ON SELECTION PROCEDURES FOR EXPONENTIAL FAMILY DISTRIBUTIONS BASED ON TYPE–I CENSORED DATA*

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Abstract

We investigate the problem of selecting the best population from exponential family distributions based on type-I censored data. A Bayes rule is derived and a monotone property of the Bayes selection rule is obtained. Following that property, we propose an early selecti on rule. Through this early selection rule, one can terminate the experiment on a few pop ulations early and possibly make the final decision before the censoring time. An example is provided in the final part to illustrate the use of the early selection rule for Weibull populations.
AMS Classification: 62F07, 62L10.

Keywords: Type–I censored data, best population, Bayes selection rule, early selection rule.

On Selection Procedures For Positive Exponential Family Distributions Based On Type-I Censored Data [1]

by

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Abstract

We investigate the problem of selecting the best population from positive exponential family distributions based on type-I censored data. A Bayes rule is derived and a monotone property of the Bayes selection rule is obtained. Following that property, we propose an early selection rule. Through this early selection rule, one can terminate the experiment on a few populations early and possibly make the final decision before the censoring time. An example is provided in the final part to illustrate the use of the early selection rule.

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1. Introduction

Consider designing and analyzing an experiment for comparing k populations π_1, $\pi_2, \cdots, \pi_k$. Suppose that m items are taken from each population and observations can be obtained from those items in time order, as for example, in a life-testing experiment. It is often desirable to terminate the test from a population as soon as there is enough statistical evidence that it is not the best population, and then this population is eliminated from further consideration.

Assume that the random observations from population π_i have a density function $f(x|\theta_i)$ of the form

$$f(x|\theta) = c(\theta)exp\{\beta(\theta)Q(x)\}h(x), \qquad x \in \mathcal{X}, \qquad (1.1)$$

where $\mathcal{X} \subset (0, \infty)$ and $h(x) > 0$ for $x \in \mathcal{X}$. Let Ω be the parameter space for each θ_i.

Let $\theta_{[1]} \leq \theta_{[2]} \leq \cdots \leq \theta_{[k]}$ denote the ordered values of the parameters $\theta_1, \theta_2, \cdots, \theta_k$. It is assumed that the exact pairing between the ordered and the unordered parameters is unknown. The population associated with the largest value $\theta_{[k]}$ is considered as the best population. The readers are referred to Gupta and Panchapakesan (1979) for a general reference on selection and ranking procedures.

The function $f(x|\theta)$ is assumed to have (nondecreasing) monotone likelihood ratio with respect to x. This assumption is equivalent to

$$(\beta(\theta_2) - \beta(\theta_1))(Q(x_2) - Q(x_1)) \geq 0 \qquad (1.2)$$

for any $\theta_2 \geq \theta_1$, $\theta_1, \theta_2 \in \Omega$, $x_2 \geq x_1$, $x_1, x_2 \in \mathcal{X}$. Without loss of generality, we assume that $Q(x)$ is a nondecreasing function of x and $\beta(\theta)$ is a nondecreasing function of θ.

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Many (one-parameter) exponential family distributions, such as *Chi-square, exponential, gamma* (α, β) with one of the two parameters known, *lognormal* (μ, σ^2) with σ known, *Weibull*(γ, β) with γ known, have the property (1.2). So our results here can be applied to them.

In an application situation of industrial life-testing experiment, m items from each of the k population $\pi_1, \cdots, \pi_k$ are independently put on test at the outset and are not replaced on failure. Due to the time restriction, the experiment terminates at a pre-specified time T. The failure time of an item is observable only if it fails before time T. Otherwise the item is said to be censored at time T. This type of time censoring is known as the type-I censoring. The type-I censoring scheme has received much attention in the statistical literature, see Spurrier and Wei (1980), and others. The ranking and selection procedures based on censored data for the exponential distribution have been considered, for example, in Berger and Kim (1985), Gupta and Liang (1993), and Huang and Huang (1980).

In this paper, we derive a Bayes selection rule for positive one-parameter exponential family distributions based on type-I censored data. A monotone property of this rule is discussed and an early selection rule is proposed. Through this early selection rule, one can terminate the experiment on a few populations early and possibly make the final decision before the given time T. The approach used here is similar to that of Gupta and Liang(1993).

