ON A DIFFERENTIAL EQUATION AND ONE STEP RECURSION FOR POISSON MOMENTS

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Abstract

Let $X \sim$ Poisson (λ), and $h(X)$ a given function of X such that $E_\lambda h(X))$ exists for all λ. We show that for each $n \geq 1$, $E_\lambda(X^n h(X))$ satisfies an nth order linear differential equation. The coefficients of this equation are explicit, and remarkably, they do not depend on the function h. A consequence is that from an expression for $E_\lambda(h(X))$, one can derive a closed form expression for $E_\lambda(X^n h(X))$ for all $n \geq 1$. In addition, these lead to exact expressions for the nth moment and the nth central moment of a Poisson random variable and in particular show that the n moment of a Poisson random variable with mean 1 is the nth Bell number B_n. These also characterize all functions $h(X)$ that are positively correlated with X.

We also present a general one step recursion formula for $E_\lambda(X^n h(X))$. These results may also facilitate computation of $E_\lambda(X^n h(X))$ as compared to direct computation from definition.

1. Introduction

The purpose of this article is to show that if X has a Poisson distribution with mean λ, and $h(X)$ is any function of X, then for all positive integers n, $E_\lambda(X^n h(X))$ admits an exact formula in terms of $f(\lambda) = E_\lambda(h(X))$ and its first n derivatives $f^{(j)}(\lambda)$, $j = 1, 2, \ldots, n$. Equivalently, one can assert that for each $n \geq 1$, $E_\lambda(h(X))$ itself satisfies an nth order linear differential equation, and it is remarkable that the coefficients of this differential equation do not depend on the function h. This exact formula also leads to a one-step recursion formula for the moment sequence $\{E_\lambda(X^n h(X))\}_{n\geq 1}$. Both of these facilitate closed form computation of $E_\lambda(X^n h(X))$ as compared to direct evaluation from

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definition. Our results, in addition, imply exact expressions for $E_\lambda(X^n)$, $E_\lambda(X-\lambda)^n$, and in particular, imply the fact that if $X \sim$ Poisson (1), then the nth moment of X equals the nth Bell number B_n.

The preceding results also have some interesting covariance implications. For instance, it follows that if X has zero correlation with $h(X)$ under every λ, then $h(X)$ must be a constant.

2. Notation and Preliminary Useful Facts

Throughout this article $S_2(n,i)$ will denote the Stirling second number defined as the number of partitions of a set of n elements into i nonempty disjoint subsets. Also, given any function h, $\Delta_1 h$ will denote the difference operator $\Delta_1 h(X) = h(X+1) - h(X)$ and for $k \geq 2$, $\Delta_k h$ will denote the kth order iterated difference operator $\Delta_k h(X) = \Delta_1(\Delta_{k-1} h(X))$. As usual, $\Delta_0 h(X) = h(X)$. With this notation, we first state some lemmas that will be subsequently used.

Lemma 1. For all $n \geq 1$, $X^n = \sum_{i=1}^{n} S_2(n,i)\{\prod_{k=0}^{i-1}(X-k)\}$.

Proof: This is well known; see, e.g., pp. 125–126 in Bryant (1993).

Lemma 2. $S_2(n,i) = S_2(n-1,i-1) + iS_2(n-1,i)$.

Proof: This is also well known; see Bryant (1993) again.

Lemma 3. For all $i \geq 0$, $h(X+i) = \sum_{j=0}^{i} \binom{i}{j} \Delta_j h(X)$

