A BOOK REVIEW OF SEVEN BOOKS ON PROBABILISTIC FINANCE THEORY

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by
Phillip Protter[1]
Purdue University

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Department of Statistics
Purdue University
West Lafayette, IN USA

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A REVIEW OF SEVEN RECENTLY PUBLISHED BOOKS CONCERNING PROBABILISTIC FINANCE THEORY

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Purdue University

Abstract

The list of books used to compile this technical report are as follows:

Introduction to Stochastic Calculus Applied to Finance. *By D. Lamberton and B. Lapeyre.* (Translated by N. Rabeau and F. Mantion.) Chapman & Hall, London, 1996. $64.95. xi+185 pp., hardcover. ISBN 0-412-71800-6.

Introduction to Mathematical Finance: Discrete Time Models. *By Stanley R. Pliska.* Blackwell, Malden MA, 1997. $52.95. x+262 pp., hardcover. ISBN 1-55786-945-6.

Financial Calculus: An Introduction to Derivative Pricing. *By Martin Baxter and Andrew Rennie.* Cambridge University Press, Cambridge, UK 1996 (reprinted with corrections, 1997, 1998), $42.95. ix+233 pp., hardcover. ISBN 0-521-55289-3.

Generalised Optimal Stopping Problems and Financial Markets. *By Dennis Wong.* Addison Wesley Longman, Ltd., Essex, UK, 1996. $40.33. vi+114 pp., paper. Pitman Research Notes in Mathematics Series. Vol 358. ISBN 0-582-30400-8.

American Put Options. *By D. M. Salopek.* Addison Wesley Longman, Ltd., Essex, UK, 1997. $54.46. 113 pp., hardcover. Pitman Monographs and Surveys in Pure and Applied Mathematics. Vol 84. ISBN 0-582-31594-8.

Exotic Options: A Guide to Second Generation Options. *By Peter G. Zhang.* World Scientific, Singapore, 1997. $96.00. xvii+675 pp., hardcover. ISBN 981-02-2222-X.

Martingale Methods in Financial Modelling. *By Marek Musiela and Marek Rutkowski.* Springer-Verlag, Berlin, 1997. $79.95. xii+512 pp., hardcover. Applications of Mathematics. Vol. 36. ISBN 3-540-61477-X.

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1 Overview

Mathematical finance is a new subject and a fascinating one. It uses recently developed mathematics in immediate applications where large sums of money are involved. Because of its applicability, there is a strong demand from the investment community for highly-trained and competent technicians. At the same time Finance Theory has created a significant research thrust in Probability Theory from a new direction, creating new problems, new ways of viewing traditional objects, and new theories.

Before discussing the books under review, let us situate the emerging literature. A probabilistic analysis of the stock market began in 1900 with the thesis of a young French mathematician, Louis Bachelier. Both subjects, Economics and Probability, were under shadows of suspicion in the world of French mathematics at the time, but the latter was more acceptable as it was motivated by physics. Bachelier essentially invented Brownian motion five years before Einstein's famous 1905 paper and decades before Kolmogorov gave mathematical legitimacy to the 300-year-old subject of Probability Theory. Bachelier's thesis was not well received and, as a result, he was blackballed by the mathematical establishment and banished to internal exile at Besançon, a small provincial capital in the mountains near Switzerland. Indeed, Poincaré remarked that his thesis "topic is somewhat remote from those our candidates are in the habit of treating" and he regretted that Bachelier did not develop further the connection to the heat equation he had discovered. Bachelier's model survived, however, and Brownian motion became the basis for models of the stock markets a half-century later, championed by P. Samuelson among others.