2. A Bayes Selection Rule

Let T be the censoring time. Let $X_{ij}, 1 \leq j \leq m$ be the failure times of the m items

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taken from population π_i. According to the censoring scheme, we observe $min(X_{ij}, T)$.

Let $N_i = \sum_{j=1}^{m} I_{[X_{ij} < T]}$ be the number of failures up to time T.

Let $Y_{i1} \leq Y_{i2} \leq \cdots \leq Y_{iN_i}$ be the ordered values of the N_i observed data given N_i.

Then $(Y_{i1}, Y_{i2}, \cdots, Y_{iN_i}, N_i)$ have a joint probability density

$$
\begin{aligned}
&f(y_{i1}, y_{i2}, \cdots, y_{in_i}, n_i | \theta_i) \\
={}& \frac{m!}{(m - n_i)!} c^{n_i}(\theta_i) e^{\beta(\theta_i) \sum_{j=1}^{n_i} Q(y_{ij})} P_{\theta_i}(X \geq T)^{(m-n_i)} \prod_{j=1}^{n_i} h(y_{ij}) \\
={}& \frac{m!}{(m - n_i)!} c^{n_i}(\theta_i) e^{\beta(\theta_i)[y_i - (m-n_i)Q(T)]} P_{\theta_i}(X \geq T)^{(m-n_i)} \prod_{j=1}^{n_i} h(y_{ij}),
\end{aligned}
\tag{2.1}
$$

where $0 \leq n_i \leq m, y_{i1} \leq y_{i2} \leq \cdots \leq y_{in_i} < T$ and

$$
y_i = \sum_{j=1}^{n_i} Q(y_{ij}) + (m - n_i)Q(T).
\tag{2.2}
$$

For convinience, we denote the expression at the right-hand-side of (2.1) by $f(y_i, n_i | \theta_i)$.

Let $\mathcal{N}$ be the sample space generated by $\widetilde{N} = (n_1, n_2, \cdots, n_k)$ and conditioned on $\widetilde{N} = \tilde{n} = (n_1, n_2, \cdots, n_k)$, let $\mathcal{Y}_n$ be the sample space generated by $\widetilde{Y} = (y_1, y_2, \cdots, y_k)$.

Let $\tilde{\theta} = (\theta_1, \theta_2, \cdots, \theta_k)$ and $\widetilde{\Omega} = \{\tilde{\theta} | \theta_i \in \Omega, 1 \leq i \leq k\}$ be the parameter space. Let $\mathcal{A}$ be the action space. Action i corresponds to the selection of population π_i as the best population. For a given $\tilde{\theta} \in \widetilde{\Omega}$ and an action i, the associated loss function is defined by

$$
L^*(\theta, i) = L(\theta_{[k]} - \theta_i)
\tag{2.3}
$$

where $L(x)$ is a nonnegative and nondecreasing function of x, $x \geq 0$, such that $L(0) = 0$.

Let $g(\tilde{\theta}) = \prod_{j=1}^{k} g_j(\theta_j)$ be the prior density over the parameter space $\widetilde{\Omega}$. It is assumed that $\int_{\widetilde{\Omega}} L(\theta_{[k]}) g(\tilde{\theta}) d\tilde{\theta} < \infty$.

A selection rule $\tilde{\delta} = (\delta_1, \delta_2, \cdots, \delta_k)$ is defined to be a measurable mapping from the sample space $(\mathcal{N}, (\mathcal{Y}_{\tilde{n}})_{\tilde{n} \in \mathcal{N}})$ to $[0, 1]^k$ such that $0 \leq \delta_i(\tilde{y}, \tilde{n}) \leq 1$ and $\sum_{j=1}^{k} \delta_i(\tilde{y}, \tilde{n}) = 1$ for

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all $\widetilde{y} \in \mathcal{Y}_n, \widetilde{n} \in \mathcal{N}$. The value of $\delta_i(\widetilde{y}, \widetilde{n})$ is the probability of selecting population π_i as the best population based on the observation $(\widetilde{y}, \widetilde{n})$.