Proof: Fix any $n \geq 1$; then, the iterated differences $\Delta_0, \Delta_1, \ldots, \Delta_{n-1}$ are linear combinations of $h(X)$, $h(X+1), \ldots, h(X+n-1)$, i.e.,

$$\begin{pmatrix} \Delta_0 h(X) \\ \Delta_1 h(X) \\ \vdots \\ \Delta_{n-1} h(X) \end{pmatrix} = A_{n\times n} \cdot \begin{pmatrix} h(X) \\ h(X+1) \\ \vdots \\ h(X+n-1) \end{pmatrix}, \tag{2.1}$$

where the elements of A are $a_{ij} = (-1)^{i+j}\binom{i-1}{j-1}, 1 \le i,j \le n$. From (2.1),

$$\begin{pmatrix} h(X) \\ h(X+1) \\ \vdots \\ h(X+n-1) \end{pmatrix} = A^{-1} \cdot \begin{pmatrix} \Delta_0 h(X) \\ \Delta_1 h(X) \\ \vdots \\ \Delta_{n-1} h(X) \end{pmatrix}. \tag{2.2}$$

One can directly verify that the elements of A^{-1} are $a^{ij} = \binom{i-1}{j-1}$ and so the lemma follows.

Lemma 4. Let $X \sim$ Poisson (λ). Then, for all $i \ge 1$, $E_\lambda(h(X) \cdot \{\prod_{k=0}^{i-1}(X-k)\} = \lambda^i E_\lambda(h(X+i))$.

Proof: See Hwang (1982).

Lemma 5. Let $X \sim$ Poisson (λ). Then for all $k \ge 1$, $E_\lambda(\Delta_k h(X)) = \frac{d^k}{d\lambda^k} E_\lambda(h(X))$.

Proof: Let $p(\lambda, x)$ denote the Poisson pmf $\frac{e^{-\lambda}\lambda^x}{x!}$. Note that

$$\frac{d}{d\lambda} p(\lambda, x) = \frac{x}{\lambda} p(\lambda, x) - p(\lambda, x) \tag{2.3}$$

Therefore,

$$\begin{aligned} \frac{d}{d\lambda} E_\lambda(h(X)) &= \sum_{x=0}^{\infty} h(x) \frac{x}{\lambda} \frac{e^{-\lambda}\lambda^x}{x!} - \sum_{x=0}^{\infty} h(x) \frac{e^{-\lambda}\lambda^x}{x!} \\ &= \sum_{x=1}^{\infty} h(x) \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!} - \sum_{x=0}^{\infty} h(x) \frac{e^{-\lambda}\lambda^x}{x!} \\ &= \sum_{x=0}^{\infty} h(x+1) \frac{e^{-\lambda}\lambda^x}{x!} - \sum_{x=0}^{\infty} h(x) \frac{e^{-\lambda}\lambda^x}{x!} \\ &= E_\lambda(\Delta_1 h(X)). \end{aligned} \tag{2.4}$$

Now the lemma follows on using the fact $\Delta_k h = \Delta_1 \Delta_{k-1} h$ and by induction.

3. A Linear Differential Equation for $Eh(X)$

For a general function $h(X)$, we will now present a linear differential equation satisfied by $E_\lambda(h(X))$ in the following sense: fix any integer $n \ge 1$, and a function $h(X)$.

Denote $E_\lambda(h(X))$ by $f(\lambda)$. Then $f(\lambda)$ satisfies an nth order linear differential equation $c_{0,n}(\lambda)f(\lambda) + c_{1,n}(\lambda)f'(\lambda) + \cdots + c_{n,n}(\lambda)f^{(n)}(\lambda) = E_\lambda(X^n h(X))$. The coefficients $c_{0,n}, c_{1,n}, \ldots, c_{n,n}$ are explicit and they do not depend on the function h. In other words, if $X \sim$ Poisson (λ), then, one has the rather remarkable fact that one can write an explicit expression for $E(X^n h(X))$ for all $n \geq 1$ by only knowing an expression for $E(h(X))$!

