ON THE INCREMENTS OF l^∞-VALUED GAUSSIAN PROCESSES*

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On The Increments Of l^∞-Valued Gaussian Processes*

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Dedicated to Professor M.Csörgő on the occasion of his sixty fifth birthday

ABSTRACT. Let $\{Y(t), -\infty < t < \infty\} = \{X_k(t), -\infty < t < \infty\}_{k=1}^\infty$ be a sequence of Gaussian processes with $\sigma_k^2(h) = E(X_k(t+h) - X_k(t))^2$. Put $\sigma^{*2}(h) = \max_{k\ge1}\sigma_k^2(h)$. M.Csörgő et al.(1994) studied moduli of continuity of $Y(\cdot)$ when it is an l^∞-valued process under the condition that $\sigma^{*2}(h)/h^\alpha$ is quasi-increasing for some $\alpha > 0$. In this paper we establish the large increment result for the l^∞-valued process $Y(\cdot)$.

1. Introduction.

Let $\{Y(t), -\infty < t < \infty\} = \{X_k(t), -\infty < t < \infty\}_{k=1}^\infty$ be a sequence of continuous Gaussian processes with stationary increments $\sigma_k^2(h) = E(X_k(t+h) - X_k(t))^2$. Csörgő and Shao(1993) and Csörgő,Lin and Shao (1994) established moduli of continuity of $Y(\cdot)$ as an l^p-valued process $(1 \le p < \infty)$ and l^∞-valued process respectively. For the former, the large increments are also studied under the condition that $\sigma(p,h)/h^\alpha$ is quasi-increasing for some $\alpha > 0$ where $\sigma(p,h) = (\sum_{k=1}^\infty \sigma_k^p(h))^{1/p}$. Lin (1996) established the large increment result when $\sigma(p,h)$ is bounded. What about the large increments for the l^∞-valued process $Y(\cdot)$? We intend to answer this question in the paper.

2. The large increments of l^∞-valued Guassian process.

Throughout this paper we assume that $EX_k(t) = 0$ for any t and every k and that $\sigma_k^2(h) = E(X_k(t+h) - X_k(t))^2$ are nondecreasing in h and strictly positive for $h > 0$. Put $\sigma^{*2}(h) = \max_{k\ge1}\sigma_k^2(h)$. In this paper, we establish the large increment results for $Y(\cdot)$ as an l^∞-valued process with both infinite $\sigma^*(h)$ and finite $\sigma^*(h)$. We pay more attention to the latter case.

As the first part, we consider the case of infinite $\sigma^*(h)$,i.e., we assume that for some $\alpha > 0$

$$\sigma^{*2}(h)/h^\alpha \text{ is quasi-increasing.} \tag{2.1}$$

Let $a_T, 0 < a_T \le T$, be a continuous function satisfying $a_T \to \infty$ as $T \to \infty$. Let y_T be the solution of the equation

$$\sum_{k=1}^\infty \Big(\frac{a_T y_T}{T\log\sigma^*(a_T)}\Big)^{\sigma^{*2}(a_T)/\sigma_k^2(a_T)} = \frac{a_T}{T\log\sigma^*(a_T)}. \tag{2.2}$$

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The following theorem is an analogue of theorem 1 of [1] in the case of large increments.

Theorem 1. *Suppose that there exist positive numbers h_0, A and B such that for any $h \geq h_0$*

$$\sum_{k=1}^{\infty}(\sigma_k(h)/\sigma^*(h))^A < B. \tag{2.3}$$

Then

(2.4)

$$\limsup_{T\to\infty} \sup_{0\leq t\leq T-a_T} \sup_{0\leq s\leq a_T} \max_{k\geq 1} \frac{|X_k(t+s)-X_k(t)|}{\sigma^*(a_T)(2\log((T\log\sigma^*(a_T))/a_T))^{1/2}} \leq 1 \quad a.s.$$

If condition (2.3) is replaced by conditions that there exist positive numbers h_1, c_1, T_0 and C so that

$$\inf_{0\leq s\leq h} \frac{\sigma^*(s)}{\sigma_k(s)} \geq c_1 \frac{\sigma^*(h)}{\sigma_k(h)} \tag{2.5}$$

for any $h \geq h_1$ and every $k \geq 1$ and

$$\sum_{k=1}^{\infty}\Big(\frac{a_T}{T\log\sigma^*(a_T)}\Big)^{\sigma^{*2}(a_T)/\sigma_k^2(a_T)} < C \tag{2.6}$$

for any $T \geq T_0$, then (2.4) remains true. If, in addition, $X_k(\cdot), k = 1, 2, \cdots$, are independent and for $0 \leq t_1 < t_2 \leq t_3 < t_4$,