Let $R(\widetilde{\delta}, g)$ denote the Bayes risk associated with the selection rule $\widetilde{\delta}$. Then by Fubini's Theorem we have

$$R(\widetilde{\delta}, g) = \sum_{\widetilde{n} \in \mathcal{N}} \int_{\mathcal{Y}_n} \sum_{i=1}^{k} \delta_i(\widetilde{y}, \widetilde{n}) \int_{\widetilde{\Omega}} L(\theta_{[k]} - \theta_i) f(\widetilde{y}, \widetilde{n} | \widetilde{\theta}) g(\widetilde{\theta}) d\widetilde{\theta} d\widetilde{y} \qquad (2.4)$$

where $f(\widetilde{y}, \widetilde{n} | \widetilde{\theta}) = \prod_{i=1}^{k} f(y_i, n_i | \theta_i)$. Now let

$$f_i(y_i, n_i) = \int_{\Omega} f(y_i, n_i | \theta_i) g_i(\theta_i) d\theta_i, \quad f(\widetilde{y}, \widetilde{n}) = \prod_{i=1}^{k} f_i(y_i, n_i),$$

$$g_i(\theta_i | y_i, n_i) = \frac{f(y_i, n_i | \theta_i) g_i(\theta_i)}{f_i(y_i, n_i)} \quad \text{and} \quad g(\widetilde{\theta} | \widetilde{y}, \widetilde{n}) = \prod_{i=1}^{k} g_i(\theta_i | y_i, n_i).$$

Then (2.4) becomes

$$R(\widetilde{\delta}, g) = \sum_{\widetilde{n} \in \mathcal{N}} \int_{\mathcal{Y}_{\widetilde{n}}} \sum_{i=1}^{k} \delta_i(\widetilde{y}, \widetilde{n}) \int_{\widetilde{\Omega}} L(\theta_{[k]} - \theta_i) g(\widetilde{\theta} | \widetilde{y}, \widetilde{n}) d\widetilde{\theta} f(\widetilde{y}, \widetilde{n}) d\widetilde{y}.$$

For each $(\widetilde{y}, \widetilde{n})$, define

$$\Delta_i(\widetilde{y}, \widetilde{n}) = \int_{\widetilde{\Omega}} L(\theta_{[k]} - \theta_i) g(\widetilde{\theta} | \widetilde{y}, \widetilde{n}) d\widetilde{\theta}, \quad i = 1, 2, \cdots, k, \qquad (2.5)$$

and let

$$A(\widetilde{y}, \widetilde{n}) = \{i | \Delta_i(\widetilde{y}, \widetilde{n}) = \min_{1 \le j \le k} \Delta_j(\widetilde{y}, \widetilde{n})\}. \qquad (2.6)$$

Then a uniformly randomized Bayes rule is $\widetilde{\delta_G} = (\delta_{G1}, \cdots, \delta_{Gk})$, where

$$\delta_{Gi} = \begin{cases} 1/|A(\widetilde{y}, \widetilde{n})| & \text{if} \quad i \in A(\widetilde{y}, \widetilde{n}) \\ 0 & \text{otherwise,} \end{cases} \qquad (2.7)$$

where $|A|$ denotes the cardinality of the set A.

3. A Monotonicity Property of $\tilde{\delta}_G$

For each fixed (y_i, n_i) in the support of f_i, $g_i(\theta_i|y_i, n_i) = 0$ if and only if $g_i(\theta_i) = 0$. Then $g_i(\theta_i|y_i^*, n_i^*)$ and $g(\theta_i|y_i, n_i)$ have a common support. Let D_i be their common support. Consider the likelihood ratio defined on D_i, by

$$r_i(\theta_i|y_i^*, n_i^*, y_i, n_i) = \frac{g_i(\theta_i|y_i^*, n_i^*)}{g_i(\theta_i|y_i, n_i)}. \tag{3.1}$$

A simple calculation shows that for some nonnegative function W

$$
\begin{aligned}
&r_i(\theta_i|y_i^*, n_i^*, y_i, n_i) \\
=\ & W(y_i^*, n_i^*, y_i, n_i)c^{n_i^*-n_i}(\theta_i)e^{\beta(\theta_i)[(y_i^*-y_i)+(n_i^*-n_i)Q(T)]}\{P_{\theta_i}(X \geq T)\}^{(n_i-n_i^*)} \\
=\ & W(y_i^*, n_i^*, y_i, n_i)e^{\beta(\theta_i)(y_i^*-y_i)}\{\int_T^\infty exp\{\beta(\theta_i)[Q(x) - Q(T)]\}h(x)dx\}^{(n_i-n_i^*)},
\end{aligned}
$$

from which we get the following lemma.

