these \bar{C}^n and \bar{D}^n as in (4.7). On the set Ω_n where each interval $(\frac{i-1}{n}, \frac{i}{n}]$ contains at most one T_j , we have $nZ^n = C^n + D^n$. Observe also that

$$\bar{D}_t^n = \sum_{j \geq 1} b_j \gamma_j^n 1_{\{T_+(n, j) \leq t\}} \quad \text{on } \Omega_n. \quad (4.12)$$

On the other hand, (4.1) shows that $\Delta \overline{C}_{T_+(j,n)}^n = \frac{1}{2n}(\gamma_j^n)^2$ on $\{T_j \leq 1\}$. Hence

$$\Delta(n\overline{Z}^n)_{T_+(n,j)} = \frac{1}{2n}(\gamma_j^n)^2 + b_j\gamma_j^n \quad \text{on } \Omega_n \cap \{T_+(n,j) \leq 1\}. \quad (4.13)$$

1) Assume (a). By (4.9) and the continuity of A , we have $\gamma_j^n/n \rightarrow 0$ for each j . We also have $\Omega_n \rightarrow \Omega$. Now, (a) implies tightness for each sequence (4.13) (j being fixed), and since $b_j \neq 0$ we deduce that each sequence γ_j^n is tight. Then (4.12) yields that the sequence \overline{D}^n is tight, which in turn yields together with (a) again that the sequence $\overline{C}_1^n = C_1^n$ is tight: At this point, Theorem 4.1 gives that A satisfies (4.2) and that C^n converges a.s. uniformly to $C_t = \frac{1}{2} \int_0^t a_s^2 ds$.

By well-known properties of stable convergence, we deduce from this, from the fact that $\Omega_n \rightarrow \Omega$ and from (a) that $\overline{D}^n \Rightarrow^{stably} Z - C$. In view of (4.12), this gives the second half of (b) and the relation (4.11).

2) Assume (b). By Theorem 4.1 C^n converges a.s. uniformly to C , as given above, so it is enough (using $\Omega_n \rightarrow \Omega$ again) to prove that the right-hand side of (4.12) stably converges: in view of (4.8) this readily follows from the second part of (b).

3) It remains to prove the last claim. Exactly as before, it is enough to prove that the processes D^n stably converge in finite-dimensional laws to the process defined by the last sum in (4.11). Since D^n is constant over each interval $[T_+(n,j), T_-(n,j+1))$, this is clearly equivalent to the stable convergence of the sequence \overline{D}^n finite-dimensionally in law along J , and this property readily follows from the stable convergence (for Skorohod topology) of \overline{D}^n to D , as seen before, because J is exactly the complements of the fixed times of discontinuity of D . \square

Now we turn to U^n and \overline{U}^n , for which we give only a sufficient condition.

Theorem 4.6. *Assume that the process A has (4.2) with a density a and that each sequence $(\overline{\beta}_j^n)_{n \geq 1}$ is tight. Then the sequences (nZ^n) and (nU^n) have UCV. If further $(\alpha_j^n, \beta_j^n)_{j \geq 1} \Rightarrow^{stably} (\alpha_j, \beta_j)_{j \geq 1}$, then for any starting point x_0 and any C^1 function f with linear growth we have:*

a) $(n\overline{U}^n) \Rightarrow^{stably} U$ where U is the unique solution of the following linear SDE:

$$\begin{aligned} U_t = & \int_0^t f'(X_{s-})U_{s-}dY_s - \frac{1}{2} \int_0^t f(X_s)f'(X_s)a_s^2 ds \\ & - \sum_{j:T_j \leq t} b_j f(X_{T_j-}) \left(\alpha_j f'(X_{T_j-}) + \beta_j \int_0^1 f'(X_{T_j-} + u\Delta X_{T_j}) du \right). \end{aligned} \quad (4.14)$$

b) (nU^n) converges stably in finite-dimensional laws along J (see Theorem 4.5) to U .

Observe that another way of writing (4.14) is as follows:

$$\begin{aligned} U_t = & \int_0^t f'(X_{s-})U_{s-}dY_s - \frac{1}{2} \int_0^t f(X_s)f'(X_s)a_s^2 ds \\ & - \sum_{j:T_j \leq t} (\Delta X_{T_j} f'(X_{T_j-})\alpha_j + (f(X_{T_j}) - f(X_{T_j-}))\beta_j). \end{aligned} \quad (4.15)$$

Before giving the proof, let us provide some comments and examples.

Remark 4.1. Equations (3.7) and (4.14) are *different*, unless $b_j\beta_j = 0$ for all j . This means in particular that the convergence of nU^n described above is *not* in the Skorohod sense, otherwise one could apply Theorem 3.2, and similarly for nZ^n in Theorem 4.5. In fact, what prevents Skorohod convergence is that in each interval $[T_j, T_+(n,j)]$ we have for these processes a “big” jump at T_j and also a continuous part with a non-vanishing increment.

In fact, the results obtained in (b) above show that the sequence nU^n converges to U in the topology of convergence in measure for function on $[0, 1]$ (also called Meyer-Zheng topology), and indeed this sequence also converges in the S-topology introduced by Jakubowski [6]. The same holds for nZ^n . For these topologies, a result like Theorem 2.3 does not hold, explaining why the limit U satisfies another equation than (3.7). \square

Remark 4.2. In a sense, the most interesting aspect of this subsection consists in “negative” results (no Skorohod convergence for nU^n , no convergence at all if the conditions above are not met...) However if one is interested in “positive” results, one can check as a by-product of the following proof that the sequence $n\bar{U}^n$ is (Skorohod)-tight and the sequence nU^{n*} is tight as soon as A satisfies (4.2) and all sequences α_j^n and β_j^n are tight, even if we do not have convergence.

Observe that we may have (4.2) for A and yet the tightness above may fail: take e.g. $T_1 = 1/2$, $T_2 = \infty$, $b_1 = 1$, and $a_s = |s - 1/2|^{-1/3}$. In this case $nZ_1^n \rightarrow \infty$, and the rate of convergence (if it exists) is not $1/n$. \square

Remark 4.3. As said before, Equation (3.1) is in principle not random. However if Y is really not random we have no chance of getting a limit in the previous theorem.

Take for example $A_t = t$ and $B_t = 1_{\{T < t\}}$ for some $T = p/q$ with p and q relative primes. The conditions of Theorem 4.5 hold, and $n\bar{Z}^n$ converges for the Skorohod topology to $Z = \frac{1}{2}A + B$. However the conditions of Theorem 4.6 do not hold because α_1^n takes successively all values $0, 1/q, \dots, (q-1)/q$ as n varies. In fact the functions nU^n and $n\bar{U}^n$ have q distinct limit functions, solutions of the (non-random) differential equations for $i = 0, 1, \dots, q-1$:

$$\begin{aligned} U(i)_t = & \int_0^t f'(X_{s-})U(i)_{s-}ds - \frac{1}{2} \int_0^t f(X_s)f'(X_s)ds \\ & + \left(f'(X_{T-})U(i)_{T-} - \frac{i}{q}f'(X_{T-})\Delta X_T - \frac{q-i}{q}(f(X_T) - f(X_{X_{T-}})) \right) 1_{\{T \leq t\}}. \quad \square \end{aligned}$$

Remark 4.4. In fact the existence of a limit for the sequences α_j^n and β_j^n is connected with the asymptotic behavior of the fractional part of the variables nT_j . This fractional part is known to converge (even stably) if T_j admits a density regular enough (see e.g. [2]), while of course it does not converge if T_j is deterministic. Another factor which might ensure convergence is enough randomness in the density a_s . In all these cases, the limit U features “more randomness” than Y , as seen from the fact that U (or sometimes even Z) is defined on genuine extensions of the original space.

Here is an example where convergence comes from the randomness of T_j : suppose that Y is as in Remark 4.3, but T is uniform on $(0, 1]$. Then $n\bar{Z}^n$ tends ω -wise to Z as above, while $n\bar{U}^n$ stably converges to the solution of the following equation, where α denotes a random variable, uniform and independent of T :

$$\begin{aligned} U_t = & \int_0^t f'(X_{s-})U_{s-}ds - \frac{1}{2} \int_0^t f(X_s)f'(X_s)ds \\ & + (f'(X_{T-})U_{T-} - \alpha f'(X_{T-})\Delta X_T - (1 - \alpha)(f(X_T) - f(X_{T-}))) 1_{\{T \leq t\}}. \end{aligned}$$

Here is another example, where convergence comes from the randomness of a_s : let W be a standard Brownian motion and set $B_t = 1_{\{t \geq 1\}}$ and $a_s = (1-s)^{-1/2}W_{1-s}1_{\{|W_{1-s}| \leq \sqrt{1-s}\}}$. Observe that $|a_s| \leq 1$. It is easy, by a scaling argument, to check that here $n\bar{Z}^n$ converges stably to $Z = \frac{1}{2}A + UB$, where U is a standard normal variable, independent of Y . \square

Remark 4.5. We have left out the case when Y has infinitely many jumps on $(0, 1]$: nothing

is known in general for this case, see however Section 6 when Y has in addition independent increments.

Proof. 1) First we prove that the sequence nZ^n is UCV. Since Z^n is of finite variation, it suffices to show that $n \int_0^1 |dZ_t^n|$ forms a tight sequence.

With the notation of the proof of Theorem 4.5, $nZ^n = C^n + D^n$ on Ω_n , so a simple computation shows that

$$n \int_0^1 |dZ_s^n| \leq 2 \int_0^1 |dC_s^n| + \sum_{i=1}^K |b_j| (|\alpha_j^n| + \bar{\beta}_j^n) \quad \text{on the set } \Omega_n. \quad (4.16)$$

Now $\Omega_n \rightarrow \Omega$, while $\int_0^1 |dC_s^n|$ is tight by Theorem 4.1, so the result immediately follows from the fact that K is a.s. finite and from the tightness of all sequences $(\alpha_j^n)_n$ and $(\bar{\beta}_j^n)_n$ (recall K is defined just after (4.6)). In view of Theorem 3.3 it then follows that nU^n is UCV, and in particular the sequence nU^{n*} is tight.

2) Recall that nU^n is the solution of (3.13), with $\alpha_n = n$, and introduce the solution V_n of the following linear equation:

$$dV_t^n = V_{t-}^n f'(X_{t-}) dY_t - k(X_t)^n d(nZ^n)_t, \quad V_0^n = 0. \quad (4.17)$$

Theorems 2.2(d), 3.1 and 2.5(b) give that $(nU^n - V^n)^* \rightarrow^P 0$, so in order to prove (a) and (b) we can replace the processes nU^n and $n\bar{U}^n$ by V^n and $\bar{V}_t^n = V_{[nt]/n}^n$ respectively.