A further breakthrough occurred in mathematical modelling when Black, Scholes and Merton proposed a method in the 1970's to price European options via an explicit formula. This brought order to a chaotic situation, where pricing had been done previously by intuition and ill-defined market forces. The ideas used by the Black–Scholes model involved a concept new to probability: that of arbitrage. Indeed, if a random variable X represents a contingent claim to be paid in the future, a fair price to pay for X based on traditional probability theory (going back to Huygens and J. Bernoulli) would be $E\{X\}$, the mathematical expectation of X. ($E\{X\} = \int_\Omega X(\omega)P(d\omega)$, where P is the underlying probability measure on the space where X is defined as a random variable.) This pricing mechanism does not work for economic models: A standard assumption in economic models is that there is an *absence of arbitrage;* that is, the model does not permit the existence of a strategy that has a positive probability of making a profit with no chance of incurring a loss. (The colloquial terms for this are "no free lunch", or "no profits without risk".) With the assumption of no arbitrage and using the ideas pioneered by Black, Scholes and Merton, one can show that pricing X by $E\{X\}$ will lead to an arbitrage opportunity: Thus $E\{X\}$ cannot be the right price.

So what is the right price? Here is where modern Probability Theory enters. The now classical model of asset prices is to model them by diffusions: solutions of stochastic differential equations. Let $(B_t)_{t\geq 0}$ represent a Brownian motion, and let X be the solution of an ordinary stochastic differential equation:

$$dX_t = \sigma(t, X_t)dB_t + b(t, X_t)dt; \quad X_0 = x. \tag{1}$$

To understand intuitively an equation such as (1), imagine a pollen grain floating down a river. A non-random ODE modelling its flow might be

$$dx(t) = f(t, x(t))dt; \tag{2}$$

here "dt" represents the force of a constant current, and $f(t, x)$ the sensitivity to the constant current at time t and place x. In addition to the current there is a much smaller force due to the constant bombardment of the pollen grain by vibrating water molecules: the Brownian force. Thus a more sophisticated model involving randomness of the flow of the pollen grain might be

$$dX(t) = f(t, X(t))dt + g(t, X(t))dB_t \tag{3}$$

where $g(t, x)$ represents the sensitivity at time t and place x to be the Brownian force B (such sensitivity would be largely due to varying water temperatures in different parts of the river). This same model seems to work for modelling asset prices (equation (3) is the same as (1)): a stock has a drift $b\,dt$ and noise σdB_t.

One is thus led to a probabilistic model $(\Omega, \mathcal{F}, P, (B_t)_{t\geq 0}(\mathcal{F}_t)_{t\geq 0})$ where B is a Brownian motion and $(\mathcal{F}_t)_{t\geq 0}$ is a filtration of σ-algebras corresponding to B. A contingent claim is then a random variable H occurring at a future time T (we say H is then $\mathcal{F}_T$-measurable). The present is time $t = 0$. The price of H is not $E\{H\}$; instead (because of the no arbitrage assumption) the price of H inevitably becomes $E_Q\{H\}$ after normalization of all values by the price of some numeraire, such as a bond, which we shall continue to do in order to simplify this review. Here Q is another probability measure equivalent to P. One can prove that Q is a Probability measure that transforms the solution X of (1) into a martingale. (Mathematically this has the effect of eliminating the drift term in the SDE (1).)

The model works well in many ways: the equivalent probability measure Q is unique, so there is a unique price for H. In addition, one can calculate explicitly the Radon-Nikodym derivative for Q. Finally, in a simple case, that of $\sigma(t,x) = \sigma x$ and $b(t,x) = \mu x$, one can compute a formula for $E_Q\{H\}$ for a certain type of H that is useful in practice. (This is the celebrated Black–Scholes formula.) More specifically, for the model

$$dX_t = \sigma X_t dB_t + \mu X_t dt, \qquad X_0 = 1;$$

if $H = (X_T - K)^+$ (a contingent claim corresponding to a standard European option), there is a closed-form expression for $E_Q\{H\}$ involving σ and K, but not μ. Thus to price the option one needs only to estimate σ from market observations.

