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ON A HOMOGENEOUS TREE

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ABSTRACT. A conjecture of Liggett [9] concerning the regime of *weak survival* for the contact process on a homogeneous tree is proved. The conjecture is shown to imply that the Hausdorff dimension of the *limit set* of such a contact process is no larger than half the Hausdorff dimension of the space of ends of the tree. The conjecture is also shown to imply that at the boundary between weak survival and strong survival, the contact process survives only weakly, a theorem previously proved by Zhang [13]. Finally, a stronger form of a theorem of Hawkes and Lyons concerning the Hausdorff dimension of a Galton-Watson tree is proved.

1. INTRODUCTION

This paper concerns the growth of an isotropic contact process on an infinite homogeneous tree. The process was introduced by Pemantle [12], and discussed at some length by Liggett in the 1996 Wald Memorial Lectures [9] (see also [10]). An *isotropic contact process* on the homogeneous tree $\mathcal{T} = \mathcal{T}_d$ of degree $d + 1$ is a continuous time Markov process A_t on the set of finite subsets of (the vertex set of) \mathcal{T} that evolves as follows. Infected sites (members of A_t) recover at rate 1 and upon recovery are removed from A_t ; healthy sites (members of A_t^c) become infected at rate ρ times the number of infected neighbors, and upon infection are added to A_t . Under the default probability measure P , the initial state A_0 is the singleton set $\{e\}$ (where e is a distinguished element of \mathcal{T} called the “root”); under P^x , the initial state A_0 is the singleton set $\{x\}$. The neighborhood system on \mathcal{T} is the usual one: each $x \in \mathcal{T}$ has exactly $d + 1$ neighbors.

The (isotropic) contact process on a homogeneous tree of degree 3 or greater differs from the contact process on an integer lattice \mathbb{Z}^d in that there exist two essentially different survival phases (cf. [12],[9]). In detail, there exist critical constants $0 < \rho_1 < \rho_2 < \infty$ such that

- (1) If $\rho < \rho_1$ then $A_t = \emptyset$ eventually, w.p.1.
- (2) If $\rho_1 < \rho < \rho_2$ then $P\{|A_t| \rightarrow \infty\} > 0$, but for each $x \in \mathcal{T}$,
$$P\{x \in A_t \text{ for arbitrarily large } t\} = 0.$$
- (3) If $\rho > \rho_2$ then with positive probability $|A_t| \rightarrow \infty$ and for all $x \in \mathcal{T}$, $x \in A_t$ for arbitrarily large values of t .

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For $\rho < \rho_1$ the process is *subcritical*; for $\rho_1 < \rho < \rho_2$ it is *weakly supercritical*; and for $\rho > \rho_2$ it is *strongly supercritical*. The main results of this paper concern only the weakly supercritical phase.

Let $x \in \mathcal{T}$ be a vertex at distance n from the root vertex e . The probability that $x \in A_t$ for *some* $t > 0$ depends only on n , by the isotropy of the process; thus, we may define

$$u_n = P\{x \in A_t \text{ for some } t > 0\}.$$

Observe that if the contact process is weakly supercritical then $u_n < 1$ for all $n \geq 1$, because if $u_n = 1$ then w.p.1 the root e would be re-infected at indefinitely large times, and so the process would not be weakly supercritical. A simple subadditivity argument shows that

$$(1) \quad \lim_{n \rightarrow \infty} u_n^{\frac{1}{n}} = \beta = \beta(\rho)$$

exists, and that $u_n \leq \beta^n$ for all n . The main result of this paper is the following theorem, conjectured by Liggett in [9].

Theorem 1. *If the contact process is weakly supercritical then*

$$(2) \quad \beta \leq \frac{1}{\sqrt{d}}.$$

Theorem 1 will be proved in sections 3-4 below. An immediate consequence of the theorem is the following corollary.