Let also U be the solution of (4.14), on an extension of the original probability space which supports the limits (α_j, β_j) . Let us introduce the processes

$$W(j)_t^n = \bar{V}_{t \wedge T_+(n,j)}^n, \quad (4.18)$$

$$W(j)_t'^n = \begin{cases} W(j)_t^n & \text{if } t \leq T_+(n,j) \\ W(j)_{T_+(n,j)}^n + V_t^n - V_{T_+(n,j)}^n & \text{if } T_+(n,j) < t < T_{j+1} \\ W(j)_{T_+(n,j)}^n + V_{T_{j+1}-}^n - V_{T_+(n,j)}^n & \text{if } T_{j+1} \leq t, \end{cases} \quad (4.19)$$

$$W(j)_t = U_{t \wedge T_j}, \quad W(j)_t' = \begin{cases} U_t & \text{if } t < T_{j+1} \\ U_{T_{j+1}-} & \text{if } t \geq T_{j+1}. \end{cases} \quad (4.20)$$

Let us also write ρ^n for the double sequence $(\alpha_j^n, \beta_j^n)_{j \geq 1}$ and $\rho = (\alpha_j, \beta_j)_{j \geq 1}$. Consider the following property:

$$(H_j) \quad (\rho^n, W(j)^n) \Rightarrow^{stably} (\rho, W(j)).$$

By hypothesis (H_0) holds. If (H_j) holds for all j then (a) follows, because $K < \infty$ a.s.

3) Suppose that (H_j) holds. First if $H(n, j)$ is the interval $(T_+(n, j), T_{j+1})$ we deduce from (4.17) that

$$W(j)_t'^n = W(j)_t^n + \int_0^t W(j)_{s-}'^n a_s f'(X_{s-}) 1_{H(n,j)}(s) ds - \int_0^t k(X_s)^n 1_{H(n,j)}(s) dC_s^n. \quad (4.21)$$

Recall that $C^n = nZ^n(A, A)$ converges a.s. uniformly to the process $C_t = \frac{1}{2} \int_0^t a_s^2 ds$. Further, set

$$J_t^n = W(j)_t^n - \int_0^t k(X_s)^n 1_{H(n,j)}(s) dC_s^n, \quad J_t = W(j)_t - \int_0^t f(X_s) f'(X_s) 1_{(T_j, T_{j+1})}(s) dC_s,$$

$$L_t^n = \int_0^t a_s f'(X_{s-}) 1_{H(n,j)}(s) ds, \quad L_t = \int_0^t a_s f'(X_s) 1_{(T_j, T_{j+1})}(s) ds,$$

so (4.21) becomes

$$W(j)_t'^n = J_t^n + \int_0^t W(j)_{s-}'^n dL_s^n. \quad (4.22)$$

Now, X is continuous on the interval (T_j, T_{j+1}) , so $(L^n - L)^* \rightarrow^P 0$ and the last term in (4.21) converges uniformly in probability to $\int_0^t f(X_s) f'(X_s) 1_{(T_j, T_{j+1})}(s) dC_s$, and since this process is continuous, as well as L , it follows from (H_j) that $(J^n, L^n, \rho^n) \Rightarrow^{stably} (J, L, \rho)$. Hence Theorem 2.5(c) yields that $(W(j)'^n, \rho^n) \Rightarrow^{stably} (W(j)', \rho)$. Using Lemma 2.2 and the fact that $W(j)'^n$ and $W(j)'$ are constant on $[T_{j+1}, 1]$, we finally deduce that if $\overline{W}(j)_t^n = W(j)_{[nt]/n}^n$, then

$$(\overline{W}(j)'_{\wedge T_-(n,j+1)}^n, \rho^n, W(j)_{T_{j+1}-}'^n) \Rightarrow^{stably} (W(j)', \rho, W(j)_{T_{j+1}-}'). \quad (4.23)$$

4) Now, set $\delta_n := V_{T_+(n,j+1)}^n - V_{T_-(n,j+1)}^n$. By (3.13) we have $\delta_n = u_n + v_n$ on the set $\Omega_n \cap \{T_{j+1} \leq 1\}$, where

$$u_n = \int_{T_-(n,j+1)}^{T_+(n,j+1)} V_{s-}^n f'(X_{s-}) a_s ds - \int_{T_-(n,j+1)}^{T_+(n,j+1)} k(X)_s^n dC_s^n,$$

$$v_n = b_{j+1} (V_{T_{j+1}-}^n - f'(X_{T_{j+1}-})) - k(X)_{T_{j+1}}^n \alpha_{j+1}^n - \int_{T_{j+1}}^{T_+(n,j+1)} k(X)_s^n a_s ds.$$

First, the sequences V^{n*} and $k(X)^{n*}$ are tight, so one deduces that $u_n \rightarrow^P 0$. Next the sequences

$$k(X)_{T_{j+1}}^n - (ff')(X_{T_{j+1}-}), \quad \int_{T_{j+1}}^{T_+(n,j+1)} k(X)_s^n a_s ds - \beta_{j+1}^n f(X_{T_{j+1}-}) \int_0^1 f'(X_{T_{j+1}-} + u \Delta X_{T_{j+1}}) du$$

converge to 0 in probability. Furthermore $V_{T_{j+1}-}^n = W(j)_{T_{j+1}-}'^n$. Therefore if

$$\delta = b_{j+1} (W(j)_{T_{j+1}-}' - f'(X_{T_{j+1}-})) - \alpha_{j+1} (ff')(X_{T_{j+1}-}) - \beta_{j+1} \int_0^1 f'(X_{T_{j+1}-} + u \Delta X_{T_{j+1}}) du,$$

one deduces from (4.23) that

$$(\overline{W}(j)'_{\wedge T_-(n,j+1)}^n, \rho^n, \delta_n) \Rightarrow^{stably} (W(j)', \rho, \delta). \quad (4.24)$$

But $W(j+1)^n = \overline{W}(j)'_{\wedge T_-(n,j+1)}^n + \delta_n 1_{[T_+(n,j+1), 1]}$, while $W(j+1) = W(j)'_{\wedge T_{j+1}-} + \delta 1_{[T_{j+1}, 1]}$. Thus (4.24) yields (H_{j+1}) , and the proof of (a) follows by induction on j .

5) Finally on the set $\{T_+(n,j) < t < T_-(n,j+1)\}$ we have $V_t^n = W(j)_t'^n$ and $U_t = W(j)_t'$. Since $W(j)'^n \Rightarrow^{stably} W(j)'$ and since U is continuous outside all T_j 's, the first claim of (b) is obvious, and we are finished. \square

5 Continuous semimartingales

1) **The martingale case.** Here we consider the case where the driving process $Y = (Y^i)_{1 \leq i \leq d}$ in (3.1) is a *continuous* d -dimensional local martingale, null at 0. We denote by $C = (C^{ij})_{1 \leq i, j \leq d}$ the quadratic variation process, i.e. $C^{ij} = \langle Y^i, Y^j \rangle$, and we write $Z^n = (Z^{n,ij})_{1 \leq i, j \leq d}$, where $Z^{n,ij} = Z^n(Y^i, Y^j)$. We introduce the d^4 -dimensional processes D^n whose components are

$$D^{n,ijkl} = n \langle Z^{n,ik}, Z^{n,jl} \rangle. \quad (5.1)$$

The main result is as follows:

Theorem 5.1. *Assume that Y is a continuous local martingale. There is equivalence between*

a) *We have (with c being a $d \times d$ symmetric nonnegative matrix-valued predictable process):*

$$C_t = \int_0^t c_s ds, \quad \int_0^1 \|c_s\|^2 ds < \infty. \quad (5.2)$$

b) *For each i the sequence of random variables $(\sqrt{n}Z^{n,ii^*})_{n \geq 1}$ is tight.*

c) *For each i the sequence of random variables $(D^{n,iiii})$ is tight.*

In this case, and if σ is a $d \times q$ matrix valued process such that $c = \sigma\sigma^\dagger$ (σ^\dagger stands for the transpose; such processes always exist, for $q \geq d$ at least), the sequence $\sqrt{n}Z^n$ stably converges in law to a process Z given by

$$Z_t^{ij} = \frac{1}{\sqrt{2}} \sum_{1 \leq k, \ell \leq q} \int_0^t \sigma_s^{ik} \sigma_s^{j\ell} dW_s^{k\ell}, \quad (5.3)$$

where $(W^{ij})_{1 \leq i, j \leq q}$ is a standard q^2 -dimensional Brownian motion defined on an extension of the space on which Y is defined and independent of Y . Moreover, we also have $(Y, \sqrt{n}Z^n) \Rightarrow (Y, Z)$ and the sequence $\sqrt{n}Z^n$ has UCV, and the following convergence holds a.s. uniformly in time:

$$D_t^{n,ijkl} \rightarrow D_t^{ijkl} := \frac{1}{2} \sum_{1 \leq u, v \leq q} \int_0^t c_s^{ik} c_s^{j\ell} ds. \quad (5.4)$$

Note that (5.2) is the minimal condition under which the process Z of (5.3) is well defined. We divide the proof into several steps.

Lemma 5.2. *Conditions (b) and (c) in Theorem 5.1 are equivalent, and they imply (a).*

Proof. Observe that $dC_t = c_t dA_t$ for some increasing continuous process A and some $d \times d$ symmetric nonnegative matrix-valued process c . This last property readily implies that (a) is equivalent to the fact that each C^{ii} satisfies (4.2). So indeed to prove our lemma we can and will suppose that Y , hence Z^n and D^n as well, are 1-dimensional.

1) Set $S(n, p) = \inf(t : \sqrt{n}|Z_t^n| \geq p)$ and $T(n, p) = \inf(t : D_t \geq p)$, and also $\bar{Z}_t^n = \sqrt{n} \sup_{s \leq t} |Z_s^n|$.

Assume first (b). Then $\sup_n P(S(n, p) < 1) \leq \sup_n P(Z^{n*} \geq p) \rightarrow 0$ as $p \rightarrow \infty$, while $E(D_{S(n, p)}^n) \leq p^2$, hence

$$P(D_1^n > q) \geq \frac{p^2}{q} + P(S(n, p) < 1)$$

goes to 0 uniformly in n as $q \rightarrow \infty$, and (c) holds. Conversely (c) yields $\sup_n P(T(n, p) < 1) \rightarrow 0$ as $p \rightarrow \infty$, while Doob's inequality yields $E((\bar{Z}_{T(n, p)}^n)^2) \leq 4p$, so we deduce (b) exactly as above.