$$E(X_k(t_2)-X_k(t_1))(X_k(t_4)-X_k(t_3)) \leq 0 \tag{2.7}$$

and

$$\lim_{T\to\infty} \frac{\log(T/a_T)}{\log\log\sigma^*(a_T)} = \infty, \tag{2.8}$$

then

$$\lim_{T\to\infty} \sup_{0\leq t\leq T-a_T} \sup_{0\leq s\leq a_T} \max_{k\geq 1} \frac{|X_k(t+s)-X_k(t)|}{\sigma^*(a_T)(2\log(T/a_T))^{1/2}} = 1 \quad a.s. \tag{2.9}$$

and

$$\limsup_{T\to\infty} \sup_{0\leq t\leq T-a_T} \max_{k\geq 1} \frac{|X_k(t+a_T)-X_k(t)|}{\sigma^*(a_T)(2\log(T/a_T))^{1/2}} = 1 \quad a.s. \tag{2.10}$$

Remark 1. Condition (2.3) is a correspondence of condition (2.2) of [1] in the case of large increments. It is understandable by comparison with (2.11) in [1] and noting that $\sigma^*(h) \to \infty$ as $h \to \infty$.

Remark 2. As an analogue of Lemma 1 in [1], we have

$$\log\frac{1}{y_T} \leq \log\sum_{k=1}^{\infty}\Big(\frac{\sigma_k(a_T)}{\sigma^*(a_T)}\Big)^A$$

for T satisfying $\sigma^*(a_T) \geq e^{A/2}$. Hence

$$y_T \geq B^{-1} > 0 \tag{2.11}$$

by (2.3) for large T. Moreover in the same way in [1], we can show the following facts: $y_T \le 1$ and the solution of equation (2.2) exists and is unique under condition (2.6), which is implied by (2.3).

Proof of Theorem 1. There is $d > 0$ such that $\sigma^{*2}(h) \ge dh^2$ for any $0 < h \le 1$ (cf. (2.9) in [1]). If $h_0 > 1$, then for any $1 < h \le h_0$, $\sigma^{*2}(h) \ge \sigma^{*2}(1) \ge h^2\sigma^{*2}(1)/h_0^2$. Hence, putting $d' = d \wedge (\sigma^{*2}(1)/h_0^2)$, we have $\sigma^{*2}(h) \ge d'h^2$ for any $h \le h_0$, which implies that for $h \le h_0$

$$\sum_{k=1}^{\infty}(\sigma_k(h)/\sigma^*(h))^A \le d'^{-A/2}h^{-A}\sum_{k=1}^{\infty}\sigma_k^A(h_0) =: d_1 h^{-A}, \tag{2.12}$$

where $d_1 = d'^{-A/2}\sum_{k=1}^{\infty}\sigma_k^A(h_0)$.

Let $\theta > 1$. Define $A_i = \{T : \theta^{i-1} \le \sigma^*(a_T) < \theta^i\}$, $A_{ij} = \{T : \theta^{j-1} \le T/a_T < \theta^j, T \in A_i\}$, $a_{ij} = \sup\{a_T, T \in A_{ij}\}$, $T_{ij} = \sup\{T : a_T = a_{ij},\ T \in A_{ij}\}$, $T'_{ij} = \sup\{T : T - a_T = \sup_{T\in A_{ij}}(T - a_T),\ T \in A_{ij}\}$ and $J = \max\{j : \theta^j \le \max_{T>0} T/a_T\}$. Then $J \le \infty$ and

(2.13)

$$\begin{aligned}
&\limsup_{T\to\infty}\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}\max_{k\ge1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(a_T)(2\log((T\log\sigma^*(a_T))/a_T))^{1/2}}\\
&\le \limsup_{i\to\infty}\sup_{1\le j<J}\sup_{0\le t\le T'_{ij}-a_{T'_{ij}}}\sup_{0\le s\le a_{ij}}\max_{k\ge1}\frac{|X_k(t+s)-X_k(t)|}{\theta^{i-1}(2\log(\theta^{j-1}\log\theta^{i-1}))^{1/2}}\\
&\le \limsup_{i\to\infty}\sup_{1\le j<J}\sup_{0\le t\le T'_{ij}-a_{T'_{ij}}}\sup_{0\le s\le a_{ij}}\max_{k\ge1}\frac{\theta^2|X_k(t+s)-X_k(t)|}{\sigma^*(a_{ij})(2\log((T_{ij}\log\sigma^*(a_{ij}))/a_{ij})^{1/2}}.
\end{aligned}$$

