DISTRIBUTION OF BROWNIAN LOCAL TIME ON CURVES

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Summary: If f is a smooth nondecreasing function on $[0,1]$, the local time L_f spent by standard Brownian motion on the graph of f satisfies $P(L_f < x) \geq P(2L_0 < x)$, $x > 0$. The 2 is best possible.

Introduction: This paper is concerned throughout with the local time of a standard Brownian motion (B_t) on the graphs of functions f defined on $[0,1]$, f smooth enough so that the paths of $B_t - f(t), 0 \leq t \leq 1$, are Brownian paths. Girsanov's theorem characterizes these functions. The local time spent by (B_t) on the graph of f up to time s is defined as the local time spent by $(B_t - f(t))$ at 0 up to time s, and designated $L_f(s)$. We shorten $L_f(1)$ to L_f.

We use $\overset{=}{\sim}$ to designate equality of distribution of random variables, and $X \overset{<}{\sim} Y$ means $P(X < t) \geq P(Y < t), t \leq r$. The purpose of this note is to answer a question raised by C. Burdzy and J. San Martin in [1], namely, is $L_f \overset{<}{\sim} L_0$ whenever f is nondecreasing. The answer is, almost. We prove

Theorem 1. *If f is nondecreasing on $[0,1]$, then $L_f \overset{<}{\sim} 2L_0$. The constant 2 is the smallest possible.*

Of course, the distribution of L_0 is the distribution of $max_{0 \leq t \leq 1} B_t$, and is explicitly known. It is easy to construct rapidly oscillating functions which show that no distributional comparison along the lines of Theorem 1 is possible, if the restriction that f be nondecreasing is entirely removed.

Proof of Theorem 1. Let $(\mathcal{F}_t)$ be the filtration of (B_t). First we give the examples which verify the last sentence of Theorem 1. The joint density of $(L_t/\sqrt{t}, B_t/\sqrt{t})$, which is the same for all t, is given on page 45 of [2], and it is easy to use this to show

(1) $$P(B_t > a\sqrt{t} | L_0(t) < \epsilon\sqrt{t}) \to 0 \text{ as } a \to \infty,$$

uniformly in $\varepsilon > 0$.

Let $\alpha_n = 1 - n^{-1}$, and define the functions g_n and h_n on $[0,1]$ by $g_n = n^2 t$, and $h_n(t) = 0$, $0 \le t \le \alpha_n, h_n(t) = n^2(t - \alpha_n), \alpha_n \le t \le 1$. Let $\varepsilon_n > 0$ be so small that

$$P(L_{g_n}(n^{-2}) > \varepsilon_n) > \alpha_n. \tag{2}$$

Put τ_n=inf$(t > \alpha_n : B_t = h_n(t))$, and define

$$A_n = \{\tau_n < 1 - n^{-2}\},$$
$$C_n = \{L_0(\alpha_n) < \varepsilon_n\}, \text{ and}$$
$$D_n = \{0 < B_{\alpha_n} < \sqrt{n}\}.$$

There are constants $z_n, n \ge 1$, such that

$$z_n < P(A_n | \mathcal{F}_{\alpha_n}) < 1 \text{ on } D_n, \text{ and } z_n \to 1. \tag{3}$$

By (2), there are constants $w_n, n \ge 1$, such that

$$w_n < P(D_n | \mathcal{F}_{\alpha_n}) < \tfrac{1}{2} \text{ on } C_n, \text{ and } w_n \to \tfrac{1}{2}. \tag{4}$$