2) From now on we assume (b) and (c). Note that $T_p = \inf(t : C_t \geq p)$ has $P(T_p < 1) \rightarrow 0$ as $p \rightarrow \infty$. If we stop the process Y at time T_p , the corresponding processes C , Z^n , D^n are also stopped at T_p . So clearly it suffices to prove (a) for the stopped processes: in other words we can and will assume that C_1 is bounded by a constant.

For every stopping time T set $\xi(n, i, T) = n \int_{T \wedge (i-1)/n}^{T \wedge i/n} (Y_s^{(n)})^2 dC_s$ (recall notation (3.5)). Observe that if N is a continuous martingale null at 0, then

$$N_t^4 = 4 \int_0^t N_s^3 dN_s + 6 \int_0^t N_s^2 d\langle N, N \rangle_s.$$

If furthermore the variable $\langle N, N \rangle_1$ is bounded, the process $\int_0^t N_s^3 dN_s$ is a martingale, hence $E(N_1^4) = 6E(\int_0^1 N_s^2 d\langle N, N \rangle_s)$. Apply this to $N_t = Y_{t \wedge T \wedge (i/n)} - Y_{t \wedge T \wedge (i-1/n)}$ to get $E(\xi(n, i, T)) = \frac{n}{6} E((Y_{t \wedge T \wedge (i/n)} - Y_{t \wedge T \wedge (i-1/n)})^4)$. On the other hand a Burkholder-Gundy inequality yields a universal constant K such that

$$E((C_{t \wedge T \wedge (i/n)} - C_{t \wedge T \wedge (i-1/n)})^2) \leq KE((Y_{t \wedge T \wedge (i/n)} - Y_{t \wedge T \wedge (i-1/n)})^4).$$

Therefore

$$E((C_{t \wedge T \wedge (i/n)} - C_{t \wedge T \wedge (i-1/n)})^2) \leq 6KE(\xi(n, i, T)). \quad (5.5)$$

Observe that $D_t^n = n \int_0^t (Y_s^{(n)})^2 dC_s$, hence $\sum_{i=1}^n \xi(n, i, T(n, p)) = D_{1 \wedge T(n, p)}^n \geq p$. Letting

$$\Gamma_t^n = n \left(\sum_{i=1}^{\lfloor nt \rfloor} (\Delta_i^n C)^2 + (C_t^{(n)})^2 \right) = n \sum_{i=1}^n (C_{t \wedge (i/n)} - C_{t \wedge (i-1/n)})^2,$$

we deduce from (5.5) that $E(\Gamma_{T(n, p)}^n) \leq 6Kp$.

Finally, let $\gamma_n = n \sum_{i=1}^n (\Delta_i^n C)^2$. Since $\gamma_n = \Gamma_{T(n, p)}^n$ on the set $\{T(n, p) \geq 1\}$, we have

$$P(\gamma_n > q) \leq P(T(n, p) < 1) + P(\Gamma_{T(n, p)}^n > q) \leq P(T(n, p) < 1) + \frac{6Kp}{q}. \quad (5.6)$$

We have $\lim_p \sup_n P(T(n, p) < 1) = 0$ by (c), hence (3.6) yields that the sequence γ_n is tight, and (a) follows from Theorem 4.1. \square

Now we assume (5.2), and we let σ be a $d \times q$ matrix-valued process such that $c = \sigma \sigma^\dagger$, for some q . Up to enlarging the space, we can assume that there is a Wiener process $W' = (W'^i)_{1 \leq i \leq q}$ such that

$$Y_t = \int_0^t \sigma_s dW'_s. \quad (5.7)$$

In virtue of [4], if we prove that for all $t \in (0, 1]$ and all $i, j \leq d$ and $k \leq q$ we have, with notation (5.1) and (5.4)

$$D_t^n \xrightarrow{P} D_t, \quad \sqrt{n} \langle Z^{n, ij}, W'^k \rangle_t \xrightarrow{P} 0, \quad (5.8)$$

then the processes $\sqrt{n}Z^n$ will converge stably in law to the process Z of (5.3): we deduce that the pair $(Y, \sqrt{n}Z^n)$ converges in law to (Y, Z) for the product topology on $\mathcal{D}(\mathbb{R}^d) \times \mathcal{D}(\mathbb{R}^{d^2})$, and since all these processes are continuous we also have convergence for the Skorohod topology on $\mathcal{D}(\mathbb{R}^{d+d^2})$. Further (5.4) follows from (5.8), and it implies UCV for the sequence $\sqrt{n}Z^n$. In other words, to prove Theorem 5.1 it remains only to show (5.8).

We begin with a lemma.

Lemma 5.3. *We have (5.8) as soon as the process σ has the form*

$$\sigma_s = \sum_{i=1}^m A_{i-1} \mathbf{1}_{(t_{i-1}, t_i]}(s), \quad (5.9)$$

where $0 = t_0 < t_1 < \dots < t_m = 1$ and where each A_i is a bounded \mathcal{F}_{t_i} -measurable random variable with values in $d \times q$ matrices.

Proof. 1) Set $\tau_n(u) = u - [nu]/n$. By a Burkholder-Gundy inequality we have for some constant K :

$$E \left((Y_t^{(n)})^4 \right) \leq K/n^2. \quad (5.10)$$

Recall that $Y^{(n)}$ is defined in (3.5). Since $Y_u^{(n)} = A_r W_u^{(n)}$ for $t_r < [nu]/n \leq u \leq t_{r+1}$, simple computations show that for $t_r < [nu]/n \leq u \leq v \leq t_{r+1}$, and with $B_r = A_r A_r^\dagger$:

$$E(Y_u^{i,(n)} Y_v^{j,(n)} | \mathcal{F}_{s_r}) = B_r^{ij} \tau_n(u) 1_{\{[nu]=[nv]\}}, \quad (5.11)$$

$$E(Y_u^{i,(n)} Y_v^{i,(n)} Y_u^{j,(n)} Y_v^{j,(n)} | \mathcal{F}_{s_r}) = \tau_n(u) \tau_n(v) (B_r^{ij})^2 + \tau_n(u)^2 ((B_r^{ij})^2 + B_r^{ii} B_r^{jj}) 1_{\{[nu]=[nv]\}}. \quad (5.12)$$

2) Let us fix r and t such that $0 < t \leq t_{r+1} - t_r$. We have

$$\begin{aligned} D_{t_r+t}^{n,ijkl} - D_{t_r}^{n,ijkl} &= n B_r^{j\ell} \int_{t_r}^{t_r+t} Y_u^{i,(n)} Y_u^{k,(n)} du, \\ D_{t_r+t}^{ijkl} - D_{t_r}^{ijkl} &= \frac{1}{2} B_r^{j\ell} B_r^{ik} t, \\ \sqrt{n} \langle Z^{n,ij}, W^{ik} \rangle_{t_r+t} - \sqrt{n} \langle Z^{n,ij}, W^{ik} \rangle_{t_r} &= \sqrt{n} B_r^{jk} \int_{t_r}^{t_r+t} Y_u^{i,(n)} du. \end{aligned}$$

So it is enough to prove that

$$n \int_{t_r}^{t_r+t} Y_u^{i,(n)} Y_u^{k,(n)} du \xrightarrow{\mathbb{L}^2} \frac{t}{2} B_r^{ik}, \quad \sqrt{n} \int_{t_r}^{t_r+t} Y_u^{i,(n)} du \xrightarrow{\mathbb{L}^2} 0.$$

3) Setting $s(n) = ([nt_r] + 1)/n$, we have $s(n) \rightarrow t_r$ and $n \int_{t_r, \vee s(n)}^{t_r+t} Y_u^{i,(n)} Y_u^{k,(n)} du \xrightarrow{\mathbb{L}^2} 0$ and $\sqrt{n} \int_{t_r, \vee s(n)}^{t_r+t} Y_u^{i,(n)} du \xrightarrow{\mathbb{L}^2} 0$ by (5.10). So it remains to prove that

$$\alpha_n := n \int_{s(n)}^{t_r+t} Y_u^{i,(n)} Y_u^{k,(n)} du - \frac{t}{2} B_r^{ik} \xrightarrow{\mathbb{L}^2} 0, \quad \beta_n := \sqrt{n} \int_{s(n)}^{t_r+t} Y_u^{i,(n)} du \xrightarrow{\mathbb{L}^2} 0.$$

Using (5.10) and $0 \leq \tau_n(u) \leq 1/n$ and the boundedness of B_r , we get

$$E(\beta_n^2) \leq n \int_{[s(n), t_r+t]^2} \frac{K}{n} 1_{\{[nu]=[nv]\}} dudv \leq \frac{2tK}{n} \rightarrow 0$$

for some constant K . Similarly

$$\begin{aligned} E(\alpha_n^2) &= n^2 \int_{[s(n), t_r+t]^2} E(Y_u^{i,(n)} Y_u^{k,(n)} Y_v^{i,(n)} Y_v^{k,(n)}) dudv \\ &\quad + \frac{t^2}{4} E((B_r^{ik})^2) - nt \int_{s(n)}^{t_r+t} E(B_r^{ik} Y_u^{i,(n)} Y_u^{k,(n)}) du. \end{aligned}$$

On the one hand, we have by (5.11):

$$nt \int_{s(n)}^{t_r+t} E(B_r^{ik} Y_u^{i,(n)} Y_u^{k,(n)}) du \rightarrow \frac{t^2}{2} E((B_r^{ik})^2).$$

On the other hand, (5.12) yields

$$|E(Y_u^{i,(n)} Y_u^{k,(n)} Y_v^{i,(n)} Y_v^{k,(n)}) - \tau_n(u) \tau_n(v) E((B_r^{ik})^2)| \leq \frac{K}{n^2} 1_{\{[nu]=[nv]\}},$$

and thus

$$n^2 \int_{[s(n), t_r + t]^2} E(Y_u^{i,(n)} Y_u^{k,(n)} Y_v^{i,(n)} Y_v^{k,(n)}) du dv \rightarrow \frac{t^2}{4} E((B_r^{ik})^2).$$

Putting these results together gives $E(\alpha_n^2) \rightarrow 0$, and we are finished. \square

Proof of Theorem 5.1. It remains to prove that (a) implies (5.8) in the general case. Let $T_p = \inf(t : \int_0^t \|c_s\|^2 ds \geq p)$. Since (a) yields $P(T_p < 1) \rightarrow 0$, by localization it is clearly enough to prove the result for the processes stopped at time T_p , which amounts to assuming that $\int_0^1 \|c_s\|^2 ds$ is bounded by a constant p .

