SOME CONSEQUENCES OF A PECULIAR COMBINATORIAL IDENTITY

by

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Abstract

We present some probability inequalities and identities for the Poisson distribution. One of these identities shows a complicated sum involving products of Poisson CDFs to be an elementary function. We also give an inequality on Poisson probabilities looking like the three term recursion formula for orthogonal polynomials. These all follow from a rather remarkable combinatorial identity and it seems likely that the combinatorial identity has other applications.

1. Introduction

In this note, we present some identities and inequalities for the Poisson distribution. They all follow from a peculiar combinatorial identity. It seems likely that this combinatorial identity has applications to other common discrete distributions.

2. Notation and A Combinatorial Identity

The following notation is used in the sequel:

$$\begin{aligned} p(\lambda, x) &= \frac{e^{-\lambda}\lambda^x}{x!} = P(\text{Poisson } (\lambda) = x) \\ F(\lambda, x) &= \sum_{y=0}^{x} p(\lambda, y) = P(\text{Poisson } (\lambda) \leq x) \\ b(\lambda, \theta, x) &= \binom{n}{x}\theta^x(1-\theta)^{n-x} = P(\text{Binomial } (n, \theta) = x) \end{aligned} \tag{2.1}$$

We will now present a special case of a more general combinatorial identity. The special case is done first because there is a probabilistic proof of the special case and this probabilistic proof is crucial for the inequalities of Section 3. A nonprobabilistic proof of the general combinatorial identity will be presented in Section 4.

Lemma 1. Let n be any positive integer and $x \leq n$ any nonnegative integer. Then

$$x! = n^x - \binom{x}{1}(n-1)^x + \binom{x}{2}(n-2)^x - \ldots + (-1)^x(n-x)^x$$

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Proof: Let $A = \{1, \ldots, x\}$ and $B = \{1, \ldots, n\}$; $A = \phi$ if $x = 0$. Let f be a random function from A to B and consider the event C that $f(A) = A$. If we let C_i be the event that $i \in f(A)$, then $C = \bigcap_{i=1}^{x} C_i$. By the inclusion-exclusion formula,

$$\begin{aligned} P(C) &= P(\bigcap_{i=1}^{x} C_i) \\ &= 1 - P(\bigcup_{i=1}^{x} C_i^c) \\ &= 1 - \sum_{i=1}^{x} P(C_i^c) + \sum\sum_{i<j=1}^{x} P(C_i^c \cap C_j^c) - \cdots \\ &= 1 - x\frac{(n-1)^x}{n^x} + \binom{x}{2}\frac{(n-2)^x}{n^x} - \cdots + (-1)^x \binom{x}{x}\frac{(n-x)^x}{n^x} \end{aligned} \tag{2.2}$$

However, C can happen in $x!$ ways and so from (2.2),

$$\begin{aligned} \frac{x!}{n^x} &= 1 - \binom{x}{1}\frac{(n-1)^x}{n^x} + \binom{x}{2}\frac{(n-2)^x}{n^x} - \cdots + (-1)^x\binom{x}{x}\frac{(n-x)^x}{n^x} \\ \Rightarrow x! &= n^x - \binom{x}{1}(n-1)^x + \binom{x}{2}(n-2)^x - \cdots + (-1)^x(n-x)^x, \end{aligned}$$

proving the lemma.

3. Probabilistic Inequality

The method of proof of Lemma 1 leads to a rather interesting probability inequality for the Poisson distribution. The following Bonferroni - type inequality will be needed in addition.

Lemma 2. Let $C_1, \ldots C_m$ be measurable events in a probability space $(\Omega, \mathcal{B}, \mathcal{P})$. Let $S_k = \sum_{i_1 < i_2 < \ldots < i_k} P(C_{i_1} C_{i_2} \ldots C_{i_k})$, $1 \le k \le m$. Then

$$P(C_1 C_2 \ldots C_m) \ge \frac{(k+1)S_{k+1} - (m-k-1)S_k}{\binom{m}{k}}$$

Proof: See pg 93 in Galambos and Simonelli (1997). Combining Lemma 2 with the method of proof of Lemma 1 leads to the following inequality for Poisson distributions. The proof of Thereom 1 actually contains a much more general inequality.