There is one more consideration. People buy and sell options. For a price to be fair, the seller of the option should be able to replicate the contingent claim by following a strategy of buying and selling shares of stock from time $t = 0$ until time $t = T$, using only the capital obtained from selling the claim. (We ignore transaction costs.) This strategy is called *hedging.* If one follows a strategy θ_t, where θ_t is the number of shares held at time t, then one's fortune at time t is:

$$F_t = F_0 + \int_0^t \theta_s dX_s,$$

where X_t represents the share price given by equation (1), and $\int_0^t \theta_s dX_s$ is a stochastic integral. Under Q, X is a martingale and so is F under mild technical assumptions on θ. We need $F_T = H$ and then $E_Q\{H\} = E_Q\{F_T\} = E_Q\{F_0\} = F_0$. Another advantage of the theory is that there always exists such a strategy θ. One can describe θ as the predictable projection of a functional derivative of H (known as a "Malliavin derivative"); in simple cases — such as European options — $\theta_t = u(X_t)$, where u can be described via a solution of a partial differential equation. It is of significant importance to know what θ is; traders actually use estimates of θ to guide their hedging strategies.

When the coefficients σ and b in (1) are no longer constant, a closed form expression for the price of a European option no longer exists; however it can be expressed as the solution of a partial differential equation and numerically approximated. There are also other kinds of options (American options and Asian options, for example) which never have simple formulas. We see that a study of this subject involves quite a few different areas:

- Advanced probability theory, including stochastic integration, stochastic differential equations, and martingales.
- Statistics (to estimate coefficients σ and b from market data).
- Partial differential equations.
- Numerical analysis of partial differential equations and also the young subject of numerical analysis of stochastic differential equations.
- Functional analysis (to understand hedging strategies which can be expressed as Malliavin derivatives — a type of extension of a Fréchet derivative).

I should point out also that the model of stock markets I have sketched above is used as a good working model by practitioners, but it is almost universally acknowledged that its major hypotheses are at best only approximately true. Much current research is underway to understand what happens when the hypotheses correspond better to reality. Unfortunately many of the model's pretty properties then disappear.

In view of the above, the natural student to learn this subject is a Ph.D. student in Probability Theory with wide-ranging interests. There is also strong demand from the Financial sector for well-trained technicians: Master's students who have a knowledge of the subject to the point where they can understand the probability models, derive and solve numerically the corresponding PDEs if appropriate, and perhaps also compute a hedging strategy.

This creates distinctly different audiences for books that give introductions to the subject, and this brings us to the books under review.

2 Pedagogic Books

The first group of three books are pedagogic in nature. The book by *Lamberton and Lapeyre* is the best one I have seen for a course aimed at the level of a Ph.D. student. A reader should have a good knowledge of post measure theoretic probability up through Brownian motion and stochastic processes, as well as some familiarity with partial differential equations. While this seems like a lot of preparation one still cannot treat the subject properly: For example when studying American options one wants to treat Snell envelopes, Novikov's theorem, the Doob–Meyer decomposition and Girsanov's theorem. On page 73 this is dismissed with the statement "to overcome technical difficulties we give only the outlines of the proof". To fill in the outline would add thirty pages to the book and take several weeks of study. Nevertheless this is a serious book as far as it goes and is quite well done. It is a translation from the French original and minor but annoying small mistakes have been added by the translators.

For example as early as page 6, Lemma 1.2.6 is placed before Theorem 1.2.7; but the notation used in the statement of Lemma 1.2.6 is defined in the proof of the following theorem. Such mistakes are minor and easily corrected by a diligent reader. The mathematics of the book is careful and well done.