The rest of the argument parallels Theorem 3.1. By a density result, and if c is such that $c = \sigma\sigma^\dagger$, there exists a sequence $\sigma(q)$ of processes of the form (5.9), such that

$$\eta_q := \int_0^1 \|\sigma_s - \sigma(q)_s\|^4 ds \rightarrow 0, \quad \int_0^1 \|\sigma(q)_s\|^4 ds \leq \int_0^1 \|\sigma_s\|^4 ds \leq p. \quad (5.13)$$

Let $Y(q)_t = \int_0^t \sigma(q)_s dW'_s$, with the associated processes $Z(q)^n$ and $D(q)^n$ (see (5.1) and $D(q)$ (see (5.4)). By Lemma 5.3, for each q we have that the following convergence for all t :

$$D(q)_t^n \xrightarrow{P} D(q)_t, \quad \sqrt{n} \langle Z(q)^{n,ij}, W^{ik} \rangle_t \xrightarrow{P} 0. \quad (5.14)$$

We have with $c(q) = \sigma(q)\sigma(q)^\dagger$:

$$\begin{aligned} |D(q)_t^{n,ijkl} - D_t^{n,ijkl}| &= n \left| \int_0^t (Y(q)_s^{i,(n)} Y(q)_s^{k,(n)} c(q)_{s,j\ell} - Y_s^{i,(n)} Y_s^{k,(n)} c_s^{j\ell}) ds \right| \\ &\leq n \int_0^t \|Y(q)_s^{(n)}\|^2 \|\sigma(q)_s - \sigma_s\| (\|\sigma(q)_s\| + \|\sigma_s\|) ds \\ &\quad + \int_0^t \|Y(q)_s^{(n)} - Y_s^{(n)}\| (\|Y(q)_s^{(n)}\| + \|Y_s^{(n)}\|) \|\sigma_s\|^2 ds. \end{aligned}$$

By combining Burkholder-Gundy and Cauchy-Schwarz inequalities, we get $E(\|Y_s^{(n)}\|^4) \leq \frac{K}{n} E(\int_{[ns]/n}^s \|\sigma(q)_u\|^4 du)$, thus $\int_0^t E(\|Y_s^{(n)}\|^4) ds \leq \frac{K}{n^2}$ by (5.13) for some constant K changing from line to line, and also $\int_0^t E(\|Y(q)_s^{(n)}\|^4) ds \leq \frac{K}{n^2}$. The same argument shows that

$$\int_0^t E(\|Y(q)_s^{(n)} - Y_s^{(n)}\|^4) ds \leq \frac{K\eta_q}{n^2}.$$

Thus (5.13) and a repeated use of the Cauchy-Schwarz inequality gives $E(|D_t^{n,ijkl} - D_t^{n,ijkl}|) \leq K\eta_q^{1/4}$, so by (5.14) we get the first part of (5.8). The second part of (5.8) is proved similarly (it is in fact a bit simpler). \square

Let us now state a corollary of the previous result, which contains Theorem 1.2 as a particular case.

Corollary 5.4. *Assume that Y is a local martingale. There is equivalence between*

- a) *We have (5.2)*
- b) *For $q = 1$ and $x_0 = 1$ and $f^i(x) = x\delta_{ij}$ (i.e. $X = \mathcal{E}(Y^j)$) the sequence $\sqrt{n}U^{n*}$ is tight, for each $j = 1, 2, \dots, d$.*
- c) *For all starting points x_0 and all C^1 functions f with linear growth, the sequences $(Y, \sqrt{n}U^n)$ and $(Y, \sqrt{n}\bar{U}^n)$ weakly converge to a limit (Y, U) .*

In this case U is the solution of the linear equation (3.2), with Z given in (5.3).

Proof. That (c) \Rightarrow (b) is obvious, and (a) implies (c) and the last claim, due to Theorems 3.2 and 5.1 (the UCV property for $\sqrt{n}Z^n$ follows from (5.4)).

Assume now (b), and fix j . Then with f, x_0 corresponding to $X = \mathcal{E}(Y^j)$, we have (as in (3.11)):

$$\sqrt{n}Z_t^{n,jj} = \int_0^t \left[(\sqrt{n}U_s^n / X_{\varphi_n(s)}^n) dY_s^j - (1/X_{\varphi_n(s)}^n) d(\sqrt{n}U_s^n) \right].$$

A sequence of continuous local martingale has UCV as soon as the sequence of their suprema is tight, so here the sequence $\sqrt{n}U^n$ has UCV. By Theorem 3.1, the fact that X does not vanish and Theorem 2.3(c), we deduce that the sequence $\sqrt{n}Z^{n,jj}$ has UCV, and (a) follows from Theorem 5.1. \square

2) The semimartingale case. Now we suppose that $Y = M + A$, where A is a continuous adapted process of finite variation and M is a continuous local martingale, both being d -dimensional and null at 0. Again $C = (C^{ij})_{1 \leq i, j \leq d}$ is the quadratic variation process, i.e. $C^{ij} = \langle Y^i, Y^j \rangle = \langle M^i, M^j \rangle$, and we write $Z^n = (Z^{n,ij})_{1 \leq i, j \leq d}$, where $Z^{n,ij} = Z^n(Y^i, Y^j)$.

We do not know what happens in the general case, and only partial results are available, when A has the form:

$$A_t^i = \int_0^t a_s^i ds, \quad \text{with} \quad \int_0^1 (a_s^i)^2 ds < \infty \quad \text{a.s.} \quad (5.15)$$

Theorem 5.5. *Assume that Y is a continuous semimartingale, and that (5.15) holds. Then there is equivalence between*

- a) *We have (5.2).*
- b) *The sequence of processes $(\sqrt{n}Z^n)$ has UCV and is tight.*

In this case, the sequence $(\sqrt{n}Z^n)$ stably converges in law to a process Z of the form (5.3) (where $c = \sigma\sigma^\dagger$) and has UCV and we also have $(Y, \sqrt{n}Z^n) \Rightarrow (Y, Z)$.

Proof 1). We set $F^{n,ij} = \sqrt{n}Z^n(M^i, M^j)$, $G^{n,ij} = \sqrt{n}Z^n(M^i, A^j)$, $H^{n,ij} = \sqrt{n}Z^n(A^i, M^j)$ and $K^{n,ij} = \sqrt{n}Z^n(A^i, A^j)$, so $\sqrt{n}Z^n = F^n + G^n + H^n + K^n$.

By Theorem 4.1 the sequence K^n tends in variation to 0, a.s., and a fortiori has UCV (more precisely, the variations of the processes $\sqrt{n}K^{n,ii}$ is bounded a.s. uniformly in n , for each i , and a simple extension of Theorem 4.1 shows that it is also true for $\sqrt{n}K^{n,ij}$).

2) Suppose that (b) holds. Each $H^{n,ij}$ is a continuous local martingale, and

$$\langle H^{n,ij}, H^{n,ij} \rangle_t = n \int_0^t (A_s^{i,(n)})^2 dC_s^{jj}. \quad (5.16)$$

By (5.15) and the Cauchy-Schwarz inequality, we obtain $n(A_s^{i,(n)})^2 \leq \int_0^1 (a_u^i)^2 du$ for all n, s . Thus the sequence of processes $(\langle H^{n,ij}, H^{n,ij} \rangle)$ is tight, and it follows (see e.g. [3]) that the sequence $(H^{n,ij})$ is tight and has UCV, and of course all its limiting processes are continuous.

Then, using (1) above and Theorem 2.3(a), we see by difference that the sequence $F^n + G^n$ is also tight and has UCV. Then the quadratic variation processes, which are the same as the quadratic variations of the processes F^n , are also tight: in other words, the local martingale M satisfies Condition (c) of Theorem 5.1, and (a) follows.

3) Now we assume that (5.2), as well as (5.15) hold. In view of (1) above and of Theorem 5.1, it remains to prove that both sequences (G^n) and (H^n) have UCV and weakly converge to 0. Since this is a componentwise property, we can and will assume that $d = 1$, and exactly as in Theorem 5.1 we also can and will assume that both random variables $\int_0^1 a_s^2 ds$ and $\int_0^1 c_s^2 ds$ are bounded by a constant α .

Let us first consider H^n . This is a continuous local martingale satisfying, by (5.16):

$$\langle H^n, H^n \rangle_1 \leq \sum_{i=1}^n \int_{(i-1)/n}^{i/n} a_s^2 ds \int_{(i-1)/n}^{i/n} c_s ds \leq \alpha \gamma_n,$$

where $\gamma_n = \sup_{1 \leq i \leq n} \int_{(i-1)/n}^{i/n} c_s ds$. We have $|\gamma_n| \leq \sqrt{\alpha}$ and $\gamma_n(\omega) \rightarrow 0$ for all ω . Thus $E(\langle H^n, H^n \rangle_1) \rightarrow 0$, and thus the sequence (H^n) has UCV and weakly converges to 0.

Next consider G^n . Let $\varepsilon > 0$ and let S, T be two stopping times such that $T \leq S \leq T + \varepsilon$ and $S \leq 1$. We have by the Cauchy-Schwarz inequality:

$$E\left(\int_T^S |dG_s^n|\right) = E\left(\int_T^S \sqrt{n} |a_s M_s^{(n)}| ds\right) \leq \left(E\left(\int_T^S a_s^2 ds\right) E\left(n \int_0^1 |M_s^{(n)}|^2 ds\right)\right)^{1/2}.$$

We also have

$$E\left(n \int_0^1 |M_s^{(n)}|^2 ds\right) = E\left(n \int_0^1 ds \int_{[ns]/n}^s c_u du\right) = E\left(\int_0^1 c_u ([nu] + 1 - nu) du\right) \leq E(C_1) \leq \sqrt{\alpha}. \quad (5.17)$$

Therefore $E(\int_T^S |dG_s^n|) \leq \alpha^{1/4} \sqrt{E(w(\varepsilon))}$, where $w(\varepsilon) = \sup_t \int_t^{t+\varepsilon} a_s^2 ds$. Since $\lim_{\varepsilon \rightarrow 0} w(\varepsilon) = 0$ and $w(\varepsilon) \leq \alpha$, we have $E(w(\varepsilon)) \rightarrow 0$ and we can apply Aldous' criterion (see e.g. [3]) and deduce that the sequence of variation processes of the processes G^n is tight, which implies in particular that the sequence (G^n) has UCV.