Corollary 1. *For $\rho = \rho_2$ the contact process is weakly supercritical.*

*Proof.*¹ For each $n \geq 1$,

$$\lim_{\rho \uparrow \rho_2} u_n(\rho) = u_n(\rho_2).$$

This may be proved by standard arguments, e.g., using the monotonicity of the “percolation structures” described in the next section. For each $\rho \in (\rho_1, \rho_2)$ and each $n \geq 1$, $u_n(\rho) \leq \beta(\rho)^n \leq d^{-n/2}$, by Theorem 1; hence, $u_n(\rho_2) \leq d^{-n/2}$. Consequently, the contact process at ρ_2 cannot be strongly supercritical, because if it were then for every $n \geq 1$ it would be the case that $u_n(\rho_2) \geq \varepsilon$, where $\varepsilon > 0$ is the survival probability for the process. \square

Our interest in Theorem 1 was prompted by its similarity to a theorem concerning the Hausdorff dimension of the limit set of branching Brownian motion in the hyperbolic plane [7]. To explain the connection, we reformulate Theorem 1 as a result about the “limit set” of the weakly supercritical contact process. Define $\partial\mathcal{T}$ to be the set of “ends” of the tree \mathcal{T} , i.e., $\partial\mathcal{T}$ is the set of all infinite paths in \mathcal{T} beginning at the root e that have no loops. There is a natural family of metrics on $\partial\mathcal{T}$ defined as follows:

$$d_\theta(\gamma, \gamma') = \theta^{-N(\gamma, \gamma')}$$

where $\theta \in (0, 1)$ and $N(\gamma, \gamma')$ is the distance to e in \mathcal{T} of the last point that γ and γ' have in common. Define the *limit set* Λ of the contact process on \mathcal{T} to be the (random) set of all $\gamma \in \partial\mathcal{T}$ such that each vertex of γ is infected at some time. It is easily seen that if the contact process is supercritical then on the event of survival Λ is nonempty and compact (relative to any of the metrics d_θ). In section 5 below we will prove the following theorem:

¹Thanks to TOM LIGGETT for pointing this out.

Theorem 2. For $\rho \in (\rho_1, \rho_2]$ and $\theta \in (0, 1)$, the Hausdorff dimension of the limit set Λ is

$$(3) \quad \delta_H^\theta(\Lambda) = -\frac{\log(d\beta)}{\log \theta}$$

almost surely on the event of survival.

It is well known, and easy to prove, that the Hausdorff dimension of the set of ends (relative to the metric d_θ) is

$$(4) \quad \delta_H^\theta(\partial\mathcal{T}) = -\frac{\log d}{\log \theta}$$

Corollary 2. In the weakly supercritical regime, the Hausdorff dimension of the limit set Λ is never greater than half the Hausdorff dimension of the space $\partial\mathcal{T}$ of ends.

In [7] a completely analogous result was proved for hyperbolic branching Brownian motion: in fact, it was shown that at the boundary of the weak survival/strong survival regimes, the Hausdorff dimension of the limit set equals half the Hausdorff dimension of $\partial\mathbb{H}$. We conjecture that this is also true for the contact process on \mathcal{T} , for *any* of the metrics d_θ : in particular, we conjecture that for $\rho = \rho_2$,

$$(5) \quad \delta_H^\theta(\Lambda) = \frac{1}{2} \delta_H^\theta(\partial\mathcal{T}).$$

2. CONTACT PROCESS: PRELIMINARIES

2.1. The Homogeneous Tree. In the subsequent discussion, we will identify the tree \mathcal{T} with its vertex set, as there will be no need to explicitly refer to the edge set. There is a natural distance function on \mathcal{T} defined as follows: For any two vertices $x, y \in \mathcal{T}$, $d(x, y)$ is the minimum number of edges in a path from x to y . There is a unique path $\gamma_{x,y}$ with this minimum number of edges – it is called the *geodesic segment* with endpoints x and y . For each $x \in \mathcal{T}$ and each integer $n \geq 0$ define $B_n(x)$ to be the set of all vertices y at distance no larger than n from x , and define $\partial B_n(x)$ to be the set of vertices y such that $d(x, y) = n$. Note that for $n \geq 1$,

$$|\partial B_n(x)| = (d+1)d^{n-1}$$

and

$$|B_n(x)| = 1 + \sum_{m=1}^n (d+1)d^{m-1} = 1 + (d+1) \frac{d^n - 1}{d - 1}.$$

As $n \rightarrow \infty$, both cardinalities grow like constant $\times d^n$. It is the exponential growth of $|B_n(x)|$ that accounts for the difference between the behavior of the contact process on \mathcal{T} and the behavior of the contact process on the euclidean lattices \mathbb{Z}^d .