Put $\Gamma_n = \{L_{hn} - L_{hn}(\alpha_n) > \epsilon_n\}$, and note, by (2) (and the strong Markov property) that

$$P(\Gamma_n | \mathcal{F}_{\tau_n}) > \alpha_n \text{ on } A_n. \tag{5}$$

Now $P(L_{h_n} < \epsilon_n) \le P(C_n) - P(C_n \cap \Gamma_n)$, and $P(C_n)/P(L_0 < \varepsilon_n) \to 1$ as $n \to \infty$, so to prove

$$\limsup_{n\to\infty} P(L_{h_n} < \varepsilon_n)/P(L_0 < \varepsilon_n) \le \tfrac{1}{2}, \tag{6}$$

it suffices to prove

$$\limsup_{n\to\infty} P(\Gamma_n | C_n) \ge \tfrac{1}{2}. \tag{7}$$

We have

$$\begin{aligned} P(\Gamma_n \cap C_n) &\ge P(\Gamma_n \cap C_n \cap A_n \cap D_n) \\ &= P(\Gamma_n | C_n \cap A_n \cap D_n) P(A_n | C_n \cap D_n) P(D_n | C_n) P(C_n). \end{aligned} \tag{8}$$

Now

$$\begin{aligned}
&P(\Gamma_n|C_n \cap A_n \cap D_n)\\
&= EP(\Gamma_n|\mathcal{F}_{\tau_n})I(C_n \cap A_n \cap D_n)/P(C_n \cap A_n \cap D_n)\\
&\geq \alpha_n,
\end{aligned}$$

using (5). Also,

$$\begin{aligned}
&P(A_n|C_n \cap D_n)\\
&= EP(A_n|\mathcal{F}_{\alpha_n})I(C_n \cap D_n)/P(C_n \cap D_n)\\
&\geq z_n,
\end{aligned}$$

using (3). In addition (4) implies $P(D_n|C_n) \geq w_n$. These three inequalities together with the estimates obtained for $\alpha_n, z_n,$ and w_n, when plugged into (8), yield (7) and thus (6). And (6), together with the fact that $\lim_{n\to\infty} P(L_0 \leq c\epsilon_n)/P(L_0 \leq \epsilon_n) = c$, for any constant c, show that the functions h_n may be used to verify that the constant 2 of Theorem 1 may not be replaced by a smaller one.

Now the first sentence of Theorem 1 will be proved. For notational convenience, we both assume $f(0) = 0$, and extend f to $[0,\infty)$ by defining $f(t) = f(1), t > 1$. Let $\epsilon > 0$. Put

$$\begin{aligned}
\tau_0^{f,\varepsilon} &= 0,\\
\tau_{2i+1}^{f,\varepsilon} &= \inf\{t \geq \tau_{2i}^{f,\varepsilon} : |B_t - f(t)| = \epsilon\}, i \geq 0, \text{ and}\\
\tau_{2i}^{f,\varepsilon} &= \inf\{t \geq \tau_{2i-1}^{f,\varepsilon} : B_t = f(t)\}, i \geq 1.
\end{aligned}$$

Let $N_{f,\epsilon} = \sup\{k : \tau_{2k} \leq 1\}$. Then $N_{f,\epsilon}$ is the sum of the downcrossings of $[0,\epsilon]$ by $B_t - f(t)$,$0 \leq t \leq 1$, plus the upcrossings of $[-\epsilon, 0]$ by $B_t - f(t)$, $0 \leq t \leq 1$, and thus (see page 222 of [2]),

(9) $$L_f = \lim_{\epsilon\to 0}(\epsilon/2)N_{f,\epsilon}.$$

Let $X \cong \tau_1^{0,1}$,$Y \cong \tau_2^{0,1} - \tau_1^{0,1}$, and let R be a random variable such that $P(R = 1) = P(R = 0) = \frac{1}{2}$. Of course, Y has the distribution of $\inf\{t > 0 : B_t = -1\}$. Let $X_i, i \geq 1, \tilde{X}_i, i \geq 1, Y_i, i \geq 1$, and $R_i, i \geq 1$, be independent sequences of independent and identically distributed random variables, the individual variables having the distributions of X, X, Y, and R respectively.

If $\Gamma_i, i \geq 1$, are nonnegative random variables, and $t > 0$, we put $\Theta_\Gamma(t) = \sup\{k > 0 : \sum_{i=1}^k \Gamma_i \leq t\}$. Put $Z_i = X_i + \tilde{X}_i + 4Y_iR_i, i \geq 1$, and $W_i = 4Y_iR_i, i \geq 1$.