Theorem 1. For all $n \geq 2, 0 \leq x \leq n$,

$$
\begin{aligned}
& p(n,x) - \frac{x}{e}p(n-1,x) + \frac{(x-1)}{e^2}p(n-2,x) \\
& \qquad \leq e^{-n} \leq p(n,x) - \frac{x}{e}p(n-1,x) + \frac{x(x-1)}{2e^2}p(n-2,x)
\end{aligned}
$$

Remark: For $x = 0, 1$, the Theorem can be proved directly. The following proof is for $2 \leq x \leq n$.

Proof: In Lemma 2, take C_i to be as in the proof of Lemma 1, identify m with x, and let n continue to be a fixed positive integer $\geq x$ as in Lemma 1.

Now,

$$
\begin{aligned}
S_k &= \sum_{i_1<\ldots<i_k} P(C_{i_1}C_{i_2}\ldots C_{i_k}) \\
&= \sum_{i_1<\ldots<i_k} P(\bigcup_{j=1}^{k} C_{i_j}^c)^c \\
&= \sum_{i_1<\ldots<i_k} \{1 - P(\bigcup_{j=1}^{k} C_{i_j})\} \\
&= \binom{x}{k}\{1 - \sum_j P(C_{i_j}^c) + \sum\sum_{j<\ell} P(C_{i_j}^c C_{i_\ell}^c) - \ldots + (-1)^k P(C_{i_1}^c \ldots C_{i_k}^c)\} \\
&= \binom{x}{k}\left\{1 - \binom{k}{1}\frac{(n-1)^x}{n^x} + \binom{k}{2}\frac{(n-2)^x}{n^x} - \ldots + (-1)^k \frac{(n-k)^x}{n^x}\right\} \\
&= \binom{x}{k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{(n-j)^x}{n^x} \qquad (3.1)
\end{aligned}
$$

Therefore, by an application of Lemma 2,

$$
\begin{aligned}
\frac{x!}{n^x} &= P(\bigcap_{i=1}^{x} C_i) \\
&\geq \frac{(k+1)S_{k+1} - (x-k-1)S_k}{\binom{x}{k}} \\
&= \frac{(k+1)\frac{x!}{(k+1)!(x-k-1)!}\sum_{j=0}^{k+1}(-1)^j\binom{k+1}{j}\frac{(n-j)^x}{n^x} - (x-k-1)\frac{x!}{k!(x-k)!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{(n-j)^x}{n^x}}{\frac{x!}{k!(x-k)!}}
\end{aligned}
$$

$$
\begin{aligned}
&= (x-k)\sum_{j=0}^{k+1}(-1)^j\binom{k+1}{j}\frac{(n-j)^x}{n^x} - (x-k-1)\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{(n-j)^x}{n^x} \\
&= (-1)^{k+1}(x-k)\frac{(n-k-1)^x}{n^x} + (x-k)\sum_{j=0}^{k}(-1)^j\left\{\binom{k+1}{j} - \binom{k}{j}\right\}\frac{(n-j)^x}{n^x} \\
&\quad + \sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{(n-j)^x}{n^x} \\
&\Rightarrow 1 \geq (-1)^{k+1}(x-k)\frac{(n-k-1)^x}{x!} + (x-k)\sum_{j=0}^{k}(-1)^j\left\{\binom{k+1}{j} - \frac{k}{j}\right\}\frac{(n-j)^x}{x!} \\
&\quad + \sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{(n-j)^x}{x!}. \qquad (3.2)
\end{aligned}
$$