It remains to prove that $G^n \Rightarrow 0$. In a first step we set $\bar{G}_t^n = \int_0^t \sqrt{n} M_s^{(n)} ds$, which is the process G^n above when $a_t = 1$ (i.e. $A_t = t$). So the sequence (\bar{G}^n) is tight, and we also have:

$$E((\bar{G}_t^n)^2) = 2n \int_0^t du \int_0^u E(M_u^{(n)} M_v^{(n)}) dv.$$

Since M is a martingale, $E(M_u^{(n)} M_v^{(n)})$ equals 0 if $[nv] \neq [nu]$ and $E(\int_{[nv]/n}^{v \wedge u} c_s ds)$ if $[nv] = [nu]$. Hence

$$E((\bar{G}_t^n)^2) = 2E\left(\int_0^t du \int_{[nu]/n}^u dv \int_{[nu]/n}^v c_s ds\right) \leq \frac{2}{n} E(C_1) \rightarrow 0.$$

It follows from all these that $\bar{G}^n \Rightarrow 0$.

In a second step, we assume that a is of the form (5.9). Then $G_t^n = \sum_{k=1}^m A_{k-1} (\bar{G}_{t \wedge t_k}^n - \bar{G}_{t \wedge t_{k-1}}^n)$, so $G^n \Rightarrow 0$ by the first step. Finally in the general case there is a sequence $a(p)$ of processes of the form (5.9) such that

$$\delta_p := E\left(\int_0^1 |a(p)_s - a_s|^2 ds\right) \rightarrow 0.$$

Setting $A(p)_t = \int_0^t a(p)_s ds$ and $G(p)^n = \sqrt{n} Z^n(M, A(p))$, we have $G(p)^n \Rightarrow 0$ for every p . On the other hand

$$\begin{aligned} E(\sup_t |G_t^n - G(p)_t^n|) &\leq E\left(\int_0^1 |\sqrt{n} M_s^{(n)}| |a_s - a(p)_s| ds\right) \\ &\leq \left(E\left(\int_0^1 |n(M_s^{(n)})^2 ds\right)\right)^{1/2} \sqrt{\delta_p} \leq \sqrt{\alpha \delta_p} \end{aligned}$$

by (5.17). Thus $G^n \Rightarrow 0$, and we are finished. \square

Remark. Theorem 5.5 puts us in the situation where Theorem 3.2 applies: if Y is a continuous semimartingale with (5.15) and (5.2), for any starting point x_0 and any C^1 function f with linear growth the processes $\sqrt{n}U^n$ and $\sqrt{n}\bar{U}^n$ weakly converge to the solution of (3.7), with Z given by (5.3). \square

6 Lévy processes

In this last section we suppose that the driving process Y is a Lévy process, and to simplify we assume that it is 1-dimensional (an extension to the multi-dimensional situation is rather straightforward). The characteristics of Y are (b, c, F) , where $b \in \mathbb{R}$, $c \geq 0$ and F is a positive measure on \mathbb{R} with $F(\{0\}) = 0$ and $\int x^2 \wedge 1 F(dx) < \infty$. We denote by μ the jump random measure of Y , and set $\nu(dt, dx) = dt \otimes F(dx)$, so Y has the form (see [3]):

$$Y_t = bt + Y_t^c + x1_{\{|x| \leq 1\}} \star (\mu - \nu) + x1_{\{|x| > 1\}} \star \mu, \quad (6.1)$$

where Y^c is the continuous martingale part of Y : it is 0 if $c = 0$ and Y/\sqrt{c} is a standard Brownian motion otherwise, and its quadratic variation process is ct . Further, the “ \star ” in (6.1) indicates the stochastic integral of a predictable function w.r.t. a random measure (see [3]). Set

$$Z^n = Z^n(Y, Y), \quad \bar{Z}_t^n = Z_{[nt]/n}^n. \quad (6.2)$$

Here, if $F = 0$ the process Y is a continuous semimartingale, to which the results of the previous section readily apply. On the contrary, when $F \neq 0$ the situation resembles that of Section 4 for discontinuous processes: we do not have convergence of $(\sqrt{n}Z^n)$ and $(\sqrt{n}U^n)$ in the Skorohod sense (unless $Y^c = 0$), but only finite-dimensional convergence in law, while the sequences $(\sqrt{n}\bar{Z}^n)$ and $(\sqrt{n}\bar{U}^n)$ weakly converge.

Let us first describe the limiting processes Z and U . We take (possibly on an extension of the space on which Y is defined) the following:

- a standard Brownian motion W ,
- two sequences $(V'_n)_{n \geq 1}$ and $(V''_n)_{n \geq 1}$ of standard normal variables,
- a sequence $(\chi_n)_{n \geq 1}$ of uniform variables on $(0, 1)$,

in such a way that all these terms are *mutually independent*, and *independent of Y* as well. We also set

$$V_n = \sqrt{\chi_n} V'_n + \sqrt{1 - \chi_n} V''_n, \quad (6.3)$$

which gives another sequence of independent standard normal variables.

Let us also denote by $(S_n)_{n \geq 1}$ an arbitrary ordering of all jump times of Y , consisting of stopping times taking values in $(0, 1] \cup \{\infty\}$: if $F(\mathbb{R}) < \infty$ we may choose the sequence S_n to be increasing and the variable $K = \inf\{n : S_n > 1\}$ is a.s. finite; otherwise the S_n 's cannot be ordered as an increasing sequence.

Now we are ready to describe the limiting processes for $\sqrt{n}Z^n$ and $\sqrt{n}U^n$. First, the limit of $\sqrt{n}Z^n$ will be

$$Z_t = \frac{c}{\sqrt{2}} W_t + \sqrt{c} \sum_{n \geq 1} V_n \Delta Y_{S_n} 1_{[S_n, 1]}(t). \quad (6.4)$$

Note that $Z = 0$ if $c = 0$. Since $\sum_{n \geq 1} (\Delta Y_{S_n})^2 < \infty$ a.s., it is not difficult to check that the last sum in (6.4) converges in \mathbb{L}^2 , conditionally on the σ -field \mathcal{F} , and so converges in probability. There is another (more abstract) way of describing Z : it is, conditionally on \mathcal{F} , a Gaussian martingale null at 0 and with angle bracket (this bracket is not an (\mathcal{F}_t) -predictable process, but conditionally on \mathcal{F} it becomes deterministic)

$$\langle Z, Z \rangle_t = \frac{c^2 t}{2} + c \sum_{s \leq t} (\Delta Y_s)^2 \quad (6.5)$$

or equivalently, it is a Gaussian centered process with covariance function $(s, t) \rightsquigarrow \langle Z, Z \rangle_{s \wedge t}$ as given in (6.5). That these two descriptions characterize the same process (conditionally on \mathcal{F}) is easy, and it shows in particular that (6.4) does not depend on the particular choice of the sequence (S_n) .

Next the limit of $\sqrt{n}U^n$ will be the unique solution of the following linear equation:

$$U_t = \int_0^t f'(X_{s-}) U_{s-} dY_s - \bar{Z}(f)_t, \quad (6.6)$$

where

$$\begin{aligned} \bar{Z}(f)_t = & \frac{c}{\sqrt{2}} \int_0^t (f f')(X_{s-}) dW_s + \sqrt{c} \sum_{n: S_n \leq t} [\sqrt{\chi_n} V_n'(f f')(X_{S_n-}) \\ & + \sqrt{1 - \chi_n} V_n'' f(X_{S_n-}) \int_0^1 f'(X_{S_n-} + u \Delta X_{S_n}) du] \Delta Y_{S_n}. \end{aligned} \quad (6.7)$$

Exactly as in (4.14), we may also write $\bar{Z}(f)$ as

$$\begin{aligned} \bar{Z}(f)_t = & \frac{c}{\sqrt{2}} \int_0^t (f f')(X_{s-}) dW_s + \sqrt{c} \sum_{n: S_n \leq t} [\sqrt{\chi_n} V_n' f'(X_{S_n-}) \Delta X_{S_n} \\ & + \sqrt{1 - \chi_n} V_n'' (f(X_{S_n}) - f(X_{S_n-}))]. \end{aligned} \quad (6.8)$$

Like in (6.4), the series on the right side of (6.7) and (6.8) are converging in measure. As for Z , another more abstract way of describing $\bar{Z}(f)$ is that, conditionally on \mathcal{F} , it is a Gaussian martingale null at 0 and with bracket

$$\langle \bar{Z}(f), \bar{Z}(f) \rangle_t = \frac{c^2}{2} \int_0^t (f f')(X_s) dW_s + \frac{c}{2} \sum_{s \leq t} (f'(X_{s-})^2 (\Delta X_s)^2 + (f(X_s) - f(X_{s-}))^2). \quad (6.9)$$

Here comes the main result of this section. We exclude the following two simple cases:

- $F = 0$ and $c = 0$ (i.e. $Y_t = bt$): then by Theorem 4.1 the sequence (nZ^n) has UCV and converges uniformly to $Z_t = b^2 t/2$, and the sequences nU^n and $n\bar{U}^n$ converge uniformly to the unique solution of $dU_t = (U_t f'(X_t) b - f(X_t) f'(X_t) b^2/2) dt$ starting at 0 (we are here in the case of an ordinary, non-random, differential equation).
- $F = 0$ and $c > 0$: then by Theorem 5.5 the sequence $(\sqrt{n}Z^n)$ has UCV and converges stably in law to $Z = cW/\sqrt{2}$ (as in (6.4) with $S_n = \infty$ for all n , i.e. the last sum in (6.4) disappears), and the sequences $\sqrt{n}U^n$ and $\sqrt{n}\bar{U}^n$ converge stably in law to the unique solution of $dU_t = U_t f'(X_t) dY_t - f(X_t) f'(X_t) dZ_t$ starting at 0 by Theorem 3.2.

Theorem 6.1. *Assume that Y is a Lévy process such that $F \neq 0$. Let x_0 be any starting point and f be any C^1 function with linear growth, and consider Equation (3.1). The sequences $(\sqrt{n}Z^n)$ and $(\sqrt{n}U^n)$ have UCV, and we have:*

- a) If $c = 0$, the sequences $(\sqrt{n}Z^n)$, $(\sqrt{n}U^n)$ and $(\sqrt{n}\bar{U}^n)$ weakly converge to 0.
- b) If $c > 0$, the sequence $(\sqrt{n}\bar{Z}^n, \sqrt{n}\bar{U}^n)$ stably converge in law to (Z, U) , as given by (6.4) and (6.6)-(6.7), and the sequence $(\sqrt{n}Z^n, \sqrt{n}U^n)$ stably converge in finite-dimensional laws to the same limit.