For any $\varepsilon > 0, r = r(\varepsilon) > 0$ will be specified later. Put $r_{ij} = a_{ij}/2^r$. For any $t > 0$, put $t_r := t_{rij} = [t/r_{ij}]r_{ij}$. Write

$$\begin{aligned}
|X_k(t+s)-X_k(t)| &\le |X_k((t+s)_r)-X_k(t_r)|\\
&+\sum_{l=0}^{\infty}|X_k((t+s)_{r+l+1})-X_k((t+s)_{r+l})| + \sum_{l=0}^{\infty}|X_k(t_{r+l+1})-X_k(t_{r+l})|.
\end{aligned}\tag{2.14}$$

Similarly to (2.14) in [1] and noting (2.11), for large T, We have

(2.15)

$$\begin{aligned}
p_0 :&= P\Big\{\sup_{0\le t\le T'_{ij}-a_{T'_{ij}}}\sup_{0\le s\le a_{ij}-r_{ij}}\max_{k\ge1}\frac{|X_k((t+s)_r)-X_k(t_r)|}{\sigma^*(a_{ij})(2\log((T_{ij}\log\sigma^*(a_{ij}))/a_{ij}))^{1/2}}\\
&\qquad \ge 1+\varepsilon\Big\}\\
&\le P\Big\{\sup_{0\le t\le T'_{ij}-a_{T'_{ij}}}\sup_{0\le s\le a_{ij}-r_{ij}}\max_{k\ge1}\frac{|X_k((t+s)_r)-X_k(t_r)|}{\sigma^*(a_{ij})(2\log((T_{ij}\log\sigma^*(a_{ij}))/(a_{ij}y_{T_{ij}})))^{1/2}}\\
&\qquad \ge 1+\varepsilon/2\Big\}\\
&\qquad \le \frac{4T'_{ij}}{a_{ij}}2^{2r}\sum_{k=1}^{\infty}\exp\Big\{-\Big(1+\frac{\varepsilon}{2}\Big)^2\Big(\log\frac{T_{ij}\log\sigma^*(a_{ij})}{a_{ij}y_{T_{ij}}}\Big)\frac{\sigma^{*2}(a_{ij})}{\sigma_k^2(a_{ij})}\Big\}\\
&\qquad \le \frac{4T'_{ij}}{a_{ij}}2^{2r}\Big(\frac{a_{ij}}{T_{ij}\log\sigma^*(a_{ij})}\Big)^{1+\varepsilon} \le c\Big(\frac{T_{ij}}{a_{ij}}\Big)^{-\varepsilon}(\log\sigma^*(a_{ij}))^{-1-\varepsilon}\\
&\qquad \le c\theta^{-\varepsilon j}(i\log\theta)^{-1-\varepsilon},
\end{aligned}$$

If $T'_{ij} \le T_{ij}$ (here, c stands for a constant whose value is irrelevant). In the contrary case we have

$$T_{ij} \le T'_{ij} \le \theta^j a_{T'_{ij}} \le \theta T_{ij},$$

and hence (2.15) is also true. Combining the lines of proofs of (2.15) in [1] and (2.15) and using condition (2.3) instead of (2.11) in [1] we have

(2.16)

$$\begin{aligned} p_1 := P\Big\{ &\sup_{0\le t\le T'_{ij}-a_{T'_{ij}}} \sup_{a_{ij}-r_{ij}\le s\le a_{ij}} \max_{k\ge 1} \frac{|X_k((t+s)_r) - X_k(t_r)|}{\sigma^*(a_{ij})(2\log((T_{ij}\log\sigma^*(a_{ij}))/a_{ij}))^{1/2}} \\ &\ge \varepsilon/2\Big\} \\ \le\ & c\frac{T'_{ij}}{a_{ij}}\Big(\frac{a_{ij}}{T_{ij}\log\sigma^*(a_{ij})}\Big)^{2+A}\Big(\frac{\sigma^*(2a_{ij}/2^r)}{\sigma^*(a_{ij})}\Big)^A\Big(\frac{2a_{ij}}{2^r}\Big)^{-A} \\ \le\ & \frac{cT'_{ij}a_{ij}}{(T_{ij}\log\sigma^*(a_{ij}))^{2+A}} \le c\theta^{-j} i^{-2-A}, \end{aligned}$$

provided r and T are large enough. Under conditions (2.5) and (2.6) we also have the same bound.