Theorem 1. Let $X \sim$ Poisson (λ) and let $h(X)$ be such that $f(\lambda) = E_\lambda(h(X))$ exists for all λ. Then, for all $n \geq 1$, $E_\lambda(X^n h(X))$ also exists, and furthermore,

$$E_\lambda(X^n h(X)) = E_\lambda(X^n)E_\lambda(h(X)) + \sum_{j=1}^{n} c_{j,n}(\lambda)\frac{d^j}{d\lambda^j}f(\lambda), \tag{3.1}$$

where

$$c_{j,n}(\lambda) = \sum_{i=j}^{n} S_2(n,i)\binom{i}{j}\lambda^i. \tag{3.2}$$

Proof:

$$\begin{aligned}
E_\lambda(X^n h(X)) &= E_\lambda(h(X)\cdot\sum_{i=1}^{n} S_2(n,i)\{\prod_{k=0}^{i-1}(X-k)\}) \qquad \text{(By Lemma 1)} \\
&= \sum_{i=1}^{n} S_2(n,i)E_\lambda(h(X)\cdot\{\prod_{k=0}^{i-1}(X-k)\}) \\
&= \sum_{i=1}^{n} S_2(n,i)\lambda^i E_\lambda(h(X+i)) \qquad \text{(By Lemma 4)} \qquad (3.3) \\
&= \sum_{i=0}^{n} S_2(n,i)\lambda^i E_\lambda(h(X+i)) \qquad (\because S_2(n,0)=0) \\
&= \sum_{i=0}^{n} S_2(n,i)\lambda^i \sum_{j=0}^{i}\binom{i}{j}E_\lambda(\Delta_j h(X)) \qquad \text{(By Lemma 3)} \\
&= \sum_{i=0}^{n}\sum_{j=0}^{i} S_2(n,i)\binom{i}{j}\lambda^i\frac{d^j}{d\lambda^j}E_\lambda(h(X)) \qquad \text{(By Lemma 5)} \\
&= \sum_{j=0}^{n}\sum_{i=j}^{n} S_2(n,i)\binom{i}{j}\lambda^i\frac{d^i}{d\lambda^j}E_\lambda(h(X)) \qquad (3.4) \\
&= \sum_{i=0}^{n} S_2(n,i)\lambda^i E_\lambda(h(X)) + \sum_{j=1}^{n}\sum_{i=j}^{n} S_2(n,i)\binom{i}{j}\lambda^i\frac{d^j}{d\lambda^j}E_\lambda(h(X)). \qquad (3.5)
\end{aligned}$$

From (3.4), on using $h(X) \equiv 1$, one gets

$$E_\lambda(X^n) = \sum_{i=0}^{n} S_2(n,i)\lambda^i. \tag{3.6}$$

Now substituting (3.6) into (3.5), for a general h,

$$E_\lambda(X^n h(X)) = E_\lambda(X^n)E_\lambda(h(X)) + \sum_{j=1}^{n} c_{j,n}(\lambda)\frac{d^j}{d\lambda^j}f(\lambda),$$

as claimed.

This derivation of (3.1) itself implies that if $E_\lambda(h(X))$ exists for all λ, then $E_\lambda(X^n h(X))$ also exists for all λ and $n \geq 1$. This proves Theorem 1.

From (3.6), one immediately gets the following fact as a corollary.

Corollary 1. Let $X \sim$ Poisson (1). Then $E(X^n) = B_n =$ the nth Bell number = Total number of partitions of a set of n elements into disjoint nonempty subsets.

4. A General Moment Recursion Formula

The expansion in (3.3) will be now used to obtain an interesting one step recursion relation for $E_\lambda(X^n h(X))$.

Theorem 2. Let $X \sim$ Poisson (λ) and let $h(X)$ be such that $E_\lambda(h(X))$ exists for all λ. Then, for all $n \geq 1$,

$$E_\lambda(X^n h(X)) = \lambda\{E_\lambda(X^{n-1}h(X)) + \frac{d}{d\lambda}E_\lambda(X^{n-1}h(X))\} \tag{3.6}$$