The tree \mathcal{T} is *homogeneous* in the sense that every vertex has exactly $d+1$ neighbors. It is also homogeneous in the sense that for any two vertices x, y there is an isometry that maps x to y . Nevertheless, it is convenient to mark a distinguished vertex e as the *root*; under the default probability measure $P = P^e$, the contact process is initiated at e . For any vertex $x \neq e$, write $|x| = d(e, x)$, and define $\mathcal{T}(x)$ to be the set of all vertices y such that the geodesic segment $\gamma_{e,y}$ passes through x . Observe that if $x, y \neq e$ are two vertices such that $x \notin \mathcal{T}(y)$ and $y \notin \mathcal{T}(x)$, then

$$\mathcal{T}(x) \cap \mathcal{T}(y) = \emptyset.$$

Note also that if $|x| = n$ then

$$\mathcal{T}(x) \cap B_n(e) = \{x\}.$$

2.2. The Contact Process. The contact process on \mathcal{T} may be constructed with the aid of a *percolation structure* on $\mathcal{T} \times (0, \infty)$ (see Harris [4] for a detailed explanation). The percolation structure is a system of independent Poisson processes attached to vertices and ordered pairs of neighboring vertices. For each $x \in \mathcal{T}$ the Poisson process attached to x has rate 1, and determines the recovery times at x : in particular, at every occurrence time site x recovers if it is infected. For each ordered pair (x, y) of neighboring vertices, the Poisson process attached to (x, y) has rate ρ ; the occurrence times are precisely those times when an infection at x may jump to y . For purposes of visualization, it is helpful to imagine that (1) for each vertex x there is a directed ray $\{x\} \times [0, \infty)$ drawn over x ; (2) at each occurrence time t of the Poisson process attached to (x, y) an arrow is drawn from (x, t) to (y, t) in $\mathcal{T} \times (0, \infty)$; and (3) at each occurrence time t of the Poisson process attached to x a mark $*$ is attached to (x, t) . The set A_t may then be described as follows: $y \in A_t$ if and only if there is a (directed) path through the percolation structure (the system of rays and arrows described above) that begins at a vertex in A_0 , ends at (y, t) , and does not pass through any marks $*$. Henceforth, we will refer to any path through the percolation structure (or its projection to \mathcal{T}) that does not pass through a mark $*$ as an *infection trail*.

We will repeatedly make use of three important properties of the contact process and the associated percolation structure. The first is *monotonicity*: if two contact processes A_t, A'_t are built over the same percolation structure, and if $A_0 \subset A'_0$, then for all $t \geq 0$ it must be the case that $A_t \subset A'_t$. In particular, if at some time t all vertices of A_t are erased except one, then the resulting subsequent process will be dominated by the original process. The second property is a *Markov property*: events determined by nonoverlapping parts of the percolation structure are necessarily independent. The third is *isotropy*: if i is an isometry of the tree \mathcal{T} that maps e to x , and if A_t is a contact process initiated by the single infected site e , then $i(A_t)$ is a contact process initiated by the single infected site x .

3. OUTWARD INFECTION TRAILS AND EMBEDDED GALTON-WATSON PROCESSES

Fix $x \in \mathcal{T}$, $x^* \in \mathcal{T}(x)$, and $y \in \mathcal{T}(x^*)$ so that $|x| = m$, $|x^*| = m + 1$, and $|y| = m + k$ (thus, x^* is the second vertex on the geodesic segment from x to y). Define an *outward infection trail* from x to y to be an infection trail that begins at x ; proceeds directly to x^* , after which it does not exit the sector $\mathcal{T}(x^*)$; and first reaches $\partial B_k(x)$ at y , where it terminates. Define

$$\mathcal{I}_{x,y}^+ = \{x \xrightarrow{d} y\} = \{\exists \text{ outward infection trail } x \rightarrow y\}.$$

The dependence on the time at which x is initially infected is suppressed. Implicit in the definition of the event is the understanding that the infection trail starts at x between time t_0 and the first recovery time at x after t_0 . Observe that $\mathcal{I}_{x,y}^+$ depends only on the percolation structure in $B_k(x) \cap (\{x\} \cup \mathcal{T}(x^*))$ after the time of initial infection at x , and hence is independent of events that depend on the complementary part of the percolation structure. By the isotropy of the contact process, $P^x(\mathcal{I}_{x,y}^+)$ depends only on the length m of the geodesic segment from x to y ; thus, for $|x| = m$ we may define

$$w_m = P(\mathcal{I}_{e,x}^+).$$

Proposition 1. $\lim_{m \rightarrow \infty} w_m^{\frac{1}{m}} = \beta$.