Consider the first series in the right hand side of (2.14). Let $b_{rij} = \sup\{b : a_{ij}/2^{r+[\log_2 ba_{ij}]+1} \ge h_0\}$ and $l_0 = [\log_2(b_{rij}a_{ij})]$. Let $D = 3/c_1$, $x_l^2 := x_{lij}^2 = 2D\log(T_{ij}\log\sigma^*(a_{ij})/a_{ij}) + 2(1+A)l$. If condition (2.3) is satisfied, then by (2.3) and (2.12)

(2.17)

$$\begin{aligned} p_2 := P\Big\{ &\sup_{0\le t\le T'_{ij}-a_{T'_{ij}}} \sup_{0\le s\le a_{ij}} \max_{k\ge 1} \sum_{l=0}^{\infty} |X_k((t+s)_{r+l+1}) - X_k((t+s)_{r+l})| \\ &\ge \sum_{l=0}^{\infty} x_l\sigma^*(a_{ij}/2^{r+l+1})\Big\} \\ \le\ & \frac{2T'_{ij}}{a_{ij}}\sum_{l=0}^{\infty}\sum_{k=1}^{\infty} 2^{r+l+1}e^{-(1+A)l}\Big(\frac{a_{ij}y_{T_{ij}}}{T_{ij}\log\sigma^*(a_{ij})}\Big)^2 \exp\Big\{-\Big(\frac{D\sigma^{*2}(a_{ij}/2^{r+l+1})}{\sigma_k^2(a_{ij}/2^{r+l+1})} - 2\Big) \\ &\cdot \log\frac{T_{ij}\log\sigma^*(a_{ij})}{a_{ij}y_{T_{ij}}}\Big\} \\ \le\ & c\cdot\frac{T'_{ij}}{a_{ij}}\Big(\frac{a_{ij}}{T_{ij}\log\sigma^*(a_{ij})}\Big)^2\Big(\sum_{l=0}^{l_0} + \sum_{l=l_0+1}^{\infty}\Big)\sum_{k=1}^{\infty} 2^l e^{-(1+A)l}\Big(\frac{\sigma_k(a_{ij}/2^{r+l+1})}{\sigma^*(a_{ij}/2^{r+l+1})}\Big)^A \\ \le\ & c\cdot\frac{T'_{ij}}{a_{ij}}\Big(\frac{a_{ij}}{T_{ij}\log\sigma^*(a_{ij})}\Big)^2\Big(\sum_{l=0}^{l_0} B2^l e^{-(1+A)l} + \sum_{l=l_0+1}^{\infty} d_1 2^l e^{-(1+A)l} \\ &\cdot (a_{ij}/2^{r+l+1})^{-A}\Big) \\ \le\ & c\cdot\frac{T'_{ij}}{a_{ij}}\Big(\frac{a_{ij}}{T_{ij}\log\sigma^*(a_{ij})}\Big)^2 \le c\theta^{-j}(i\log\theta)^{-2} \end{aligned}$$

for large r and T. Also, if conditions (2.5) and (2.6) are satisfied, we have the same bound.

For the second series in the right hand side of (2.14), we also have the same conclusion.

Moveover, it follows from condition (2.1) that

$$\sum_{l=0}^{\infty} x_l \sigma^*(a_{ij}/2^{r+l+1}) \le \frac{\varepsilon}{4}\sigma^*(a_{ij})\Big(\log \frac{T_{ij}\log\sigma^*(a_{ij})}{a_{ij}}\Big)^{1/2} \tag{2.18}$$

provided r is large enough (cf. (2.20) in [1]). Combining these estimators, we obtain

$$\sum_{i=1}^{\infty}\sum_{j=1}^{J} P\Big\{\sup_{0\le t\le T'_{ij}-a_{T'_{ij}}}\ \sup_{0\le s\le a_{ij}}\ \max_{k\ge 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(a_{ij})(2\log((T_{ij}\log\sigma^*(a_{ij}))/a_{ij}))^{1/2}} \ge 1+2\varepsilon\Big\} < \infty.$$

Therefore by the Borel-Cantelli lemma, the right hand side of (2.13) is bounded by one almost surely. (2.4) is proved.

Now we prove (2.10). Having (2.4), it is enough to show

$$\limsup_{T\to\infty}\ \sup_{0\le t\le T-a_T}\ \max_{k\ge 1}\frac{|X_k(t+a_T)-X_k(t)|}{\sigma^*(a_T)(2\log(T/a_T))^{1/2}} \ge 1 \quad a.s. \tag{2.19}$$

Using (2.8), Slepian's lemma and independence of $\{X_k(\cdot)\}_{k=1}^{\infty}$, we have for $T_n(\uparrow\infty)$ large enough