In case (b) above, the sequence $\sqrt{n}Z^n$ is *not* tight: if it were, by taking a convergent subsequence we could apply Theorem 3.2, but then the limit of $(\sqrt{n}U^n)$ would be given by (3.7), which is *not* the same equation as (6.6).

The proof of Theorem 6.1 will go through several steps.

Step 1: Suppressing big jumps. Here we assume that Theorem 6.1 holds for Lévy processes having bounded jumps. Let Y be an arbitrary Lévy process, and set for any $p > 1$:

$$Y(p)_t = bt + Y_t^c + x1_{\{|x| \leq 1\}} \star (\mu - \nu) + x1_{\{1 < |x| \leq p\}} \star \mu.$$

Let $X(p)$ be the solution of (3.1) relative to $Y(p)$, and let $Z(p)^n$, $U(p)^n$, $Z(p)$ and $U(p)$ the processes associated with $Y(p)$ and $X(p)$. Observe that in the definition of $Z(p)$ and $U(p)$ we can use the same sequence of stopping times (S_n) and the same terms (W, V_n', V_n'', χ_n) as for Z and U .

Let also $\Omega_p = \{\omega : |\Delta Y_s| \leq p \forall s \in (0, 1]\}$. Then Ω_p increases to Ω , while on Ω_p we have the following equalities between processes: $X(p) = X$, $Z(p)^n = Z^n$, $U(p)^n = U^n$, $Z(p) = Z$, $U(p) = U$. Since our theorem holds for each $Y(p)$, it follows that it also holds for Y .

Therefore, from now on we assume that *the jumps of Y are bounded by a constant*, i.e. the measure F has compact support.

Step 2: The UCV property. For a moment, let M, N be two martingales with angle brackets $\langle M, M \rangle_t = \alpha t$ and $\langle N, N \rangle_t = \beta t$, and set $A_t = \alpha t$. Then

$$E(\langle Z^n(M, N), Z^n(M, N) \rangle_1) = \beta \int_0^1 E((M_s^{(n)})^2) ds = \alpha\beta \int_0^1 (s - [ns]/n) ds = \frac{\alpha\beta}{2n}, \quad (6.10)$$

$$E(\langle Z^n(A, N), Z^n(A, N) \rangle_1) = \beta a^2 \int_0^1 (s - [ns]/n)^2 ds = \frac{\beta a^2}{3n^2}, \quad (6.11)$$

$$E\left(\int_0^1 |dZ^n(M, A)_s|\right) = |a| \int_0^1 E(|M_s^{(n)}|) ds \leq |a| \int_0^1 (\alpha(s - [ns]/n))^{1/2} ds \leq \frac{2|a|\sqrt{\alpha}}{3\sqrt{n}}, \quad (6.12)$$

$$E\left(\int_0^1 |dZ^n(A, A)_s|\right) = a^2 \int_0^1 (s - [ns]/n) ds = \frac{a^2}{2n}. \quad (6.13)$$

Let us come back to Y . Since F has compact support, we can set

$$b' = b + \int_{|x| > 1} xF(dx), \quad \alpha = c + \int x^2 F(dx), \quad B_t = b't, \quad M = Y^c + x \star (\mu - \nu),$$

so that $Y = B + M$, while $\langle M, M \rangle_t = \alpha t$. Then $Z^n = Z^n(M, M) + Z^n(M, B) + Z^n(B, M) + Z^n(B, B)$. The two sequences of local martingales $(\sqrt{n}Z^n(M, M))$ and $(\sqrt{n}Z^n(B, M))$ have UCV by (6.10) and (6.11), and the two sequences of processes with finite variation $(\sqrt{n}Z^n(M, B))$ and $(\sqrt{n}Z^n(B, B))$ have UCV by (6.12) and (6.13). Hence the sequence $(\sqrt{n}Z^n)$ has UCV, as well as $(\sqrt{n}U^n)$ by Theorem 3.3.

Step 3: Suppressing small jumps. 1) For $\varepsilon > 0$ we set

$$M^\varepsilon = x1_{\{|x| \leq \varepsilon\}} \star (\mu - \nu), \quad N^\varepsilon = x1_{\{|x| > \varepsilon\}} \star (\mu - \nu), \quad A^\varepsilon = x1_{\{|x| > \varepsilon\}} \star \mu,$$

$$b_\varepsilon = b' - \int_{|x|>\varepsilon} xF(dx), \quad \rho_\varepsilon = \int_{|x|>\varepsilon} x^2F(dx), \quad B_t^\varepsilon = b_\varepsilon t.$$

Then (6.1) readily yields

$$Y = B + Y^c + M^\varepsilon + N^\varepsilon = B^\varepsilon + Y^c + M^\varepsilon + A^\varepsilon.$$

A simple computation, using the bilinearity of $(U, V) \rightsquigarrow Z^n(U, V)$, gives

$$\sqrt{n}Z^n = F^{n,\varepsilon} + G^{n,\varepsilon}, \tag{6.14}$$

where

$$F^{n,\varepsilon} = H^{n,\varepsilon} + I^{n,\varepsilon}, \quad G^{n,\varepsilon} = J^{n,\varepsilon} + K^{n,\varepsilon} + L^{n,\varepsilon},$$

and

$$\begin{aligned} H^{n,\varepsilon} &= \sqrt{n}Z^n(B^\varepsilon + Y^c, B^\varepsilon + Y^c), \\ I^{n,\varepsilon} &= \sqrt{n}(Z^n(A^\varepsilon, Y^c) + Z^n(Y^c, A^\varepsilon)), \\ J^{n,\varepsilon} &= \sqrt{n}(Z^n(M^\varepsilon, M^\varepsilon + N^\varepsilon + Y^c) + Z^n(N^\varepsilon + Y^c + B, M^\varepsilon)), \\ K^{n,\varepsilon} &= \sqrt{n}Z^n(N^\varepsilon + B, N^\varepsilon + B), \quad L^{n,\varepsilon} = \sqrt{n}(Z^n(M^\varepsilon, B) - Z^n(B^\varepsilon, B^\varepsilon)). \end{aligned}$$

2) Observe that $\langle N^\varepsilon, N^\varepsilon \rangle_t = \rho_\varepsilon t$ and $\langle Y^c, Y^c \rangle_t = ct$ and $\langle M^\varepsilon, M^\varepsilon \rangle_t = (\rho_0 - \rho_\varepsilon)t$. We deduce from (6.12) and (6.13) that $E(\int_0^1 |dL_s^{n,\varepsilon}|) \leq 2|b'|\sqrt{\rho_0 - \rho_\varepsilon}/3 + b_\varepsilon^2/\sqrt{n}$, so

$$\lim_{\varepsilon \rightarrow 0} \limsup_n E(\int_0^1 |dL_s^{n,\varepsilon}|) = 0. \tag{6.15}$$

Next, use (6.10) and (6.11) to obtain that the local martingale $J^{n,\varepsilon}$ has $\langle J^{n,\varepsilon}, J^{n,\varepsilon} \rangle_1 \leq 6(\rho_0 - \rho_\varepsilon)(\rho_0 + c + b'^2)t$. Therefore, using Doob's inequality,

$$\lim_{\varepsilon \rightarrow 0} \limsup_n E(\langle J^{n,\varepsilon}, J^{n,\varepsilon} \rangle_1) = 0. \tag{6.16}$$

Next, the process $B + N^\varepsilon$ is the sum of a continuous process with finite variation having (4.2) and a process of the form (4.6), and the associated variables α_i^n and $\bar{\beta}_i^n$ (see (4.9) and (4.10)) are bounded by a constant independent of n , and the number of jumps of this process is a Poisson random variable. Then $\sup_n E(\sqrt{n} \int_0^1 |dK_t^{n,\varepsilon}|) < \infty$, and

$$\lim_n E(\int_0^1 |dK_s^{n,\varepsilon}|) = 0 \quad \forall \varepsilon > 0. \tag{6.17}$$

Putting together (6.15), (6.16) and (6.17), we readily deduce that if δ^n is a sequence of predictable processes such that δ^{n*} is tight, then

$$\lim_{\varepsilon \rightarrow 0} \limsup_n P((\delta^n \cdot G^{n,\varepsilon})^* > \eta) = 0 \quad \forall \eta > 0. \tag{6.18}$$

3) We can now prove (a): suppose that $c = 0$. Then $Y^c = 0$, so $I^{n,\varepsilon} = 0$ and $H^{n,\varepsilon} = \sqrt{n}Z^n(B^\varepsilon, B^\varepsilon)$ converges weakly to 0 by Theorem 4.1 for each $\varepsilon > 0$. Then $F^{n,\varepsilon} \Rightarrow 0$ as $n \rightarrow \infty$, and combining this with (6.14) and (6.18) with $\delta^n = 1$ yields that $\sqrt{n}Z^n \Rightarrow 0$. Then we can apply Theorem 3.2 to obtain $\sqrt{n}U^n \Rightarrow 0$, and thus (a) holds.

4) From now on we suppose that $c > 0$. Recall that $\sqrt{n}U^n$ is the solution of (3.13), with $\alpha_n = \sqrt{n}$, and introduce the solution V_n of the following linear equation:

$$dV_t^n = V_{t-}^n f'(X_{t-}) dY_t - k(X_t^n) d(\sqrt{n}Z^n)_t, \quad V_0^n = 0. \tag{6.19}$$

As in the proof of Theorem 4.6, Theorems 2.2(d), 3.1 and 2.5(b) and the UCV property of $\sqrt{n}Z^n$ give that $(\sqrt{n}U^n - V^n)^* \xrightarrow{P} 0$.