Proof: From (3.3),

$$\begin{aligned}
&E_\lambda(X^n h(X)) \\
&= \sum_{i=1}^{n} S_2(n,i)\lambda^i E_\lambda(h(X+i)) \\
&= \sum_{i=1}^{n} S_2(n-1,i-1)\lambda^i E_\lambda(h(X+i)) + \sum_{i=1}^{n} iS_2(n-1,i)\lambda^i E_\lambda(h(X+i)) \qquad \text{(By Lemma 2)} \\
&= \lambda\sum_{i=0}^{n-1} S_2(n-1,i)\lambda^i E_\lambda(h(X+i+1)) + \lambda\sum_{i=1}^{n} S_2(n-1,i)(\lambda^i)' E_\lambda(h(X+i))
\end{aligned}$$

$$= \lambda \sum_{i=1}^{n-1} S_2(n-1,i)\lambda^i E_\lambda(h(X+i+1)) + \lambda \sum_{i=1}^{n-1} S_2(n-1,i)(\lambda^i)' E_\lambda(h(X+i))$$

$$(\because S_2(n-1,0) = S_2(n-1,n) = 0)$$

$$= \lambda \sum_{i=1}^{n-1} S_2(n-1,i)\lambda^i E_\lambda(h(X+i+1)) + \lambda \sum_{i=1}^{n-1} S_2(n-1,i)\frac{d}{d\lambda}(\lambda^i E_\lambda(h(X+i)))$$

$$- \lambda \sum_{i=1}^{n-1} S_2(n-1,i)\lambda^i \frac{d}{d\lambda} E_\lambda(h(X+i))$$

$$= \lambda \sum_{i=1}^{n-1} S_2(n-1,i)\lambda^i E_\lambda(h(X+i+1)) + \lambda \frac{d}{d\lambda} E_\lambda(X^{n-1}h(X))$$

$$- \lambda \sum_{i=1}^{n-1} S_2(n-1,i)\lambda^i \frac{d}{d\lambda} E_\lambda(h(X+i)) \qquad \text{(By (3.3)}$$

$$= \lambda \sum_{i=1}^{n-1} S_2(n-1,i)\lambda^i E_\lambda(h(X+i+1)) + \lambda \frac{d}{d\lambda} E_\lambda(X^{n-1}h(X))$$

$$- \lambda \sum_{i=1}^{n-1} S_2(n-1,i)\lambda^i \{E_\lambda(h(X+i+1)) - E_\lambda(h(X+i))\} \qquad \text{(By Lemma 5)}$$

$$= \lambda \frac{d}{d\lambda} E_\lambda(X^{n-1}h(X)) + \lambda \sum_{i=1}^{n-1} S_2(n-1,i)\lambda^i E_\lambda(h(X+i))$$

$$= \lambda\{\frac{d}{d\lambda} E_\lambda(X^{n-1}h(X)) + E_\lambda(X^{n-1}h(X))\}. \qquad \text{(By (3.3) again)}$$

This proves Theorem 2.

5. Some Applications

We close this article with three specific applications of the results given in the previous sections.

Theorem 3. Let $X \sim$ Poisson (λ) and $h(X)$ is such that $f(\lambda) = E_\lambda(h(X))$ exists for all λ. Then,

(a) $\mathrm{Cov}_{\lambda_0}(X, h(X)) \geq 0$ at a specified λ_0 if and only if $h(X)$ has an increasing expectation *locally* at λ_0, i.e., $f'(\lambda_0) \geq 0$;

(b) There is no nontrivial function h such that $\mathrm{Cov}_\lambda(X, h(X)) = 0$ for all λ.

Proof: (a) By Theorem 1, $E_\lambda(Xh(X)) = E_\lambda(X)E_\lambda(h(X)) + c_{1,1}(\lambda)f'(\lambda)$, where $c_{1,1}(\lambda) = \lambda S_2(1,1) = \lambda$. Thus,

$$\mathrm{Cov}_\lambda(X, h(X)) = \lambda f'(\lambda) \geq 0 \tag{5.1}$$

if and only if $f'(\lambda) \geq 0$.

(b) From (5.1), $\mathrm{Cov}_\lambda(X, h(X)) = 0 \ \forall \lambda$

$$\begin{aligned} &\Leftrightarrow f'(\lambda) = 0 \quad \forall \lambda \\ &\Leftrightarrow f(\lambda) = \text{ constant} \\ &\Leftrightarrow h(X) = \text{ constant}, \end{aligned}$$

by the completeness of the Poisson family (see Lehmann and Casella (1998)). This proves Theorem 3.