(2.20)

$$\begin{aligned}
&P\Big\{\sup_{0\le t\le T_n-a_{T_n}}\ \max_{k\ge 1}\frac{|X_k(t+a_{T_n})-X_k(t)|}{\sigma^*(a_{T_n})(2\log(T_n/a_{T_n}))^{1/2}} < 1-\varepsilon\Big\}\\
&\le P\Big\{\max_{0\le j\le T_n/a_{T_n}}\ \max_{k\ge 1}\frac{X_k((j+1)a_{T_n})-X_k(ja_{T_n})}{\sigma^*(a_{T_n})(2\log((T_n\log\sigma^*(a_{T_n}))/(a_{T_n}y_{T_n})))^{1/2}} < 1-\varepsilon\Big\}\\
&\le \prod_{j=0}^{[T_n/a_{T_n}]}\prod_{k=1}^{\infty}\Big\{1-\exp\Big\{-\frac{(1-\varepsilon)\sigma^{*2}(a_{T_n})}{\sigma_k^2(a_{T_n})}\log\frac{T_n\log\sigma^*(a_{T_n})}{a_{T_n}y_{T_n}}\Big\}\Big\}\\
&\le \prod_{j=0}^{[T_n/a_{T_n}]}\exp\Big\{-\sum_{k=1}^{\infty}\Big(\frac{a_{T_n}y_{T_n}}{T_n\log\sigma^*(a_{T_n})}\Big)^{(1-\varepsilon)\sigma^{*2}(a_{T_n})/\sigma_k^2(a_{T_n})}\Big\}\\
&\le \exp\Big\{-\frac{T_n}{a_{T_n}}\Big(\frac{a_{T_n}}{T_n\log\sigma^*(a_{T_n})}\Big)^{1-\varepsilon}\Big\}\le \exp\Big\{-\Big(\frac{T_n}{a_{T_n}}\Big)^{\varepsilon/2}\Big\}\to 0
\end{aligned}$$

as $n\to\infty$. Hence (2.10) is proved.

Finally we prove (2.9). Recalling (2.4), it suffices to show that the " lim inf " is not less than one almost surely. Let $a'_{ij}=\inf\{a_T, T\in A_{ij}\}$, $t'_{ij}=\inf\{T: a_T = a'_{ij}, T\in A_{ij}\}$, $t_{ij}=\inf\{T: T-a_T=\inf_{T\in A_{ij}}(T-a_T), T\in A_{ij}\}$. We have

$$\begin{aligned}
&\liminf_{T\to\infty}\ \sup_{0\le t\le T-a_T}\ \sup_{0\le s\le a_T}\ \max_{k\ge 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(a_T)(2\log(T/a_T))^{1/2}}\\
&\ge \liminf_{i\to\infty}\ \inf_{1\le j\le J}\ \sup_{0\le t\le t_{ij}-a_{t_{ij}}}\ \sup_{0\le s\le a'_{ij}}\ \max_{k\ge 1}\frac{|X_k(t+s)-X_k(t)|}{\theta^i(2\log\theta^j)^{1/2}}\\
&\ge \liminf_{i\to\infty}\ \inf_{1\le j\le J}\ \max_{1\le l\le t_{ij}/a'_{ij}}\ \max_{k\ge 1}\frac{|X_k((l+1)a'_{ij})-X_k(la'_{ij})|}{\theta^2\sigma^*(a'_{ij})(2\log((t'_{ij}\log\sigma^*(a'_{ij}))/(a'_{ij}y_{t'_{ij}})))^{1/2}}.
\end{aligned}$$

Then, similary to (2.20) we have

$$P\Big\{\max_{0\le l\le t_{ij}/a'_{ij}}\max_{k\ge 1}\frac{|X_k((l+1)a'_{ij})-X_k(la'_{ij})|}{\sigma^*(a'_{ij})(2\log((t'_{ij}\log\sigma^*(a'_{ij}))/(a'_{ij}y_{t'_{ij}})))^{1/2}}<1-\varepsilon\Big\}$$
$$\le \exp\Big\{-\frac{t_{ij}}{a'_{ij}}\Big(\frac{a'_{ij}}{t'_{ij}\log\sigma^*(a'_{ij})}\Big)^{1-\varepsilon}\Big\}$$
$$\le \exp\{-(t'_{ij}/a'_{ij})^{\varepsilon/2}\log\sigma^*(a'_{ij})\}\le \exp\{-\theta^{(j-1)\varepsilon/2}(i\log\theta)\}.$$

The second inequality is due to (2.8) again. Hence by the Borel-Cantelli lemma, (2.9) is proved. This completes the proof of theorem 1.