Lemma 3.1 *Let $r_i(\theta_i|y_i^*, n_i^*, y_i, n_i)$ be defined by (3.1). Then*

(a) for $n_i^* = n_i, y_i^* > y_i$, $r_i(\theta_i|y_i^*, n_i^*, y_i, n_i)$ is a nondecreasing function of θ_i in D_i and

(b) for $y_i^* = y_i, n_i^* > n_i$, $r_i(\theta_i|y_i^*, n_i^*, y_i, n_i)$ is a nonincreasing function of θ_i in D_i.

The following lemma is used in the proof of lemma 3.3.

Lemma 3.2 *If $g(\theta)$ and $h(\theta)$ are probability density functions such that $g(\theta)/h(\theta)$ is nondecreasing function of θ in Ω, then for any nonincreasing function $f(\theta)$ of θ in Ω,*

$$\int_\Omega f(\theta)h(\theta)d\theta \geq \int_\Omega f(\theta)g(\theta)d\theta.$$

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Lemma 3.3 *Let $\Delta_i(\widetilde{y}, \widetilde{n})$ be defined in (2.5). For each $i(1 \leq i \leq k)$, $\Delta_i(\widetilde{y}, \widetilde{n})$ is nonincreasing in y_i and also in $n_j, j \neq i$ when all the other variables are kept fixed, and nondecreasing in n_i and also in $y_j, j \neq i$, when all the other variables are kept fixed.*

Proof. We only prove that $\Delta_i(\widetilde{y}, \widetilde{n})$ is nonincreasing in y_i when all the other variables are kept fixed. The other parts can be proved in a similar way.

Define

$$
\begin{aligned}
\widetilde{\theta}^i &= (\theta_1, \cdots, \theta_{i-1}, \theta_{i+1}, \cdots, \theta_k) \\
\widetilde{\Omega}^i &= \{\widetilde{\theta}^i : \theta_j \in \Omega, j = 1, 2, \cdots, k, j \neq i\} \\
\widetilde{y} &= (y_1, \cdots, y_k) \\
\widetilde{y}^* &= (y_1, \cdots, y_{i-1}, y_i^*, y_{i+1}, \cdots, y_k).
\end{aligned}
$$

Then

$$
\Delta_i(\widetilde{y}, \widetilde{n}) = \int_{\widetilde{\Omega}^i} \Big[\int_\Omega L(\theta_{[k]} - \theta_i) g_i(\theta_i | y_i, n_i) d\theta_i \Big] \prod_{j \neq i} g_j(\theta_j | y_j, n_j) d\widetilde{\theta}^i.
$$

Since for each fixed $\widetilde{\theta}^i$ and $\widetilde{n}$, $L(\theta_{[k]} - \theta_i)$ is nonincreasing in θ_i and by Lemma 3.1, $r_i(\theta_i | y_i^*, n_i^*, y_i, n_i)$ is a nondecreasing function of θ_i for $y_i^* > y_i$. So Lemma 3.2 implies that

$$
\int_\Omega L(\theta_{[k]} - \theta_i) g_i(\theta_i | y_i, n_i) d\theta_i \geq \int_\Omega L(\theta_{[k]} - \theta_i) g_i(\theta_i | y_i^*, n_i) d\theta_i
$$

and hence $\Delta_i(\widetilde{y}, \widetilde{n}) \geq \Delta_i(\widetilde{y}^*, \widetilde{n})$.

Now, from Lemma 3.3, we obtain a monotone property for $\widetilde{\delta}_G$ in the following theorem.

Theorem 3.4 *For each* $i = 1, 2, \cdots, k$, $\delta_{Gi}(\widetilde{y}, \widetilde{n})$ *is nondecreasing in* y_i *and nonincreasing in* n_i, *when all the other variables are kept fixed.*

Proof. We only prove that $\delta_{Gi}(\widetilde{y}, \widetilde{n})$ is nondecreasing in y_i when all other variables are kept fixed. The monotone property of $\widetilde{\delta}_G$ in n_i can be proved in a similar way.

Use the notation in Lemma 3.3. Assume $y_i^* > y_i$.