Proof. Isotropy, monotonicity, and the strong Markov property imply that $w_{m+k} \geq w_m w_k$, so by the subadditivity lemma, $\lim_{m \rightarrow \infty} w_m^{\frac{1}{m}}$ exists. Obviously, $w_m \leq u_m$, so the limit is no greater than β . It remains to prove that the limit is no *smaller* than β .

For each $x \in \mathcal{T}$ and each integer $n \geq m = |x|$, define H_x^n to be the event that there is an infection trail starting from the root vertex e and terminating at x that does not exit the ball $B_{n-1}(e)$ before reaching x . Define

$${}_n v_m = P^e(H_x^n).$$

The probability cited depends only on n and $|x|$, by the isotropy of the contact process. For each $n \geq |x|$ the event H_x^n is contained in the event that site x is *ever* infected, and so ${}_n v_m \leq u_m$. Moreover, as $n \rightarrow \infty$ the events H_x^n increase to the event that site x is ever infected, so

$$(6) \quad \lim_{n \rightarrow \infty} \uparrow {}_n v_m = u_m.$$

Fix $\varepsilon > 0$, and choose m, k sufficiently large that

$${}_k v_m > \beta^m (1 - \varepsilon)^m.$$

That such a choice is possible follows from the preceding paragraph and the definition of β . Let y be a vertex at distance k from the root e , and define α_k to be the probability that there is a *direct* infection trail starting at e (i.e., one that follows the geodesic segment from e to y). By the isotropy of the process, this probability depends only on $|y| = k$. We claim that for every integer $n \geq 1$,

$$(7) \quad w_{nm+2k} \geq \alpha_k^2 ({}_k v_m)^n.$$

This will imply that

$$\liminf_{n \rightarrow \infty} w_{nm+2k}^{1/(nm+2k)} \geq \beta(1 - \varepsilon).$$

Since w_n is nonincreasing in n and $\varepsilon > 0$ is arbitrary, it will then follow that

$$\liminf_{n \rightarrow \infty} w_n^{1/n} \geq \beta.$$

Choose $x \in \mathcal{T}$ such that $|x| = nm + 2k$, and let x_0, x_1, \dots, x_n be the vertices on the geodesic segment from e to x such that $|x_i| = k + im$. Suppose that (a) there is a direct infection trail from e to x_0 ; (b) for each $i = 0, 1, \dots, n-1$, there is an infection trail from x_i to x_{i+1} , beginning in the initial infection epoch of x_i , that does not exit $B_k(x_i)$; and (c) there is a direct infection trail from x_n to x , beginning in the initial infection epoch of x_n . Then the concatenation of the infection trails (a), (b), and (c) is an outward infection trail from e to x . Consequently, w_{nm+2k} is no smaller than the probability that there are infection trails satisfying (a), (b), and (c). By the Markov property, isotropy, and monotonicity of the contact process, this probability is $\alpha_k ({}_k v_m)^n \alpha_k$. This proves (7). \square

If there is an outward infection trail from an infected site x (beginning at x during the *first* infection epoch of x) to a site $x^* \in \mathcal{T}(x)$, say that x^* is a *descendant* of x . Fix an integer $L \geq 1$ and a vertex y such that $|y| = 1$, and define

$$(8) \quad \mathcal{Z}_0 = \mathcal{Z}_0^L = \{e\},$$

$$(9) \quad \mathcal{Z}_n = \mathcal{Z}_n^L = \cup_{x \in \mathcal{Z}_{n-1}} \{\text{descendants of } x \text{ in } \partial B_{nL}(e)\} - \mathcal{T}(y).$$

Let $Z_n = Z_n^L$ denote the cardinality of \mathcal{Z}_n^L .

Note: Removing the subtree $\mathcal{T}(y)$ compensates for the asymmetry between the vertex e and the other vertices of the tree: in particular, every vertex $x \in \mathcal{T} - \mathcal{T}(y)$ (including e) has exactly d neighbors at distance $|x| + 1$ from e . This is needed for the following result.