The second part is to consider the case that $\sigma^{*2}(h)\to\sigma^{*2}<\infty$ as $h\to\infty$. For example, if $X_k(\cdot)$,k=1,2,$\cdots$, are Ornstein-Uhlenbeck processes, $\sigma^*(h)$ is bounded. Let $\sigma_k^2=\lim_{h\to\infty}\sigma_k^2(h)$ and z_T be the solution of the equation

$$\sum_{k=1}^{\infty}\Big(\frac{a_T z_T}{T}\Big)^{\sigma^{*2}/\sigma_k^2}=\frac{a_T}{T}. \tag{2.21}$$

Let $\alpha>1$ be given, m a positive integer. Put

$$L_m x=\underbrace{\log_\alpha\cdots\log_\alpha}_{m\,fold}x,\ d(m,r)=\underbrace{\alpha^{\alpha^{\cdot^{\cdot^{\cdot^{\alpha^{r}}}}}}}_{m\,fold}.$$

Theorem 2. *Suppose that $\sigma^*(h)\to\sigma^*$ as $h\to\infty$ and that there exists $A>0$ such that*

$$\sum_{k=1}^{\infty}\sigma_k^A<\infty. \tag{2.22}$$

And suppose that there exist $2\le\alpha<e, 0<\delta<1-1/\alpha$ and integer $m\ge 1$ such that

$$\int_1^\infty \sigma^*(\alpha^{-x^2})dx<\infty, \tag{2.23}$$

$$a_T\le Td(m,(L_mT)^{\delta+1/\alpha})^{-1}. \tag{2.24}$$

Then

$$\limsup_{T\to\infty}\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}\max_{k\ge 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(2\log(T/a_T))^{1/2}}\le 1\quad a.s. \tag{2.25}$$

If, in addition, $X_k(\cdot), k=1,2,\cdots$, are independent and (2.7) is satisfied, then

$$\lim_{T\to\infty}\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}\max_{k\ge 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(2\log(T/a_T))^{1/2}}=1\quad a.s. \tag{2.26}$$

and

$$\limsup_{T\to\infty}\sup_{0\le t\le T-a_T}\max_{k\ge 1}\frac{|X_k(t+a_T)-X_k(t)|}{\sigma^*(2\log(T/a_T))^{1/2}}=1\quad a.s. \tag{2.27}$$

Remark 3. It is easy to see that

$$d(m,(L_mT)^{\delta+1/\alpha})=T^{(L_1T)^{-1+(L_2T)^{\cdot^{\cdot^{-1+(L_mT)^{-1+\delta+1/\alpha}}}}}}.$$

Put $d(m, (L_mT)^{\delta+1/\alpha}) = T^{(L_1T)^{-1+o(T,m)}}$. Then $o(T,m) \to 0$ and $\frac{o(T,m+1)}{o(T,m)} \to 0$ as $T \to \infty$. Hence, as $T \to \infty$,

$$d(m, (L_mT)^{\delta+1/\alpha}) = \alpha^{(L_1T)^{o(T,m)}} \le \alpha^{(L_1T)^\varepsilon} \quad \text{for any } \varepsilon > 0,$$

$$d(m+1, (L_{m+1}T)^{\delta+1/\alpha})/d(m, (L_mT)^{\delta+1/\alpha}) \to 0.$$

Proof of Theorem 2. Without loss of generality, we assume $m \ge 3$. Similarly to Theorem 1, condition (2.22) guarantees that the solution of equation (2.21) exists and is unique. Moreover, there exists $0 < b < 1$ such that $z_T \ge b$.

It is easy to see from (2.24) that

$$\frac{\log(T/a_T)}{\log\log T} \to \infty \quad as \quad T \to \infty. \tag{2.28}$$

For $\theta > 1$, define $A_j = \{T : \theta^{j-1} \le T/a_T < \theta^j\}$, $a_j = \sup\{a_T : T \in A_j\}$, $T_j = \sup\{T : a_T = a_j, T \in A_j\}$ and $T_j' = \sup\{T : T - a_T = \sup_{T\in A_j}(T - a_T), T \in A_j\}$. Given $r > 0$ for any $t > 0$ put $t_r^j := t_r(a_j) = [td(m+2, r)/a_j](a_j/d(m+2, r))$. Write

(2.29)

$$\begin{aligned}|X_k(t+s) - X_k(t)| \le& |X_k((t+s)_r^j) - X_k(t_r^j)| \\ &+ \sum_{l=0}^{\infty} |X_k((t+s)_{r+l+1}^j) - X_k((t+s)_{r+l}^j)| \\ &+ \sum_{l=0}^{\infty} |X_k(t_{r+l+1}^j) - X_k(t_{r+l}^j)|.\end{aligned}$$