The book by *Pliska* is also an excellent book from a mathematician's standpoint. It covers quite different topics from *Lamberton and Lapeyre.* To wit, *Lamberton and Lapeyre* treat continuous time Finance, which necessitates a treatment of Brownian motion and stochastic calculus (Itô calculus). *Pliska* instead restricts his subject to discrete time models, which significantly lowers the barrier to entry. This does not mean, however, that Pliska has written a book that contains easy material. It could be argued that discrete time models, while needing less technical knowledge, become sufficiently complicated as to be ultimately more difficult. Nevertheless a working understanding of discrete time models can provide a good basis for a future study of continuous time models. (This is implicitly recognized by both *Lamberton–Lapeyre* and *Baxter–Rennie* who begin their books with discrete time models before treating continuous time ones.) *Pliska* begins by treating finite sample spaces both for single and multiperiod models. The First Fundamental Theorem of Asset Pricing (i.e., there is no arbitrage if and only if there is an equivalent risk neutral probability measure — also known as an equivalent martingale measure) is proved in the usual way using the Hahn–Banach theorem. Later, in Chapter 7, he allows sample spaces to be infinite and proves the same theorem. He provides careful treatments of European and American options, but due perhaps to his desire for accessibility, he does not develop the tools he needs. For example, on page 131 he says "we need something called the optional sampling theorem which says...". It would have been preferable to carefully state and prove the theorems of this type that are needed. As it stands, no references are given for these results, and the reader is on his own to try to find an accessible source. This is the serious defect of the book.

The book of *Baxter and Rennie* is quite different from *Lamberton–Lapeyre* and *Pliska.* It makes no attempt to be rigorous. Instead *Baxter–Rennie* gives heuristic descriptions of mathematics for students with insufficient backgrounds to study the subject properly. It is, of course, a tricky business to explain not just the Itô calculus but also equivalent martingale measures and the like on a heuristic level; in this I feel *Baxter–Rennie* is surprisingly effective. They begin with discrete models (as do *Lamberton–Lapeyre*) with very clear explanations, and then try to use this to motivate continuous time models. While essentially nothing is proved and the word "Theorem" does not appear, the authors are nevertheless careful and take care to state results with precise (and correct) hypotheses.

3 Research Books

The four research books covered in this review are quite different. Let us begin with the two small books published by Pitman. The book by *Wong* states in the preface that "the aim of this book is...to provide an introduction to the mathematics used to describe financial markets". This is an excellent idea as such a book is acutely needed. The book as it stands, however, is ill conceived and poorly executed. I have no idea who the audience is for this book. In the preliminaries the topics are treated so briefly that the reader who does not already know them will not be able to grasp them, and the reader who does already know them will have no need of such a short and superficial review. Then later, for example, Wong assumes without special mention that the reader is familiar with the Doob–Meyer decomposition theorem; is it then necessary to review the Itô integral for such a reader? The heart of the book is Chapter 2 on Optimal Stopping Problems, where the author treats topics related to Snell envelopes (useful in studying American options); perhaps this book will prove interesting for that reason, as the treatment is quite readable.

The book by *Salopek* is devoted uniquely to American options in the continuous time setting. A European option gives the buyer the right to purchase the asset at an agreed price and at an agreed time in the future. An American option give the buyer the right to purchase the asset at an agreed price but at any time between the present and an agreed time in the future. In general it is more difficult to analyze mathematically American options. After a brief introduction, Chapter 2 provides a fairly nice and well-done partial review of some of the fundamental papers concerning American options in the Black–Scholes setting; in particular the work of Jacka and Karatzas. Here the connection between the probability models and the pricing of the option via a free boundary partial differential equation is made clear. This material is not new to book format; one could alternatively consult the wonderful book of D. Duffie (for example: "Dynamic Asset Pricing Theory" (Second Edition), 1996). Nevertheless *Salopek's* treatment of American options, complete with analytic approximations and a taste of numerical methods, gives a nice overview. An annoying feature of the book is that the reference numbers in the text do not agree with those of the bibliography.

Up to now we have referred only to the two principal types of options: the European and American options. While these are the most important and most common, they are but the tip of the iceberg, and the next book — the one of *Peter Zhang* — attempts to treat (in a partial survey style) many of the less common options known as exotic options. This includes the fairly common Asian options, as well as more unusual options such as Bermuda options, so-named because they combine features of both European and American options. Except for *Baxter–Rennie,* this is the only book under review not written by an academic, but rather by a practitioner (Zhang is a vice president of Chase Manhattan Bank). In the language of Zhang and of most practitioners the European and American options already discussed are "plain vanilla options". A European option has a price expressed as $E_Q\{(X_T - K)^+\}$, where Q is the risk neutral measure (also called the equivalent martingale measure), X_T is the asset price at time T, and K is the strike price — that is, the agreed upon price for which the asset can be purchased at time T. Note that the price involves the stock price only at time T. A large class of exotic options can be described as being path dependent options: options for which the price can abstractly be written as $E_Q\{f(X_s; 0 \le s \le T)\}$.