Proposition 2. $(Z_n)_{n \geq 0}$ is a Galton-Watson process with mean offspring number $d^L w_L$.

Proof. The event \mathcal{I}_{x,x^*}^+ depends only on the percolation structure in $B_L(x) \cap \mathcal{T}(x)$ after the time of initial infection at x . For distinct vertices x in $\partial B_{nL}(e)$ these regions of the percolation structure do not overlap with each other or with the contact structure inside $B_{nL}(e)$. Consequently, if $N(x)$ is the number of descendants of x in $B_{|x|+L}(e)$, then the random variables $N(x)$, where x ranges over \mathcal{Z}_n , are conditionally independent given $\cup_{k \leq n} \mathcal{Z}_k$, each with the same distribution as $N(e)$. This proves that $(Z_n)_{n \geq 0}$ is a Galton-Watson process. For each vertex $x \in \mathcal{T} - \mathcal{T}(y)$ there are d^m vertices in $\mathcal{T}(x)$ at distance L from x , and for each such vertex x^* , the probability that x^* is a descendant of x is w_L . It follows that the mean offspring number is $d^L w_L$. \square

4. BACKSCATTERING

In this section we will use the existence of embedded Galton-Watson processes to prove the following proposition.

Proposition 3. *Suppose that $d\beta^2 > 1$. Then there exists $\varepsilon > 0$ such that for every $n \geq 1$,*

$$P(F_n) \geq \varepsilon,$$

where F_n is the event that there is an infection trail that starts at e , exits $B_n(e)$, and then returns to e .

Corollary 3. *If $d\beta^2 > 1$ then the contact process is not weakly supercritical.*

Proof. Proposition 3 implies that $P(F) \geq \varepsilon$, where $F = \cap_{n \geq 1} F_n$. On the event F there are infection trails starting at e that wander arbitrarily far away from e and then return to e . But a contact process initiated by the single infected site e can only reach finitely many sites in finite time. Consequently, on the event F , the root vertex e is re-infected at arbitrarily large times, and therefore the contact process is strongly supercritical. \square

Proof of Proposition 3: Assume that $d\beta^2 > 1$, and fix β_* such that $d^{-\frac{1}{2}} < \beta_* < \beta$. Choose an integer L sufficiently large that all of the following are true:

$$(10) \quad u_L \geq w_L > \beta_*^L;$$

$$(11) \quad p_L \triangleq P\{Z_n^L \geq (d\beta_*)^{nL} \forall n \geq 0\} > 0;$$

$$(12) \quad (1 - p_L \beta_*^L / 2)^{(d\beta_*)^L} < 1/2.$$

The existence of such an integer follows from Proposition 1, Proposition 2, and elementary considerations: The definition of w_L implies that $u_L > w_L$, and Proposition 1 implies that for all large L , $w_L > \beta_*^L$. Proposition 2 implies that $(Z_n^L)_{n \geq 0}$ is a Galton-Watson process with mean offspring number $d^L w_L > d^L \beta_*^L$ (for sufficiently large L), so $p_L > 0$, by an elementary fact about Galton-Watson processes. In fact, as $L \rightarrow \infty$ through powers of 2, the probabilities p_L increase, because for $x_* \in \mathcal{T}(x)$ at distance $2L$ from x , doubling L makes it easier for x_* to be a descendant of x . Finally, since $d\beta_*^2 > 1$ and $p_L > \varepsilon > 0$ for all large $L = 2^k$,

$$(d\beta_*)^L \log(1 - p_L \beta_*^L / 2) < -(d\beta_*)^L p_L \beta_*^L / 2 < -\log 2$$

for all large $L = 2^k$, which implies (12).