For $\delta > 0$ in (2.24), let δ_1 satisfy $1 - 1/\alpha - \delta < \delta_1 < 1 - 1/\alpha$. And let $\varepsilon(a_T) = d(m-1, (L_m a_T)^{1-\delta_1})/\log_\alpha a_T$, $r := r(a_j) = L_{m+2}a_j^{\varepsilon(a_j)}$, $r' := r'(a_j) = L_{m+2}a_j$. Then $d(m+2, r) = a_j^{\varepsilon(a_j)}$ and $d(m+2, r') = a_j$, moreover,

$$\begin{aligned}0 < r' - r &= L_{m+2}a_j - L_m[(L_2a_j)(1 + \frac{L_1\varepsilon(a_j)}{L_2a_j})] \\ &= -L_1(1-\delta_1) < 1.\end{aligned} \tag{2.30}$$

Similaryly to (2.15), we have

(2.31)

$$\begin{aligned}p_0' :=& P\Big\{\sup_{0\le t\le T_j' - a_{T_j'}} \sup_{0\le s\le a_j} \max_{k\ge1} \frac{|X_k((t+s)_r^j) - X_k(t_r^j)|}{\sigma^*(2\log(T_j/a_j))^{1/2}} \\ &\ge 1+\varepsilon\Big\} \\ \le& \frac{4T_j'}{a_j} d(m+2, r)^2 (\frac{a_j}{T_j})^{1+\varepsilon} \le 4a_j^{2\varepsilon(a_j)+\varepsilon}(T_j'/T_j)T_j^{-\varepsilon} \le 5a_j^{2\varepsilon(a_j)+\varepsilon}T_j^{-\varepsilon}\end{aligned}$$

if $T_j'/T_j \le 1$. In the case of $T_j'/T_j > 1$, we have

$$T_j < T_j' \le \theta^j a_{T_j'} \le \theta^j a_j \le \theta T_j.$$

Hence (2.31) holds true in any case provided $\theta < 5/4$. By condition (2.24) we obtain

$$a_j^{2\varepsilon(a_j)+\varepsilon/2} \le T_j^{2\varepsilon(a_j)+\varepsilon/2} d(m, (L_mT_j)^{\delta+1/\alpha})^{-\varepsilon/2} \le T_j^{\varepsilon/2}.$$

Inserting it into (2.31) yields

$$p_0' \le 5(T_j/a_j)^{-\varepsilon/2} \le 5\theta^{-(j-1)\varepsilon/2}.$$

Let $x_l'^2 := x_{lj}'^2 = 4\log(T_j/a_j) + 2(1+A)d(m+1, r+l+1)$. Then we have

(2.32)

$$\begin{aligned}
p_2' : &= P\Big\{ \sup_{0\le t\le T_j'-a_{T_j'}} \sup_{0\le s\le a_j} \max_{k\ge 1} \Big|\sum_{l=0}^{\infty}\Big(X_k((t+s)_{r+l+1}^j) - X_k((t+s)_{r+l}^j)\Big)\Big| \\
&\quad \ge \sum_{l=0}^{\infty} x_l'\sigma^*(a_j/d(m+2, r+l+1))\Big\} \\
&\le \frac{4T_j'}{a_j}(\frac{a_j}{T_j})^2 \sum_{l=0}^{\infty}\sum_{k=1}^{\infty} d(m+2, r+l+1)e^{-(1+A)d(m+1,r+l+1)} \\
&\quad \cdot \exp\Big\{-\Big(\frac{2\sigma^{*2}(a_j/d(m+2, r+l+1))}{\sigma_k^2(a_j/d(m+2, r+l+1))} - 2\Big)\log\frac{T_j}{a_j}\Big\} \\
&\le \frac{4a_j}{T_j}\frac{T_j'}{T_j}\sum_{l=0}^{\infty} d(m+2, r+l+1)e^{-(1+A)d(m+1,r+l+1)} \\
&\quad \cdot \sum_{k=1}^{\infty}\Big(\frac{\sigma_k(a_j/d(m+2, r+l+1))}{\sigma^*(a_j/d(m+2, r+l+1))}\Big)^A.
\end{aligned}$$

Noting $0 < r' - r < 1$ (see (2.30)), we have

$$a_j/d(m+2, r+l+1) \le d(m+2, r')/d(m+2, r+1) \le 1$$

for $l \ge 0$. Hence, using the fact that $\sigma^{*2}(h) \ge dh^2$ for $0 < h \le 1$, we obtain

$$\sigma^*(a_j/d(m+2, r+l+1))^{-A} \le d^{-A/2}(a_j/d(m+2, r+l+1))^{-A}$$

and further

$$p_2' \le \frac{5d^{-A/2}a_j^{1-A}}{T_j}\sum_{l=0}^{\infty}(\alpha/e)^{(1+A)d(m+1,r+l+1)} \le c\theta^{-j}.$$

For the second series of the right hand side of (2.29), we have the same estimator.