If $i \notin A(\widetilde{y}, \widetilde{n})$, then $\delta_{Gi}(\widetilde{y}, \widetilde{n}) = 0$. Since $\delta_{Gi}(\widetilde{y}^*, \widetilde{n})$ is nonnegative, $\delta_{Gi}(\widetilde{y}^*, \widetilde{n}) \geq \delta_{Gi}(\widetilde{y}, \widetilde{n})$.

If $i \in A(\widetilde{y}, \widetilde{n})$, then $\Delta_i(\widetilde{y}, \widetilde{n}) \leq \min_{j \neq i} \Delta_j(\widetilde{y}, \widetilde{n})$. Using Lemma 3.3,

$$\Delta_i(\widetilde{y}^*, \widetilde{n}) \leq \Delta_i(\widetilde{y}, \widetilde{n}) \leq \min_{j \neq i} \Delta_j(\widetilde{y}, \widetilde{n}) \leq \min_{j \neq i} \Delta_j(\widetilde{y}^*, \widetilde{n}).$$

And hence $i \in A(\widetilde{y}^*, \widetilde{n})$.

To get $\delta_{Gi}(\widetilde{y}^*, \widetilde{n}) \geq \delta_{Gi}(\widetilde{y}, \widetilde{n})$, we still need to show

$$A(\widetilde{y}^*, \widetilde{n}) \subset A(\widetilde{y}, \widetilde{n}). \tag{3.2}$$

For each $h \in A(\widetilde{y}^*, \widetilde{n})$, $\Delta_h(\widetilde{y}^*, \widetilde{n}) = \Delta_i(\widetilde{y}^*, \widetilde{n})$. Using Lemma 3.3,

$$\Delta_h(\widetilde{y}, \widetilde{n}) \leq \Delta_h(\widetilde{y}^*, \widetilde{n}) = \Delta_i(\widetilde{y}^*, \widetilde{n}) \leq \Delta_i(\widetilde{y}, \widetilde{n}) = \min_{1 \leq j \leq k} \Delta_j(\widetilde{y}, \widetilde{n})$$

and hence $h \in A(\widetilde{y}, \widetilde{n})$. So (3.2) is proved and $\delta_{Gi}(\widetilde{y}^*, \widetilde{n}) \geq \delta_{Gi}(\widetilde{y}, \widetilde{n})$.

4. An Early Selection Rule

In this section, we consider the following linear loss function: $L(\theta_{[k]} - \theta_i) = \theta_{[k]} - \theta_i$, the difference between the parameters of the best and the selected populations. Thus the set $A(\widetilde{y}, \widetilde{n})$ given by (2.6) turns out to be:

$$A(\widetilde{y}, \widetilde{n}) = \{i \mid \int \theta_i g_i(\theta_i | y_i, n_i) d\theta_i = \max_{1 \leq j \leq k} \int \theta_j g_j(\theta_j | y_j, n_j) d\theta_j\}. \tag{4.1}$$

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Note that $E[\theta_i|y_i, n_i] = \int \theta_i g_i(\theta_i|y_i, n_i)d\theta_i$. This can be calculated for the given model and the prior. Now, similar to the proof of Lemma 3.3, we can prove the following result.

Lemma 4.1 *For each fixed i, $E[\theta_i|y_i, n_i]$ is nondecreasing in y_i and nonincreasing in n_i.*

Now, we will use Lemma 4.1 to derive an early selection rule.

At time t, $0 < t < T$, let $N_i(t)$ denote the number of failures from population π_i upto time t. That is, $N_i(t) = |\{X_{ij} : 1 \leq j \leq m, X_{ij} < t\}|$. Also, let $Y_{i1} \leq Y_{i2} \leq \cdots \leq Y_{iN_i(t)}$ denote the observed failure times given $N_i(t)$. At time t, we can make an early decision as follows:

Declare population π_i as a non-best population and exclude it from further experiment if there exists some population π_h such that

$$N_h(t) < m \quad \text{and} \quad E[\theta_h|y_h(t), m] \geq E[\theta_i|y_i(t, T), N_i(t)] \tag{4.2a}$$

or

$$N_h(t) = m \quad \text{and} \quad E[\theta_h|y_h(t), m] > E[\theta_i|y_i(t, T), N_i(t)] \tag{4.2b}$$

where

$$y_h(t) = \sum_{j=1}^{N_h(t)} Q(y_{hj}) + (m - N_h(t))Q(t) \tag{4.3a}$$

and

$$y_i(t, T) = \sum_{j=1}^{N_i(t)} Q(y_{ij}) + (m - N_i(t))Q(T). \tag{4.3b}$$

Let $S(t)$ denote the indices of the (non-eliminated) contending populations for the best at time t, where $0 < t < T$. That is,

$$S(t) = \{i : \text{ either } (N_h(t) < m \text{ and } E[\theta_h|y_h(t), m] < E[\theta_i|y_i(t, T), N_i(t)]) \quad \text{or}$$

$$(N_h(t) = m \text{ and } E[\theta_h|y_h(t), m] \leq E[\theta_i|y_i(t, T), N_i(t)]) \quad \text{for all } h \neq i\}. \tag{4.4}$$

The following lemma shows that for any t, $0 < t < T$, $S(t)$ is not empty.

Lemma 4.2 *For any $0 < t < T$, the set $S(t)$ defined by (4.4) is not empty.*

Proof. Let

$$S'(t) = \{i : E[\theta_i|y_i(t), N_i(t)] = \max_{1 \leq h \leq k} E[\theta_h|y_h(t), N_h(t)]\}.$$

Then $S'(t)$ is not empty. We prove that $S'(t)$ is a subset of $S(t)$. For $i \in S'(t)$ and any $h \neq i$,

if $N_h(t) < m$, then

$$E[\theta_i|y_i(t, T), N_i(t)] \geq E[\theta_i|y_i(t), N_i(t)] \geq E[\theta_h|y_h(t), N_h(t)] > E[\theta_h|y_h(t), m];$$

if $N_h(t) = m$, then

$$E[\theta_i|y_i(t, T), N_i(t)] \geq E[\theta_i|y_i(t), N_i(t)] \geq E[\theta_h|y_h(t), N_h(t)] \geq E[\theta_h|y_h(t), m].$$

In either situation, we see that $i \in S(t)$. Hence $S'(t) \subset S(t)$.

Now, the experiment terminates as soon as there is a time t, $0 < t < T$, such that $|S(t)| = 1$ and in this case, we select the population with its index in $S(t)$ as the best population. Otherwise, the experiment goes on until time T. Let

$$S(T) = \{i : E[\theta_i|y_i, N_i] = \max_{j \in S(T^-)} E[\theta_j|y_j, N_j]\}, \tag{4.5}$$

where $S(T^-)$, which is not empty by Lemma 4.2, denotes the set of the indices of those populations having not been eliminated before time T. Then, a uniformly randomized selection is made from $S(T)$.

From the above description, we see that the early selection rule can possibly make a final selection earlier than the termination time T. Denote this early selection rule by $\widetilde{\delta}_G^* = (\delta_{G1}^*, \cdots, \delta_{Gk}^*)$. Then, we have the following theorem.

Theorem 4.3 *Under the loss function $L(\theta)$, $\widetilde{\delta}_{Gi}^* = \widetilde{\delta}_{Gi}(\widetilde{y}, \widetilde{n})$ for all $1 \leq i \leq k$, $\widetilde{y} \in \mathcal{Y}_n$ and $\widetilde{n} \in \mathcal{N}$, where $\widetilde{\delta}_{Gi}(\widetilde{y}, \widetilde{n})$ is defined by (4.1) and (2.7).*

Let $t_1 = inf\{t : {}^{\#}S(t) = 1, 0 < t \leq T\} \wedge T$, where $a \wedge b = min(a, b)$. Then Theorem 4.3 is equivalent to the following theorem.

Theorem 4.4 $S(t_1) = A(\widetilde{y}, \widetilde{n})$ *for all* $(\widetilde{y}, \widetilde{n})$.