To finish the proof, we will show that, if $x \rightarrow y$ denotes the event that there is an infection trail from x to y beginning during the first infection epoch at x , then for every integer $n \geq 1$,

$$(13) \quad r_n \triangleq P\{\exists x \in \mathcal{Z}_n = \mathcal{Z}_n^L : x \rightarrow e\} > \frac{p_L \beta_*^L}{2}.$$

Observe that on this event, there exists an infection trail that starts at e , extends to some x at distance nL from e , and then extends back to e , so (13) does in fact prove the proposition. The proof of (13) is by induction on n . For $n = 1$, the event occurs if $Z_1 \geq 1$ and, for a single randomly chosen $x \in \mathcal{Z}_1$, there is an infection trail from x to e beginning at some time between the first infection and the first recovery at x . By (11), the probability that $Z_1 \geq 1$ is at least p_L . Given that $Z_1 \geq 1$, the conditional probability that $x \rightarrow e$ for a randomly chosen $x \in \mathcal{Z}_1$ is u_L , which by (10) is at least β_*^L . Thus,

$$r_1 \geq p_L \beta_*^L.$$

Now suppose that inequality (13) is true for every positive integer smaller than n , where $n \geq 2$. There will be a vertex $x^* \in \mathcal{Z}_n$ such that $x^* \rightarrow e$ if (i) the cardinality Z_1 of \mathcal{Z}_1 is at least $(d\beta_*)^L$; (ii) for some $x \in \mathcal{Z}_1$ there is a vertex $x^* \in \mathcal{Z}_n \cap \mathcal{T}(x)$ such that $x^* \rightarrow x$; and (iii) there is an infection trail from x to e beginning at some time between the time of first re-infection from \mathcal{Z}_n and the time of first recovery thereafter. Fix x , and let G_x be the event that both (ii) and (iii) occur for this x . Given that $x \in \mathcal{Z}_1$, (a) (ii) and (iii) are conditionally independent, because the events in question involve nonoverlapping (in time) parts of the percolation structure; (b) the conditional probability of (ii) is r_{n-1} , by isotropy; and, consequently, (c) the conditional probability of G_x is $r_{n-1}u_L$. Moreover, conditional on the composition of the set \mathcal{Z}_1 , the events G_x , $x \in \mathcal{Z}_1$, are independent, once again because these events involve nonoverlapping parts of the percolation structure. Consequently, by (10)-(12) and the induction hypothesis,

$$\begin{aligned} r_n &\geq P\{Z_1 \geq (d\beta_*)^L\}P(\cup_{x^* \in \mathcal{Z}_n} \{x^* \rightarrow e\} \mid Z_1 \geq (d\beta_*)^L) \\ &\geq p_L P(\cup_{x \in \mathcal{Z}_1} \cup_{x^* \in \mathcal{Z}_n \cap \mathcal{T}(x)} \{x \rightarrow e \ \& \ x^* \rightarrow x\} \mid Z_1 \geq (d\beta_*)^L) \\ &\geq p_L (1 - (1 - r_{n-1})^{(d\beta_*)^L}) w_L \\ &\geq p_L w_L (1 - (1 - \frac{p_L (\beta_*)^L}{2})^{(d\beta_*)^L}) \\ &\geq p_L \beta_*^L / 2 \end{aligned}$$

□

5. THE LIMIT SET

In this section we use the existence of the embedded Galton-Watson processes to identify the Hausdorff dimension of the limit set Λ .

Proof of Theorem 2. For each integer $n \geq 1$ define \mathcal{Y}_n to be the set of all vertices in \mathcal{T} at distance n from the root e that are *ever* infected, and define Y_n to be the cardinality of \mathcal{Y}_n . Then

$$EY_n = (d+1)d^n u_n \leq (d+1)d^n \beta^n,$$

since there are exactly $(d+1)d^n$ vertices at distance n from e . Consequently, by the Borel-Cantelli lemma, for any $\varepsilon > 0$ it is P -almost surely the case that eventually

$$Y_n \leq d(\beta + \varepsilon)^n.$$

The sets \mathcal{Y}_n provide a sequence of open covers of Λ : in particular, if \mathcal{E}_x is the set of all ends of \mathcal{T} that pass through x , then

$$\Lambda \subset \cup_{x \in \mathcal{Y}_n} \mathcal{E}_x.$$

Note that for each $x \in \mathcal{Y}_n$, the diameter (in the d_θ metric) of \mathcal{E}_x is θ^n , which becomes small as $n \rightarrow \infty$. Hence, by the result of the preceding paragraph, for any $\varepsilon > 0$,

$$\sum_{x \in \mathcal{Y}_n} \text{diameter}_\theta(\mathcal{E}_x)^{-\log(d\beta+d\varepsilon)/\log\theta} \leq 1$$