That is, the price is allowed to depend on the entire path the price history has taken up to time T. An important example arising from the oil industry are Asian options, where prices at time T are based on weighted averages of prices before time T. This is intended to prevent manipulation of options prices at time T where there are a few very large dominant players, as is the case with oil commodity prices and oil companies.

A serious limitation of *Zhang's* book is his desire to have "closed form solutions" to his options. This leads him to assume that asset prices follow the classic stochastic differential equation

$$dX_t = \sigma X_t dB_t + bX_t dt \tag{4}$$

with σ and b constants. As previously discussed, almost no one believes that (4) is a realistic model of asset prices. Instead the book would have been more interesting if it included a serious discussion of how to approximate general solutions numerically without resorting to what *Zhang* refers to as closed form solutions. That said, there is a (perhaps unintentional) advantage of the limitation to equation (4) for asset prices: The author is able to focus on describing and explaining the myriad types of exotic options that currently exist, without a large number of distracting discussions about varying hypotheses and situations. The mathematical contributions of *Zhang's* book are rather minimal, but what this reviewer found rewarding was his descriptions, explanations, and insightful discussions of various types of exotic options. The import of the book is its clear expository treatment of path dependent options, whether they be Asian options, Lookback options, or different types of barrier options (knock-ins, knock-outs, digital, moving barrier, etc.). The focus is always the pricing of options, with almost no mention of hedging strategies. Also not all exotic option type financial derivatives are mentioned: swaptions, for example, are excluded.

Last we consider the book of *Musiela and Rutkowski.* Despite its 500 page length and its appearance in a Springer series, this book can be thought of as a text for approximately the same students who might use *Lamberton–Lapeyre:* namely students at the Ph.D. level who already have studied measure theoretic Probability Theory through at least Brownian motion.

Musiela–Rutkowski begin with a one hundred page treatment of the discrete time case. This is more detailed than that of *Lamberton–Lapeyre* (for example, they include a discussion of transaction costs), but this reviewer prefers the treatment by *Pliska,* which not only goes into more detail, but asks (and answers) more interesting questions. That said, the heart of *Musiela–Rutkowski's* book is the treatment of continuous time models. Here again there seems to be an overdependence on the very simplified Black–Scholes framework of equation (4). Even in Chapter 6, where the Black–Scholes equation is modified, the modification is still linear but with time dependent coefficients. Nevertheless, within these constraints the authors do a reliable job of explaining the standard issues at an advanced level. A particular strength of the book is its many non-technical discussions of the relevant literature; if the reader feels a certain treatment is too elementary, he (or she) is given many references in the literature to consult. Also when the authors apparently feel a given topic is too advanced, they occasionally give a nice synopsis of what one will find if one consults the literature. An example is their discussion of one of Delbaen–Schachermayer's versions of the Second Fundamental Theorem of Asset Pricing. The *Musiela–Rutkowski* book is the closest approximation of the books reviewed to a research level text on the subject.

4 Projections

The area of Probabilistic Finance Theory is currently undergoing intense mathematical development. New tools, such as backward and forward–backward stochastic differential equations are being developed. New techniques in the numerical analysis of stochastic differential equations can allow one to abandon the simplifying assumptions of the classical Black–Scholes model. Alternative models are being proposed when one no longer has complete markets. Much new research is being done concerning transaction costs. Finally the fundamental assumptions of most models are being questioned: that of continuous sample paths, and that of the Markov nature of asset prices. Given this exciting and rich research going on, hopefully we can look forward to more advanced books on the subject in the future, as the interplay between the worlds of Finance and Probability Theory continues.

Philip Protter
Pudue University