The next result is on a formula for the central moments of a Poisson distribution.

Theorem 4. Let $X \sim$ Poisson (λ). Then,

(a) For any $n \geq 1$,

$$E_\lambda(X - \lambda)^n = \sum_{k=0}^{n} a_{k,n}\lambda^k,$$

where

$$a_{k,n} = \sum_{i=0}^{k} (-1)^i \binom{n}{i} S_2(n-i, k-i). \tag{5.2}$$

in the above, $S_2(0,0) = 1$;

(b) For any $n \geq 1$, the leading coefficient $a_{n,n} = 0$;

(c) For $n \geq 3$, $a_{n-1,n}$ is also 0.

Proof:

(a) By Theorem 1 and Binomial expansion,

$$
\begin{aligned}
E_\lambda(X-\lambda)^n &= \sum_{k=0}^{n}\sum_{i=0}^{k}(-1)^{n-k}\binom{n}{k}S_2(k,i)\lambda^{n-k+i} \\
&= \sum_{i=0}^{n}\sum_{k=i}^{n}(-1)^{n-k}\binom{n}{k}S_2(k,i)\lambda^{n-k+i} \\
&= \sum_{i=0}^{n}\sum_{k=0}^{n-i}(-1)^{n-k-i}\binom{n}{k+i}S_2(k+i,i)\lambda^{n-k} \qquad \text{(write } k \text{ for } k-i\text{)} \\
&= \sum_{i=0}^{n}\sum_{k=i}^{n}(-1)^{k-i}\binom{n}{n-k+i}S_2(n-k+i,i)\lambda^{k} \qquad \text{(write } k \text{ for } n-k\text{)} \\
&= \sum_{i=0}^{n}\sum_{k=i}^{n}(-1)^{k-i}\binom{n}{k-i}S_2(n-k+i,i)\lambda^{k} \\
&= \sum_{k=0}^{n}\sum_{i=0}^{k}(-1)^{k-i}\binom{n}{k-i}S_2(n-k+i,i)\lambda^{k} \\
&= \sum_{k=0}^{n}\sum_{i=0}^{k}(-1)^{i}\binom{n}{i}S_2(n-i,k-i)\lambda^{k} \qquad \text{(write } i \text{ for } k-i\text{)} \\
&= \sum_{k=0}^{n}a_{k,n}\lambda^k,
\end{aligned}
$$

as claimed.

(b) From (5.2), the coefficient of λ^n is

$$
\begin{aligned}
a_{n,n} &= \sum_{i=0}^{n}(-1)^i\binom{n}{i}S_2(n-i,n-i) \\
&= \sum_{i=0}^{n}(-1)^i\binom{n}{i} \\
&= 0.
\end{aligned}
$$

(c) Again, from (5.2), the coefficient of λ^{n-1} is

$$
\begin{aligned}
a_{n-1,n} &= \sum_{i=0}^{n-1}(-1)^i \binom{n}{i} S_2(n-i, n-i-1) \\
&= \sum_{i=0}^{n-1}(-1)^i \binom{n}{i}\binom{n-i}{2} \\
(\because S_2(m, m-1) &= \binom{m}{2}; \text{ see pp. 18 in Tomescu (1985))} \\
&= \sum_{i=0}^{n-2}(-1)^i \binom{n}{i}\binom{n-i}{2} \\
&= \frac{n(n-1)}{2}\sum_{i=0}^{n-2}(-1)^i \binom{n-2}{i} \\
&= 0 \text{ if } n > 2,
\end{aligned}
$$

as claimed

It turns out that for the Poisson distribution, the third central moment is also λ. This is stated below.

Corollary 2. If $X \sim$ Poisson (λ),

$$E_\lambda(X-\lambda)^2 = E_\lambda(X-\lambda)^3 = \lambda, \quad E_\lambda(X-\lambda)^4 = \lambda + 3\lambda^2.$$

Proof: Follows from (5.2)

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