for all sufficiently large n . This implies that with probability 1,

$$(14) \quad \delta_H^\theta(\Lambda) \leq \frac{-\log(d\beta)}{\log\theta}.$$

The proof of the reverse inequality uses the existence of the embedded Galton-Watson processes $Z_n = Z_n^L$ constructed in section 3. Recall that Z_n is the cardinality of the set \mathcal{Z}_n of vertices at distance nL from e that are *descendants* of vertices in \mathcal{Z}_{n-1} , and that the mean offspring number is $d^L w_L$. Now any infinite sequence of vertices $x_n \in \mathcal{Z}_n$ such that each x_n is a descendant of x_{n-1} determines a unique end of \mathcal{T} that must be included in the limit set Λ , since all of the vertices x_n are eventually infected. Thus, Λ contains the set of ends of the Galton-Watson tree for the Galton-Watson process Z_n . By a theorem of Hawkes [5] (see also Lyons [11]), the Hausdorff dimension (in the metric d_θ) of this Galton-Watson tree is

$$\frac{\log(dw_L^{1/L})}{\log\theta}.$$

It follows that, on the event of survival of (Z_n^L) , this is a lower bound for the Hausdorff dimension of Λ .

Since $w_L^{1/L} \rightarrow \beta$ as $L \rightarrow \infty$, to complete the proof it suffices to show that the Hausdorff dimension of Λ is almost surely constant on the event that the contact process survives. This is routine. Let δ_* be the essential supremum of the random variable $\delta_H^\theta(\Lambda)$. Then for any $\delta < \delta_*$, there is positive probability p that the limit set of a contact process initiated at e has Hausdorff dimension at least δ . Since the Hausdorff dimension of a subset of $\partial\mathcal{T}$ is unchanged by application of an isometry of \mathcal{T} , it follows that for any vertex x , there is positive probability p that the limit set of a contact process initiated at x has Hausdorff dimension at least δ . But on the event that the contact process (started at e) survives, infinitely many vertices are infected, and at each initial infection a new contact process is initiated (contained in the original, of course). Birkhoff's ergodic theorem implies that infinitely many of these contact processes have limit sets with Hausdorff dimension at least δ . (Note: Ergodicity of the implied stationary process is a consequence of the tail triviality of the percolation structure.) \square

6. APPENDIX: PRUNING A GALTON-WATSON TREE

In this appendix, we provide a new and simple proof of the theorem of Hawkes and Lyons cited in the proof of Theorem 2 above. The result we obtain is in fact stronger than the results of Hawkes and Lyons: we prove that a Galton-Watson tree must contain infinite homogeneous subtrees of any degree smaller than the growth rate prescribed by the mean offspring number. Proposition 4 contains a precise statement. The proof is similar in spirit to that of Proposition 3 above.

Let $(Z_n)_{n \geq 0}$ be a supercritical Galton-Watson process with mean offspring number $\mu > 1$, and let \mathcal{Z} be the Galton-Watson tree associated with the process $(Z_n)_{n \geq 0}$. Thus, \mathcal{Z} has vertices arranged in "levels" $n = 0, 1, 2, \dots$, with exactly one n th level vertex for each particle counted in Z_n ; and \mathcal{Z} has directed edges from n th level vertices to $(n+1)$ th level

vertices, one for each parent-child pair. For each integer $L \geq 1$ let \mathcal{Z}^L be the Galton-Watson tree associated with the Galton-Watson process $(Z_{nL})_{n \geq 0}$. Fix a real number $1 < \lambda < \mu$ and an integer $L \geq 1$. Define the (L, λ) -pruned tree $\mathcal{Z}^L(\lambda)$ as follows: (1) Set $\mathcal{Z}_0^L = \mathcal{Z}^L$. (2) For each integer $n \geq 1$, define \mathcal{Z}_{n+1}^L by removing from \mathcal{Z}_n^L every vertex with fewer than λ^L offspring, and all of that vertex's descendants. (3) Set

$$\mathcal{Z}^L(\lambda) = \bigcap_{n=1}^{\infty} \mathcal{Z}_n^L$$

Observe that the sequence of trees \mathcal{Z}_n^L is nested, so the intersection is well-defined and is a tree (albeit possibly empty). By construction, every vertex of $\mathcal{Z}^L(\lambda)$ has at least λ^L offspring, so if $\mathcal{Z}^L(\lambda) \neq \emptyset$ then it is infinite, and in fact contains an embedded tree in which every vertex has exactly $\lfloor \lambda^L \rfloor$ offspring.