We claim

(10) $$\Theta_Z(\varepsilon^{-2}) \overset{=}{\sim} N_{0,\varepsilon},$$

and

(11) $$\Theta_W(\varepsilon^{-2}) \overset{>}{\sim} N_{f,2\varepsilon}.$$

Before proving (10) and (11), we show how they can be used to prove the first statement of Theorem 1. It is not difficult to use $EX_i = E\tilde{X}_i < \infty$, and $EY_iR_i = \infty$, to show

$$\lim_{t\to\infty} \Theta_Z(t)/\Theta_W(t) = 1 \text{ a.s..}$$

From this, scaling, (9), (10), and (11), we have

$$\begin{aligned} P(L_f < x) &= \lim_{\epsilon\to 0} P(\epsilon N_{f,2\epsilon} < x) \\ &\geq \lim_{\epsilon\to 0} P(\epsilon\Theta_W(\epsilon^{-2}) < x) \\ &= \lim_{\epsilon\to 0} P(\epsilon\Theta_Z(\epsilon^{-2}) < x) \\ &= P(L_0 < x/2), \end{aligned}$$

the second equality since $\lim_{\varepsilon\to 0} \varepsilon\Theta_Z(\varepsilon^{-2})$ has a continuous distribution, namely, the distribution of L_0.

Now we prove (10). Let

$$v_i^{0,\varepsilon} = \inf\{t \geq \tau_{2i-1}^{0,\varepsilon}\colon\ |B_t - B_{\tau_{2i-1}^{0,\varepsilon}}| = \varepsilon\}.$$

Then $\varepsilon^{-2}(\tau_{2i-1}^{0,\varepsilon} - \tau_{2i-2}^{0,\varepsilon}) \overset{\sim}{=} X$, $\varepsilon^{-2}(v_i^{0,\varepsilon} - \tau_{2i-1}^{0,\varepsilon}) = X$, and $\varepsilon^{-2}(\tau_i^{0,\varepsilon}) \overset{\sim}{=} 4Y_iR_i$, the last equality since $\tau_{2i}^{0,\varepsilon} = v_i^{0,\varepsilon}$ with probability $\frac{1}{2}$, and $|B_{v_i^{0,\varepsilon}}| = 2\varepsilon$ with probability $\frac{1}{2}$. Thus $\varepsilon^{-2}(\tau_{2i}^{0,\varepsilon} - \tau_{2i-2}^{0,\varepsilon}) \overset{\sim}{=} X_i + \tilde{X}_i + 4R_iY_i$, all the independence being provided by the strong Markov property. The strong Markov property also provides the additional independence necessary to extend this last equality to (10).

Finally (11) is proved. Since f is nondecreasing, the conditional probability, given $\mathcal{F}_{\tau_{2i}^{f,2\varepsilon}}$, that $B_{\tau_{2i+1}^{f,2\varepsilon}} = f(B_{\tau_{2i+1}^{f,\varepsilon}}) - 2\varepsilon$, exceeds $\frac{1}{2}$. Furthermore, again because f is nondecreasing, if this happens, then $\tau_{2i+2}^{f,2\varepsilon} \geq \inf\{t > \tau_{2i+1}^{f,2\varepsilon}\colon\ B_t = f(B_{\tau_{2i-2}^{f,2\varepsilon}}) + 2\varepsilon\}$, the only possibility of equality being if f is linear on some interval. Thus $\varepsilon^{-2}(\tau_{2i+2}^{f,2\varepsilon} - \tau_{2i}^{f,2\varepsilon}) \gtrsim 4Y_i R_i$. Furthermore this distributional inequality holds conditionally given $\mathcal{F}_{\tau_{2i}^{f,2\varepsilon}}$, and now (11) follows.

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References

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[2] Itô, K., and McKean, H.P. Diffusion Processes and their sample paths. Springer Verlag, Berlin, 1965.