Next, we introduce the solution $V^{n,\varepsilon}$ of the following linear equation:

$$dV_t^{n,\varepsilon} = V_{t-}^{n,\varepsilon} f'(X_{t-}) dY_t - k(X_t)^n dF_t^{n,\varepsilon}, \quad V_0^{n,\varepsilon} = 0. \quad (6.20)$$

Set also $R_t^n = \int_0^t k(X_s)^n d(\sqrt{n}Z^n)_t$ and $R_t^{n,\varepsilon} = \int_0^t k(X_s)^n dF_s^{n,\varepsilon}$. Lemma 2.4 yields

$$P((V^n - V^{n,\varepsilon})^* > \eta) \leq \varepsilon' + P(f(X)^* > A) + P(R^{n*} > u) + P((R^n - R^{n,\varepsilon})^* > w) + \frac{w}{\eta} K_{A,\varepsilon'}$$

for a constant $K_{A,\varepsilon'}$ depending on A, ε' . Since $\sqrt{n}Z^n$ has UCV, and $k(X^n)^*$ is tight, Theorem 2.2(c) shows that R^{n*} is tight. On the other hand, $R_t^n - R_t^{n,\varepsilon} = \int_0^t k(X_s)^n dG_s^{n,\varepsilon}$, so (6.18) implies that $\lim_{\varepsilon \rightarrow 0} \limsup_n P((R^n - R^{n,\varepsilon})^* > w) = 0$ for all $w > 0$. Thus one readily deduces, by taking ε' arbitrary, then A and u big, then w small, that

$$\lim_{\varepsilon \rightarrow 0} \limsup_n P((V^n - V^{n,\varepsilon})^* > \eta) = 0$$

for all $\eta > 0$. In view of what precedes, we thus obtain

$$\lim_{\varepsilon \rightarrow 0} \limsup_n P((\sqrt{n}U^n - V^{n,\varepsilon})^* > \eta) = 0 \quad \forall \eta > 0. \quad (6.21)$$

On the other hand, define $Z(\varepsilon)$ and $\bar{Z}(\varepsilon, f)$ by (6.4) and (6.7) (or (6.8)), except that in the sum of the right side we add the indicator function of the set $\{|\Delta Y_{S_n}| > \varepsilon\}$. It is easy to check that

$$\lim_{\varepsilon \rightarrow 0} P((Z - Z(\varepsilon))^* + (\bar{Z}(f) - \bar{Z}(\varepsilon, f))^* > \eta) = 0 \quad \forall \eta > 0. \quad (6.22)$$

Then if $U(\varepsilon)$ is the solution of (6.6) with $\bar{Z}(\varepsilon, f)$ instead of $\bar{Z}(f)$, we also have by (6.22) and Theorem 2.5:

$$\lim_{\varepsilon \rightarrow 0} P((U - U(\varepsilon))^* > \eta) = 0 \quad \forall \eta > 0. \quad (6.23)$$

Putting together (6.18), (6.21), (6.22) and (6.23), we see that in order to obtain (a), it is enough to prove that for each $\varepsilon > 0$, and if $\bar{F}_t^{n,\varepsilon} = F_{[nt]/n}^{n,\varepsilon}$ and $\bar{V}_t^{n,\varepsilon} = V_{[nt]/n}^{n,\varepsilon}$, then

$$(\bar{F}^{n,\varepsilon}, \bar{V}^{n,\varepsilon}) \Rightarrow^{stably} (Z(\varepsilon), U(\varepsilon)), \quad (6.24)$$

$$(F^{n,\varepsilon}, V^{n,\varepsilon}) \text{ stably converges in finite-dimensional law to } (Z(\varepsilon), U(\varepsilon)). \quad (6.25)$$

Step 4: From now on we fix $\varepsilon > 0$. We denote by $0 < T_1 < \dots < T_n < \dots$ the successive jump times of Y with size bigger than ε . The number of T_n with $T_n < \infty$ (or equivalently $T_n \leq 1$) is a Poisson random variable K , with parameter $F(\mathbb{R} \setminus [-\varepsilon, \varepsilon])$. The processes $Z(\varepsilon)$ and $Z(\varepsilon, f)$ are given by

$$Z(\varepsilon)_t = \frac{c}{\sqrt{2}} W_t + \sqrt{c} \sum_{n \geq 1} \zeta_n \Delta Y_{T_n} 1_{[T_n, 1]}(t), \quad (6.26)$$

$$\begin{aligned} \bar{Z}(\varepsilon, f)_t = & \frac{c}{\sqrt{2}} \int_0^t (f f')(X_{s-}) dW_s + \sqrt{c} \sum_{n: T_n \leq t} [\sqrt{\xi_n} \zeta'_n (f f')(X_{T_n-}) \\ & + \sqrt{1 - \xi_n} \zeta''_n f(X_{T_n-}) \int_0^1 f'(X_{T_n-} + u \Delta X_{T_n}) du] \Delta Y_{T_n}, \end{aligned} \quad (6.27)$$

where the family $(W, \xi_n, \zeta'_n, \zeta''_n)$ has the same properties as (W, χ_n, V'_n, V''_n) and

$$\zeta_n = \sqrt{\xi_n} \zeta'_n + \sqrt{1 - \xi_n} \zeta''_n \quad (6.28)$$

(this comes from relabelling the χ_n, V'_n, V''_n 's).

Now, we associate with each T_j the times $T_+(n, j)$ and $T_-(n, j)$ by (4.8), and replace (4.9) by

$$\alpha_j^n = \sqrt{n} \Delta Y_{T_j} (Y_{T_j}^c - Y_{T_-(n, j)}^c), \quad \beta_j^n = \sqrt{n} \Delta Y_{T_j} (Y_{T_+(n, j)}^c - Y_{T_j}^c), \quad \gamma_j^n = \alpha_j^n + \beta_j^n, \quad (6.29)$$

these quantities being 0 if $T_j = \infty$. We also write $\rho_n = (\alpha_j^n, \beta_j^n)_{j \geq 1}$. Finally, set

$$\alpha_j = \sqrt{c \xi_j} \zeta_j' \Delta Y_{T_j}, \quad \beta_j = \sqrt{c(1 - \xi_j)} \zeta_j'' \Delta Y_{T_j}, \quad \gamma_j = \alpha_j + \beta_j = \sqrt{c} \zeta_j \Delta Y_{T_j}, \quad (6.30)$$

and $\rho = (\alpha_j, \beta_j)_{j \geq 1}$.

Lemma 6.2. *We have $(H^{n, \varepsilon}, \rho_n) \Rightarrow^{stably} (\frac{c}{\sqrt{2}} W, \rho)$.*

Proof. For simplicity we write $H = \frac{c}{\sqrt{2}} W$. Let $Y' = Y - Y^c - A^\varepsilon$, and

$$R_t^n = Y_t^c - \sum_{j \leq 1} (Y_{t \wedge T_+(n, j)}^c - Y_{t \wedge T_+(n, j)}^c), \quad S_t^n = H_t^{n, \varepsilon} - \sum_{j \leq 1} (H_{t \wedge T_+(n, j)}^{n, \varepsilon} - H_{t \wedge T_+(n, j)}^{n, \varepsilon}).$$

In order to prove the result, we need to show that

$$E(h(Y)g(H^{n, \varepsilon}, \rho_n)) \rightarrow E(h(Y)g(H, \rho))$$

for all bounded functions h and uniformly continuous bounded functions g . By the same density argument as in Lemma 2.1, it is enough to prove this when $h(Y) = u(Y')v(Y^c)w(A^\varepsilon)$, where u, v, w are bounded functions, with in addition v continuous. Now, clearly $(R^n - Y^c)^* \rightarrow 0$, and Theorem 5.5 yields $H^{n, \varepsilon} \Rightarrow^{stably} H$, so $(S^n - H^{n, \varepsilon})^* \rightarrow^P 0$. Hence it suffices to prove that

$$E(u(Y')v(R^n)w(A^\varepsilon)g(S^n, \rho_n)) \rightarrow E(u(Y')v(Y^c)w(A^\varepsilon)g(H, \rho)) \quad (6.31)$$

In fact it is even enough to prove (6.31) when w depends only on the k first jump times and sizes of A^ε , and g depends only on S^n and on the k first variables α_j^n and β_j^n . Further, the set Ω_n (depending also on k) on which each interval $(\frac{i-1}{n}, \frac{i}{n}]$ contains at most one T_j tends to Ω , hence we can put the indicator function in the left expectation of (6.31). So it remains to prove that

$$E[u(Y')v(R^n)w((T_j, \Delta Y_{T_j})_{1 \leq j \leq k})g(S^n, (\alpha_j^n, \beta_j^n)_{1 \leq j \leq k})1_{\Omega_n}] \rightarrow E[u(Y')v(Y^c)w((T_j, \Delta Y_{T_j})_{1 \leq j \leq k})g(H, (\alpha_j, \beta_j)_{1 \leq j \leq k})]. \quad (6.32)$$

Now, (6.29) and the independence of the increments of Y^c over all the intervals $(T_-(n, j), T_+(n, j)]$ from all the other random terms appearing in the left side of (6.32) yield that in this left side we can replace α_j^n and β_j^n by $\sqrt{c \xi_j^n} \zeta_j' \Delta Y_{T_j}$ and $\sqrt{c(1 - \xi_j^n)} \zeta_j'' \Delta Y_{T_j}$, where $\xi_j^n = n(T_j - T_-(n, j))$ is the fractional part of nT_j . Using once more $(R^n - Y^c)^* \rightarrow 0$ and $(S^n - H^{n, \varepsilon})^* \rightarrow^P 0$, we then see that this left side has the same limit as

$$E[u(Y')v(Y^c)w((T_j, \Delta Y_{T_j})_{1 \leq j \leq k})g(H^{n, \varepsilon}, (\sqrt{c \xi_j^n} \zeta_j' \Delta Y_{T_j}, \sqrt{c(1 - \xi_j^n)} \zeta_j'' \Delta Y_{T_j})_{1 \leq j \leq k})1_{\Omega_n}].$$

Now if F_k and G_k denote the laws of the k -uplets $(\Delta Y_{T_j})_{1 \leq j \leq k}$ and $(T_j)_{1 \leq j \leq k}$ (which are independent), the previous expression becomes (with $\{u\}$ denoting the fractional part of u):

$$\int F_k(dx_1, \dots, dx_k) G_k(dt_1, \dots, dt_k) 1_{\cap_{1 \leq i \leq k} \{[nt_i] < [nt_{i+1}]\}} \\ E[u(Y')v(Y^c)w((t_j, x_j)_{1 \leq j \leq k})g(H^{n, \varepsilon}, (\sqrt{c\{t_j/n\}} \zeta_j' x_j, \sqrt{c(1 - \{t_j/n\})} \zeta_j'' x_j)_{1 \leq j \leq k})].$$

Now we can use the property $H^{n,\varepsilon} \Rightarrow^{stably} H$ and the uniform continuity of g to get that the above has the same limit as

$$\begin{aligned} & \int F_k(dx_1, \dots, dx_k) G_k(dt_1, \dots, dt_k) \mathbf{1}_{\cap_{1 \leq i < k} \{[nt_i] < [nt_{i+1}]\}} \\ & E[u(Y')v(Y^c)w((t_j, x_j)_{1 \leq j \leq k})g(H, (\sqrt{c\{t_j/n\}}\zeta'_j x_j, \sqrt{c(1 - \{t_j/n\})}\zeta''_j x_j)_{1 \leq j \leq k})] \\ & = E[u(Y')v(Y^c)w((T_j, \Delta Y_{T_j})_{1 \leq j \leq k})g(H, (\sqrt{c\xi_j^n}\zeta'_j x_j, \sqrt{c(1 - \xi_j^n)}\zeta''_j x_j)_{1 \leq j \leq k})]. \end{aligned}$$

At this point, we need to prove that the above converges to the right side of (6.31). Because T_1, \dots, T_k is independent of $Y^c, Y', H, \zeta'_j, \zeta''_j, \Delta Y_{T_j}$, this amounts to prove that $(\xi_j^n)_{1 \leq j \leq k} \Rightarrow^{stably} (\xi_j)_{1 \leq j \leq k}$. But, since G_k has a regular density on its support, a trivial extension of 1-dimensional results of Tukey [15], see also [2], gives this property.