Let $\beta > 0$ satisfy $r' - r + \beta < 1$. Then

(2.33)

$$\begin{aligned}
&\sum_{l=0}^{\infty} d(m+1, r+l+1)^{1/2}\sigma^*(a_j/d(m+2, r+l+1)) \\
&\le \sum_{l=0}^{\infty}\Big(1 - \Big(\frac{d(m+1, r+l+1-\beta)}{d(m+1, r+l+1)}\Big)^{1/2}\Big)^{-1}\int_{d(m+1,r+l+1-\beta)^{1/2}}^{d(m+1,r+l+1)^{1/2}} \sigma^*(a_j\alpha^{-y^2})dy \\
&\le \Big(1 - \Big(\frac{d(m+1, r+1-\beta)}{d(m+1, r+1)}\Big)^{1/2}\Big)^{-1}\int_{d(m+1,r+1-\beta)^{1/2}}^{\infty} \sigma^*(\alpha^{d(m+1,r')-y^2})dy \\
&\le \Big(1 - \Big(\frac{d(m+1, r+1-\beta)}{d(m+1, r+1)}\Big)^{1/2}\Big)^{-1}\int_{d(m+1,r+1-\beta)^{1/2}-d(m+1,r')^{1/2}}^{\infty} \sigma^*(\alpha^{-y^2})dy \\
&\le \varepsilon
\end{aligned}$$

provided T is large enough by condition (2.23). Furthermore, obviously, (2.33) implies

$$\sum_{l=0}^{\infty} \sigma^*(a_j/d(m+2, r+l+1)) \leq \frac{\varepsilon}{8}\sigma^*$$

for large T. Then

$$\sum_{l=0}^{\infty} x_l' \sigma^*(a_j/d(m+2, r+l+1)) \leq \frac{\varepsilon}{2}\sigma^*\Big(2\log\frac{T_j}{a_j}\Big)^{1/2}.$$

Combining these results, we obtain

$$\sum_{j=1}^{\infty} P\Big\{\sup_{j\geq 1} \sup_{0\leq t\leq T_j'-a_{T_j'}} \sup_{0\leq s\leq a_j} \max_{k\geq 1} \frac{|X_k(t+s)-X_k(t)|}{\sigma^*(2\log(T_j/a_j))^{1/2}} \leq 1+2\varepsilon\Big\} < \infty,$$

which implies (2.25).

The proofs of (2.26) and (2.27) are similar to that of (2.9) and (2.10) respectively when we use $(1+\varepsilon)\sigma^*(a_T)$ instead of σ^* (noting $\sigma^* \leq (1+\varepsilon)\sigma^*(a_T)$ for large T), so they are omitted. Theorem 2 is proved.

Remark 4. Employing the similar method we can relax the restriction (2.3) for a_T in [2],i.e., we can use α close to e enough instead of $\alpha = 2$.

As an application of Theorem 2, we establish the large increment result for l^∞-valued Ornstein-Uhlenbeck process. Let $\{Y(t), -\infty < t < \infty\} = \{X_k(t), -\infty < t < \infty\}_{k=1}^{\infty}$ be a sequence of independent Ornstein-Uhlenbeck processes with coefficients $\gamma_k \geq 0$ and $\lambda_k > 0$. We have $\sigma_k^2(h) = \frac{2\gamma_k}{\lambda_k}(1-e^{-\lambda_k h})$ and $\sigma_k^{*2} = \frac{2\gamma_k}{\lambda_k}$, $k = 1, 2, \cdots$. It is well-known that (2.7) is satisfied.

Corollary 1. *Suppose that there exists $A > 0$ such that*

$$\sum_{k=1}^{\infty}\Big(\gamma_k/\lambda_k\Big)^A < \infty$$

and that there exist $2 \leq \alpha < e, 0 < \delta < 1-1/\alpha$ and integer $m \geq 1$ such that (2.23) and (2.24) are satisfied. Then

$$\lim_{T\to\infty} \sup_{0\leq t\leq T-a_T} \sup_{0\leq s\leq a_T} \max_{k\geq 1} \frac{|X_k(t+s)-X_k(t)|}{\sigma^*(2\log(T/a_T))^{1/2}} = 1 \quad a.s.$$

$$\limsup_{T\to\infty} \sup_{0\leq t\leq T-a_T} \max_{k\geq 1} \frac{|X_k(t+a_T)-X_k(t)|}{\sigma^*(2\log(T/a_T))^{1/2}} = 1 \quad a.s.$$

References

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