Choosing, in particular, k to be 1 in (3.2), we have

$$
\begin{aligned}
1 &\geq (x-1)\frac{(n-2)^x}{x!} - (x-1)\frac{(n-1)^x}{x!} + \frac{n^x}{x!} - \frac{(n-1)^x}{x!} \\
&= \frac{n^x}{x!} - x\frac{(n-1)^x}{x!} + (x-1)\frac{(n-2)^x}{x!} \\
&= e^n p(n,x) - e^n\frac{x}{e}p(n-1,x) + e^n\frac{(x-1)}{e^2}p(n-2,x) \\
&\Rightarrow p(n,x) - \frac{x}{e}p(n-1,x) + \frac{(x-1)}{e^2}p(n-2,x) \leq e^{-n},
\end{aligned}
$$

as claimed in the Theorem. The reverse inequality follows easily.

4. The Generalized Combinatorial Identity

The combinatorial identity of Lemma 1 has the following strong generalization. We are unable to present a probabilistic proof of this generalization. Here is the generalized combinatorial identity.

Lemma 3. Let λ be any real number and x any nonnegative integer. Then

$$
x! = \lambda^x - \binom{x}{1}(\lambda-1)^x + \binom{x}{2}(\lambda-2)^x - \ldots + (-1)^x(\lambda-x)^x
$$

Proof: For a general λ, the proof consists of treating the rhs of the identity as a polynomial in λ and showing that the constant term equals $x!$ and the coefficient of λ^j for $1 \leq j \leq x$ is 0. Tomescu (1985) gives a proof using this technique in Exercise 1.38 for λ equal to a positive integer. But that proof applies for every real number λ.

5. Probabilistic Identities:

We will now present three identities that follow from Lemma 3. The first expresses Poisson probabilities in terms of binomial probabilities; the other two show a complicated function involving Poisson CDFs to work out to an elementary function. This is quite mysterious.

Theorem 2. With the notation of Section 2,

(a) $p(\lambda, x) = \frac{e^{-\lambda}}{1+\sum_{j=1}^{x}(\frac{j-\lambda}{j})^j \; b(x, \frac{j}{\lambda}, j)}$, for any $\lambda > 0$ and $x \leq [\lambda]$

(b) For every even integer $m \geq 0$ and every real $\varepsilon \geq 0$, $\sum_{j=0}^{m}(-1)^j e^{2j} F(j+\varepsilon, j) F(j+\varepsilon, m-j)$ $= \frac{e^{-2\varepsilon}(m+2)}{2}$

(c) For every odd integer $m \geq 1$, and every real $\varepsilon \geq 0$, $\sum_{j=0}^{m}(-1)^j e^{2j} F(j+\varepsilon, j) F(j+\varepsilon, m-j)$ $= -\frac{e^{-2\varepsilon}(m+1)}{2}$

Proof:

(a) By Lemma 3,

$$
\begin{aligned}
\frac{e^{-\lambda}}{p(\lambda, x)} = \frac{x!}{\lambda^x} &= 1 + \sum_{j=1}^{x}(-1)^j \binom{x}{j}(1 - \frac{j}{\lambda})^x \\
&= 1 + \sum_{j=1}^{x}(-1)^j \binom{x}{j}(1 - \frac{j}{\lambda})^{x-j}(\frac{j}{\lambda})^j(\frac{\lambda}{j})^j(1 - \frac{j}{\lambda})^j \\
&= 1 + \sum_{j=1}^{x}(-1)^j(\frac{\lambda}{j} - 1)^j b(x, \frac{j}{\lambda}, j) \\
&= 1 + \sum_{j=1}^{x}(\frac{j-\lambda}{j})^j b(x, \frac{j}{\lambda}, j) \qquad (5.1)
\end{aligned}
$$

and so part (a) follows:

(b) Again, by Lemma 3, for all $\lambda > 0$ and $0 \leq x \leq [\lambda]$,

$$
\begin{aligned}
1 &= \frac{\lambda^x}{x!} - \frac{(\lambda-1)^x}{(x-1)!1!} + \frac{(\lambda-2)^x}{(x-2)!2!} - \cdots + (-1)^x \frac{(\lambda-x)^x}{0!x!} \\
&= \sum_{j=0}^{x}(-1)^j \frac{(\lambda-j)^{x-j}}{(x-j)!} \frac{(\lambda-j)^j}{j!} \\
&= \sum_{j=0}^{x}(-1)^j e^{2(\lambda-j)} p(\lambda-j, x-j) p(\lambda-j, j) \qquad (5.2)
\end{aligned}
$$

Therefore, for all $\lambda > 0$, and $0 \le x \le [\lambda]$,

$$e^{-2\lambda} = \sum_{j=0}^{x} (-1)^j e^{-2j} p(\lambda - j, x - j) p(\lambda - j, j) \tag{5.3}$$

Summing (5.3) over x,

$$\begin{aligned}
([\lambda]+1)e^{-2\lambda} &= \sum_{x=0}^{[\lambda]} \sum_{j=0}^{x} (-1)^j e^{-2j} p(\lambda - j, x - j) p(\lambda - j, j) \\
&= \sum_{j=0}^{[\lambda]} \sum_{x=j}^{[\lambda]} (-1)^j e^{-2j} p(\lambda - j, x - j) p(\lambda - j, j) \\
&= \sum_{j=0}^{[\lambda]} (-1)^j e^{-2j} p(\lambda - j, j) F(\lambda - j, [\lambda] - j) \\
&= (-1)^{[\lambda]} e^{-2[\lambda]} \sum_{j=0}^{[\lambda]} (-1)^j e^{2j} p(\lambda - [\lambda] + j, [\lambda] - j) \\
&\qquad F(\lambda - [\lambda] + j, j)
\end{aligned} \tag{5.4}$$

Hence, for all $\lambda > 0$,

$$(-1)^{[\lambda]}([\lambda]+1)e^{2[\lambda]-2\lambda} = \sum_{j=0}^{[\lambda]} (-1)^j e^{2j} p(\lambda - [\lambda] + j, [\lambda] - j) F(\lambda - [\lambda] + j, j) \tag{5.5}$$

Writing n for the integer part of λ and ε for the fractional part of λ, from (5.5) one has, for all $n \ge 0$ and $\varepsilon \ge 0$,

$$(-1)^n (n+1) e^{-2\varepsilon} = \sum_{j=0}^{n} (-1)^j e^{2j} p(j + \varepsilon, n - j) F(j + \varepsilon, j) \tag{5.6}$$

If we now sum (5.6) over n, from $n = 0$ to any specified integer m, we get

$$\begin{aligned}
e^{-2\varepsilon} \sum_{n=0}^{m} (-1)^n (n+1) &= \sum_{n=0}^{m} \sum_{j=0}^{n} (-1)^j e^{2j} p(j + \varepsilon, n - j) F(j + \varepsilon, j) \\
&= \sum_{j=0}^{m} \sum_{n=j}^{m} (-1)^j e^{2j} p(j + \varepsilon, n - j) F(j + \varepsilon, j) \\
&= \sum_{j=0}^{m} (-1)^j e^{2j} F(j + \varepsilon, m - j) F(j + \varepsilon, j)
\end{aligned} \tag{5.7}$$

However, for even m, $\sum_{n=0}^{m}(-1)^n(n+1)$ is $1+\frac{m}{2}$, and so the identity in (b) follows.

For part (c), one uses that for odd m, $\sum_{n=0}^{m}(-1)^n(n+1)$ is $-\frac{m+1}{2}$.

References

Galambos, J. and Simonelli, I. (1997). Bonferroni-type Inequalities with Applications, Springer - Verlag, New York.

Tomescu, I. (1985). Problems in Combinatorics and Graph Theory, John Wiley, New York.