Proof. Case 1. If $t_1 < T$, then $|S(t_1)| = 1$. Without loss of generality, we let π_k be the population with index in the set $S(t_1)$. Since $A(\widetilde{y}, \widetilde{n})$ contains at least one element, it suffices to show that $i \notin A(\widetilde{y}, \widetilde{n})$ for all $i \neq k$. Since $i \notin S(t_1)$, it means that population π_i is eliminated at some prior time, say t_0, That is, at time t_0, for some π_h, either

$$N_h(t_0) < m \quad \text{and} \quad E[\theta_h|y_h(t_0), m] \geq E[\theta_i|y_i(t_0, T), N_i(t_0)] \tag{4.6a}$$

or

$$N_h(t_0) = m \quad \text{and} \quad E[\theta_h|y_h(t_0), m] > E[\theta_i|y_i(t_0, T), N_i(t_0)]. \tag{4.6b}$$

Now, note that $N_i(t)$ is a nondecreasing function of $t \in (0, T]$ and $N_i(t) \leq m$. Also, by (4.3b), $y_i(t, T)$ is nonincreasing in t. For any s and t with $0 < s < t < T$, $N_h(s) \leq N_h(t)$. By (4.3a)

$$y_h(t) - y_h(s) = \sum_{j=N_h(s)+1}^{N_h(t)} Q(y_{hj}) + (m - N_h(t))Q(t) - (m - N_h(s))Q(s).$$

If the jth item from population π_h is censored (not failed) at time s, then $y_{hj} \geq s$ and $Q(y_{hj}) \geq Q(s)$. Therefore

$$y_h(t) - y_h(s) \geq (N_h(t) - N_h(s))Q(s) + (m - N_h(t))Q(t) - (m - N_h(s))Q(s) \geq 0.$$

This shows that $y_h(t)$ is nondecreasing in t. Note that

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$$N_h = N_h(T) \le m, \quad N_i(t) \le N_i(T) = N_i, \quad y_i(t_0, T) \ge y_i$$

and

$$y_h = \begin{cases} > y_h(t_0) & \text{if } N_h(t_0) < m, \\ = y_h(t_0) & \text{if } N_h(t_0) = m. \end{cases}$$

Thus, when $N_h(t_0) = m$, then $N_h = m$. Then by Lemma 4.1 and (4.6b),

$$
\begin{aligned}
E[\theta_h | y_h, N_h] &= E[\theta_h | y_h(t_0), m] \\
&> E[\theta_i | y_i(t_0, T), N_i(t_0)] \\
&\ge E[\theta_i | y_i, N_i].
\end{aligned}
$$

When $N_h(t_0) < m$, then $y_h > y_h(t_0)$ and $N_h \le m$, Therefore, by Lemma 4.1 and (4.6a),

$$
\begin{aligned}
E[\theta_h | y_h, N_h] &= E[\theta_h | y_h(t_0), m] \\
&> E[\theta_i | y_i(t_0, T), N_i(t_0)] \\
&\ge E[\theta_i | y_i, N_i].
\end{aligned}
$$

In either situation, we see that $i \notin A(\widetilde{y}, n)$.

Case 2. If $t_1 = T$, we need to prove that

(a) $i \notin S(T)$ implies $i \notin A(\widetilde{y}, n)$, and (b) $i \in S(T)$ implies $i \in A(\widetilde{y}, n)$.

We prove (a) first. Suppose $i \notin S(T)$. Then, π_i is eliminated at a time $t_0 \le T$ by some other π_h.

If $t_0 \le T$, this reduces to the situation discussed in Case 1.

If $t_0 = T$, then by (4.5), $E[\theta_h | y_h, N_h] > E[\theta_i | y_i, N_i]$. Therefore, by the definition of $A(\widetilde{y}, \widetilde{n})$, $i \notin A(\widetilde{y}, \widetilde{n})$.

For (b), we have firstly $A(\widetilde{y}, \widetilde{n}) \subset S(T) \subset S(T^-)$ by (a) and definition of $S(T)$ and $S(T^-)$. If $i \in S(T^-)$,

$$
\begin{aligned}
E[\theta_i | y_i, N_i] &= \max_{j \in S(T^-)} E[\theta_j | y_j, N_j] \\
&\geq \max_{j \in S(T)} E[\theta_j | y_j, N_j] \\
&\geq \max_{j \in A(\widetilde{y}, \widetilde{n})} E[\theta_j | y_j, N_j].
\end{aligned}
$$

This means $i \in A(\widetilde{y}, \widetilde{n})$. The proof now is completed.