Step 5: Now we turn to the proof of (6.24) and (6.25), which will reproduce the proof of Theorem 5.1 in a more complicated situation.

1) The sets Ω_n on which each interval $(\frac{i-1}{n}, \frac{i}{n}]$ contains at most one T_j tend to Ω . Then similarly to (4.18)-(4.20) we set

$$\begin{aligned} W(j)_t^n &= \overline{V}_{t \wedge T_+(n,j)}^{n,\varepsilon}, \\ W(j)_t'^n &= \begin{cases} W(j)_t^n & \text{if } t \leq T_+(n,j) \\ W(j)_{T_+(n,j)}^n + V_t^{n,\varepsilon} - V_{T_+(n,j)}^{n,\varepsilon} & \text{if } T_+(n,j) < t < T_{j+1} \\ W(j)_{T_+(n,j)}^n + V_{T_{j+1}-}^{n,\varepsilon} - V_{T_+(n,j)}^{n,\varepsilon} & \text{if } T_{j+1} \leq t, \end{cases} \\ W(j)_t &= U(\varepsilon)_{t \wedge T_j}, \quad W(j)_t' = \begin{cases} U(\varepsilon)_t & \text{if } t < T_{j+1} \\ U(\varepsilon)_{T_{j+1}-} & \text{if } t \geq T_{j+1}. \end{cases} \end{aligned}$$

We also denote by $F(j)_t^n, F(j)_t'^n, F(j)_t$ and $F(j)_t'$ the processes obtained by replacing above $(\overline{V}^{n,\varepsilon}, V^{n,\varepsilon}, U(\varepsilon))$ by $(\overline{F}^{n,\varepsilon}, F^{n,\varepsilon}, Z(\varepsilon))$. We consider the property (recall that $H = \frac{c}{\sqrt{2}}W$):

$$(H_j) \quad (\rho^n, H^{n,\varepsilon}, (F(j)_t^n, W(j)_t^n)) \Rightarrow^{stably} (\rho, H, (F(j)_t, W(j)_t)).$$

Observe that (H_0) holds by Lemma 6.2. If we have (H_j) for all j then (6.24) holds, because $K < \infty$ a.s.

2) Suppose that (H_j) holds. Let $H(n, j)$ be the interval $(T_+(n, j), T_{j+1})$. Then (see (6.21)):

$$F(j)_t'^n = F(j)_t^n + \int_0^t \mathbf{1}_{H(n,j)}(s) dH_s^{n,\varepsilon}, \quad (6.33)$$

$$W(j)_t'^n = W(j)_t^n + \int_0^t W(j)_{s-}'^n f'(X_{s-}) \mathbf{1}_{H(n,j)}(s) dY_s^c - \int_0^t k(X)_s^n \mathbf{1}_{H(n,j)}(s) dH_s^{n,\varepsilon}. \quad (6.34)$$

Then (4.22) holds with

$$\begin{aligned} J_t^n &= W(j)_t^n - \int_0^t k(X)_s^n \mathbf{1}_{H(n,j)}(s) dH_s^{n,\varepsilon}, \quad J_t = W(j)_t - \int_0^t f(X_{s-}) f'(X_{s-}) \mathbf{1}_{(T_j, T_{j+1})}(s) dH_s, \\ L_t^n &= \int_0^t f'(X_{s-}) \mathbf{1}_{H(n,j)}(s) dY_s^c, \quad L_t = \int_0^t f'(X_{s-}) \mathbf{1}_{(T_j, T_{j+1})}(s) dY_s^c. \end{aligned}$$

Clearly $(L^n - L)^* \rightarrow^P 0$, while the sequence $k(X)_t^n$ is bounded (in n and t) by a finite random variable and converges to $(ff')(X_{s-})$ at each continuity point of Y , while the sequence

$H^{n,\varepsilon}$ has UCV and these processes are continuous: so $(J^n - J'^n)^* \rightarrow^P 0$ if $J'_t^n = W(j)_t^n - \int_0^t (ff')(X_{s-}) 1_{H(n,j)}(s) dH_s^{n,\varepsilon}$. Then (H_j) yields $(J^n, L^n, \rho^n, H^{n,\varepsilon}) \Rightarrow^{stably} (J, L, \rho, H)$, so Theorem 2.5(c) and (6.33) give $((F(j)^{n,\varepsilon}, W(j)^{n,\varepsilon}), \rho^n, H^{n,\varepsilon}) \Rightarrow^{stably} ((F(j)', W(j)'), \rho, H)$. Therefore if $\overline{F}(j)_t^n = F(j)_{[nt]/n}^n$ and $\overline{W}(j)_t^n = W(j)_{[nt]/n}^n$ we get

$$\begin{aligned} & ((\overline{F}(j)^n_{\cdot \wedge T_-(n,j+1)}, \overline{W}(j)^n_{\cdot \wedge T_-(n,j+1)}), F(j)^n_{T_{j+1}-}, W(j)^n_{T_{j+1}-}, \rho^n, H^{n,\varepsilon}) \\ & \Rightarrow^{stably} ((F(j)', W(j)'), F(j)'_{T_{j+1}-}, W(j)'_{T_{j+1}-}, \rho, H). \end{aligned} \quad (6.35)$$

3) Set $\mu_n := F_{T_+(n,j+1)}^{n,\varepsilon} - F_{T_-(n,j+1)}^{n,\varepsilon}$ and $\delta_n := V_{T_+(n,j+1)}^{n,\varepsilon} - V_{T_-(n,j+1)}^{n,\varepsilon}$. On the set $\Omega_n \cap \{T_j \leq 1\}$ we have $\mu_n = a_n + \gamma_{j+1}^n$ and $\delta_n = u_n + v_n$, where (with $\overline{Y} = Y - A^\varepsilon$):

$$\begin{aligned} a_n &= H_{T_+(n,j+1)}^{n,\varepsilon} - H_{T_-(n,j+1)}^{n,\varepsilon}, \quad u_n = \int_{T_-(n,j+1)}^{T_+(n,j+1)} V_{s-}^{n,\varepsilon} f'(X_{s-}) d\overline{Y}_s - \int_{T_-(n,j+1)}^{T_+(n,j+1)} k(X)_s^n dH_s^{n,\varepsilon}, \\ v_n &= \Delta Y_{T_{j+1}} (V_{T_{j+1}-}^{n,\varepsilon} f'(X_{T_{j+1}-}) - k(X)_{T_{j+1}}^n \alpha_{j+1}^n) - \int_{T_{j+1}}^{T_+(n,j+1)} k(X)_s^n dY_s^c. \end{aligned}$$

First, the sequences V^{n*} and $k(X)^{n*}$ are tight, and \overline{Y} is continuous at time T_j , and $H^{n,\varepsilon} \Rightarrow H$ with H continuous, so one deduces that $a_n \rightarrow^P 0$ and $u_n \rightarrow^P 0$. Next the sequences

$$k(X)_{T_{j+1}}^n - (ff')(X_{T_{j+1}-}), \int_{T_{j+1}}^{T_+(n,j+1)} k(X)_s^n dY_s^c - \beta_{j+1}^n f(X_{T_{j+1}-}) \int_0^1 f'(X_{T_{j+1}-} + u\Delta X_{T_{j+1}}) du$$

converge to 0 in probability. Furthermore $F_{T_{j+1}-}^{n,\varepsilon} = F(j)'_{T_{j+1}-}$ and $V_{T_{j+1}-}^{n,\varepsilon} = W(j)'_{T_{j+1}-}$. Thus if

$$\delta = \Delta Y_{T_{j+1}} (W(j)'_{T_{j+1}-} f'(X_{T_{j+1}-}) - \alpha_{j+1} (ff')(X_{T_{j+1}-}) - \beta_{j+1} \int_0^1 f'(X_{T_{j+1}-} + u\Delta X_{T_{j+1}}) du),$$

one deduces from (6.35) that

$$\begin{aligned} & ((\overline{F}(j)^n_{\cdot \wedge T_-(n,j+1)}, \overline{W}(j)^n_{\cdot \wedge T_-(n,j+1)}), \mu_n, \delta_n, \rho^n, H^{n,\varepsilon}) \\ & \Rightarrow^{stably} ((F(j)', W(j)'), \gamma_{j+1}, \delta, \rho, H). \end{aligned} \quad (6.36)$$

But $F(j+1)^n = \overline{F}(j)^n_{\cdot \wedge T_-(n,j+1)} + \mu_n 1_{[T_+(n,j+1), 1]}$ and $W(j+1)^n = \overline{W}(j)^n_{\cdot \wedge T_-(n,j+1)} + \delta_n 1_{[T_+(n,j+1), 1]}$, while $F(j+1) = F(j)'_{\cdot \wedge T_{j+1}-} + \gamma_{j+1} 1_{[T_{j+1}, 1]}$ and $W(j+1) = W(j)'_{\cdot \wedge T_{j+1}-} + \delta 1_{[T_{j+1}, 1]}$. Thus (6.36) yields (H_{j+1}) , and the proof of (6.24) follows by induction on j .

4) Finally on the set $\{T_+(n,j) < t < T_-(n,j+1)\}$ we have $F_t^{n,\varepsilon} = F(j)'_t^n$, $Z(\varepsilon)_t = F(j)'_t$, $V_t^{n,\varepsilon} = W(j)'_t^n$ and $U(\varepsilon)_t = W(j)'_t$. Since $(F(j)^n, W(j)^n) \Rightarrow^{stably} (F(j)', W(j)')$ and since $Z(\varepsilon)$ and $U(\varepsilon)$ have no fixed times of discontinuity, we deduce (6.25), and the proof of Theorem 6.1 is finished.

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