Proposition 4. *For every $1 < \lambda < \mu$ there exists $L \geq 1$ sufficiently large that, with positive probability, the (L, λ) -pruned tree $\mathcal{Z}^L(\lambda)$ contains the root vertex v_0 of \mathcal{Z}^L . On this event, the Galton-Watson tree \mathcal{Z}^L contains an infinite homogeneous tree of degree $\lfloor \lambda^L \rfloor + 1$.*

Lemma 1. *For every $\nu < \mu$ there exists $\gamma_1 = \gamma_1(\nu) < 1$ such that for all sufficiently large $n \geq 1$,*

$$P\{0 < Z_n < \nu^n\} \leq \gamma_1^n.$$

Proof. This is fairly well known, and may be proved by a relatively standard generating function argument. \square

Lemma 2. *Let ξ_1, ξ_2, \dots be independent, identically distributed Bernoulli- r random variables, and let $S_n = \sum_{i=1}^n \xi_i$. For any pair r, s of real numbers satisfying $0 < s < r < 1$, there exists $\gamma_2 = \gamma_2(r, s) < 1$ such that for all sufficiently large $n \geq 1$,*

$$P\{S_n \leq ns\} \leq \gamma_2^n$$

Proof. This is a standard result. \square

Proof of Proposition 4. It suffices to prove that for every $1 < \lambda < \mu$ there exist an integer $L \geq 1$ and a real number $\alpha > 0$ such that for all $n \geq 1$,

$$(15) \quad p_n \stackrel{\text{def}}{=} P\{v_0 \in \mathcal{Z}_n^L\} > \alpha.$$

Denote by p_* the survival probability for the Galton-Watson process $(Z_n)_{n \geq 0}$ (i.e., $q_* = 1 - p_*$ is the probability of eventual extinction, given that $Z_0 = 1$). Choose γ_2 so that the conclusion of Lemma 2 holds with $r = p_*/4$ and $s = p_*/8$. Fix $\nu \in (\lambda, \mu)$, and choose $\gamma_1 < 1$ so that the conclusion of Lemma 1 holds. Then for all sufficiently large integers L , all of the following inequalities will hold:

$$\begin{aligned} \gamma_1^L &< p_*/4; \\ \gamma_2^{\lambda^L} &< p_*/4; \\ \lambda^L &< s\nu^L. \end{aligned}$$

We will show that for any such L , equation (15) must be true for $\alpha = r = p_*/4$.

The proof is by induction on n . The probability that v_0 has fewer than ν^L offspring in \mathcal{Z}^L is less than $q_* + \gamma_1^L$, by Lemma 1. Consequently,

$$p_1 > p_* - \gamma_1^L > p_*/2.$$

Assume now that $p_n > p_*/4$; we must show that $p_{n+1} > p_*/4$. Observe that $v_0 \in \mathcal{Z}_{n+1}^L$ occurs if and only if v_0 has at least λ^L offspring v in \mathcal{Z}_n^L . Consider *all* of the offspring v of

v_0 in \mathcal{Z}^L . The event that there are at least ν^L of these has probability greater than $p_*/2$, by the preceding paragraph. For each offspring v , there is probability at least $p_n > p_*/4 = r$ that $v \in \mathcal{Z}_n^L$, by the induction hypothesis. Hence, by Lemma 1 and our choice of γ_2 and s , the conditional probability that v_0 has fewer than $s\nu^L$ offspring in \mathcal{Z}_n^L , given that v_0 has at least ν^L offspring in \mathcal{Z}^L , is smaller than

$$\gamma_2^{\nu^L} < p_*/4.$$

Since $\lambda^L < s\nu^L$, it follows that the event that v_0 has fewer than λ^L offspring in \mathcal{Z}_n^L has probability smaller than $(1 - p_*/2) + p_*/4$. Consequently, the probability p_{n+1} that v_0 has at least λ^L offspring in \mathcal{Z}_n^L is greater than $p_*/2 - p_*/4 = \alpha$. □

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