5. An Example

We use simulated data to illustrate how the early selection rule works. Suppose that we have five populations π_i, $i = 1, 2, 3, 4, 5$. The lifetime of the population π_i follows a lognormal distribution with density

$$
f(x | \theta_i) = \frac{1}{\sqrt{2\pi}x} \exp\left\{ -\frac{[\ln(x) - \theta_i]^2}{2} \right\}, \qquad x > 0.
$$

We assume that $\theta_1, \cdots, \theta_5$ are independent and identically distributed with a uniform prior distribution $U(0, 1)$. Thus $\theta_1, \cdots, \theta_5$ are simulated independently from $U(0, 1)$. Then ten observations are simulated independently from each lognormal population. The data are listed in the following table.

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π_1	π_2	π_3	π_4	π_5
4.67	2.37	1.56	0.92	0.89
7.55	1.04	1.43	2.66	2.08
10.06	2.81	3.93	0.66	2.13
1.11	3.32	2.50	2.22	1.48
0.86	3.26	0.60	0.95	0.13
5.39	6.20	0.53	0.94	2.89
1.00	1.48	1.08	8.52	0.34
2.03	8.14	1.51	4.03	0.32
2.88	6.68	1.09	14.36	0.47
5.09	4.07	1.53	1.47	2.46

We want to select a population with the largest mean lifetime. Since the mean lifetime of the population π_i is an increasing function of θ_i, what we need to do is to find a population with the parameter $\theta_{[k]}$. Suppose that the type-I censoring scheme is planned before the life-testing experiment and the censoring time is set to be $T = 3$ (in the same unit as the failure time data). After some simple mathematical calculation, we have

$$
\begin{aligned}
E[\theta_i|y_i, n_i] &= \int_0^1 \theta_i g_i(\theta_i|y_i, n_i) d\theta_i \\
&= \frac{\int_0^1 \theta_i e^{-[n_i \theta_i^2/2 + y_i \theta_i]} \times [1 - \Phi(\ln T - \theta_i)]^{10-n_i} T^{-\theta_i(10-n_i)} d\theta_i}{\int_0^1 e^{-[n_i \theta_i^2/2 + y_i \theta_i]} \times [1 - \Phi(\ln T - \theta_i)]^{10-n_i} T^{-\theta_i(10-n_i)} d\theta_i},
\end{aligned}
$$

where $\Phi(x)$ is the cumulative distribution function of the standard normal distribution. According to the selection rule $\tilde{\delta}_G$, the population with the largest value of $E[\theta_i|y_i, n_i]$ is selected as the best. The numerical value of $E[\theta_i|y_i, n_i]$ can be computed through Monte Carlo integration method. For the data listed above, we obtain the following table.

	π_1	π_2	π_3	π_4	π_5
$E[\theta_i\|y_i, n_i]$	0.82	0.87	0.42	0.66	0.21

So at the end of the experiment, we select π_2 as the best population.

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However, if the early selection rule $\tilde{\delta}_G^*$ is applied, we can make the selection before $T = 3$ and end the experiment earlier. According to the selection rule $\tilde{\delta}_G^*$, at time t, $0 < t < T = 3$, exclude the population π_i as a non-best population and remove it from further experiment if there exists some population π_h such that

$$N_h(t) < m \quad \text{and} \quad E[\theta_h | y_h(t), m] \geq E[\theta_i | y_i(t, T), N_i(t)]$$

or

$$N_h(t) = m \quad \text{and} \quad E[\theta_h | y_h(t), m] > E[\theta_i | y_i(t, T), N_i(t)].$$

According to this rule, π_5 is excluded at $t = 0.8$, π_3 at $t = 1.3$, π_4 at $t = 1.4$ and π_1 at $t = 2.1$. That is, at $t = t_1 = 2.1$, all the populations π_1, π_3, π_4 and π_5 are removed from the experiment and the population π_2 is selected as the best. So the experiment can be terminated at $t_1 = 2.1$ and the time saved is 0.9 or 30%.

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13. ABSTRACT (Maximum 200 words)

We investigate the problem of selecting the best population from exponential family distributions based on type-I censored data. A Bayes rule is derived and a monotone property of the Bayes selection rule is obtained. Following that property, we propose an early selection rule. Through this early selection rule, one can terminate the experiment on a few populations early and possibly make the final decision before the censoring time. An example is provided in the final part to illustrate the use of the early selection rule for Weibull populations.

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