DECISION THEORY FOR SPATIAL COMPETITION

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Abstract

The Hotelling Beach model of spatial competition in economics asks what is the optimum location for the first firm if customers visit the firm geographically closest to them, and if there are n future competitors. We study this problem from a statistical perspective for both one and two dimensional markets. The minimax formulation of Hotelling (1929) as well as a Bayesian formulation are considered.

Suppose S denotes the market in which customers and the firms are located. The first firm assigns a prior distribution F with density f on the location Z of a customer. In the minimax formulation, no prior distributions on the locations of the future competitors are needed. In the Bayesian formulation, the first firm assumes that the locations $Y_1, \ldots, Y_n$ of its future competitors are i.i.d. with some distribution G having a density g, and are independent of Z.

The results show that the decisive factors are the number of future competitors n, and whether or not f and g have appropriate symmetry or unimodality. Specifically, when the number of future competitors is large, the first firm's optimum location is proved to often move towards the boundary of S, even if the customers are distributed according to a density f having a mode in the interior of S. This is true in both one and two dimensions, and in one dimension, we are actually

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able to give an asymptotic representation for the optimum location of the first firm. Additionally, a very nice dichotomy is often seen to hold : for n less than a threshold value, the first firm should choose a central location and for all n larger, it should choose a noncentral location asymptotically converging to the boundary.

The results also suggest that for two dimensional markets, the shape of S does not much affect the qualitative nature of the first firm's optimum action. This is in contrast to results on equilibrium in competition where the results very much depend on the exact shape of a two dimensional market.

All the results are illustrated with examples and computation.

1 Introduction

In this article, we give a mathematical analysis of the problem of optimum location for the first vendor in a competitive market, taking into account information about future vendors. In a classic article, Hotelling (1929) addressed this problem in the simple setting of one future vendor; Hotelling showed that if customers always visited the store closest to them and purchased a constant amount of the merchandise, then any median of the distribution of the location of customers is a minimax location for the first vendor. Due to this historical connection to Hotelling, the problem has popularly come to be known as the Hotelling beach problem. The goal of this article is to present a series of results and examples that illustrate the very significant effect of the number of future competitors and the effect of the dimension on the optimal location of the first vendor. We present results under both the minimax formulation of Hotelling and the natural Bayes formulation in which the first vendor imposes a prior distribution on the location and/or the number of the future competitors. The emphasis is statistical rather than large scale numerical optimization.

The Hotelling beach model has been the subject of many studies in urban and spatial economics. If vendors (or firms) sell a certain product at the same price but transportation costs are proportional to the distance between the customer and the firm, then customers buy from the geographically closest firm and the Hotelling beach model applies. Most of the studies in the area of spatial economics have considered the question of eventual stability in competition, when firms relocate from time to time to increase their profit. The literature shows that stability depends very much on the number of firms; theoretical results and simulations can be seen in Eaton and Lipsey (1975a,b, 1980), Okabe and Suzuki (1987), and Gabszewicz et al. (1986), among others. The case of a two dimensional market is technically a lot harder, and the stability questions seem to depend on the shape of the market; see, for instance, Okabe and Aoyagi (1991). Our aim in this article is to study the locational optimization problem, in one as well as two dimensions, and we do not address the equilibrium problem here. We also do not consider optimum locations for subsequent competitors; see Steele and Zidek (1980) for some results on that problem.

Throughout the article, S denotes the region in either one or two dimensions where the customers and all vendors are located, n stands for the number of future competitors and F stands for the CDF of the distribution of the location of a customer. We always assume that F has a density f. We also assume as in Hotelling (1929) that the customer visits the business closest to him, but we do not always assume that the amount purchased is a pure constant. We let the amount purchased be a function $h(d)$ of the customer's distance d from the closest vendor. In general, the minimax results are under a general h and the Bayes results are under a constant h.

A common feature of many of the results is that symmetry and unimodality of f plays a very important role. For instance, we show that one can have a general h function and still have the median m of F to be the first vendor's minimax location provided f is symmetric and unimodal around m. However, the minimax location can behave in erratic ways and even the uniqueness of the minimax location becomes false if the unimodality assumption is removed. These results and other examples on the first vendor's minimax location for the case of one dimensional S are presented and proved in Section 2.

The Bayes formulation requires the first vendor to assign a prior distribution on the locations of the future rivals. A strictly realistic formulation would have the prior distribution for any competitor depend on the known locations of all the preceding competitors. The cost of such a puritanical formulation is that clean results that show some useful structure seem to be basically impossible in this formulation and computation specialized to a given set up seems to be the only resolution. We consider the set up in which the customer has a location $Z \sim F$ and n future rivals have i.i.d. locations $Y_1, Y_2, \ldots, Y_n \sim G$, independent of Z. We always assume that G has a density g.

It turns out that in this formulation, the Bayes solution of the first vendor shows a number of interesting properties. First, again symmetry and unimodality of f plays a central role. And n, the number of future competitors also plays a very central role. We show that if n is 1, and f is symmetric and unimodal and g is just symmetric, then the center of symmetry m is the first vendor's Bayes solution. This seems intuitively very plausible. But very intriguingly, if n is large, even with all of these assumptions on f and g, the center of symmetry m is often not the first vendor's Bayes solution. Specifically, if the region S is a compact interval ,say $[-1,1]$, and f, g are both uniform on S, then the midpoint of the interval is the first vendor's Bayes solution if and only if n is less than or equal 4, and there are two distinct Bayes solutions $\pm x_n$ if n is greater than or equal to 5. In fact, as $n \to \infty$, the Bayes solutions move towards the boundary and admit the asymptotic representation $x_n = 1 - \frac{4}{n} + o(\frac{1}{n})$. There is a definitive threshold in terms of n. And interestingly, the threshold phenomenon persists even if we treat n to be a random variable itself; for example, we show that if n has a Poisson(λ) distribution, then the midpoint of S is the first vendor's Bayes solution if and only if λ is less than or equal to 6, and there are two Bayes solutions $\pm x_\lambda$ if $\lambda > 6$. Moreover, x_λ has the asymptotic representation $x_\lambda = 1 - \frac{4}{\lambda} + o(\frac{1}{\lambda})$.

We are actually able to show a general structure as regards the asymptotic behavior of any sequence of Bayes solutions $\{x_n\}$ under a general f and g. Under certain smoothness assumptions on f and g, we are able to show that any accumulation point of the sequence $\{x_n\}$ must either be one of the boundary points of S or an interior mode of the ratio f/g. We also show that there is, in a certain sense, an inherent tendency for the Bayes solution to converge to a boundary point of S. For example, for ease of exposition, consider the case when g is uniform. Then if a certain boundary point, say 1, has a density $f(1)$ not much less than the density of f at an overall interior mode m, then the Bayes solutions x_n will in fact converge to 1, and not to the overall mode m. For m to asymptotically remain the first vendor's best location, m will have to be a pronounced mode relative to the boundary points of S. Remark 1 on pp 23 describes this phenomenon precisely. These results for the case of one dimensional S are stated and proved in Section 3.

In Section 4, we proceed to the interesting case of a two dimensional market. And as before, we consider the minimax as well as the Bayes formulation. If the function h (the amount of purchase made by a customer) is a constant, then parallel to Hotelling's result for one dimension one has the result that any halfspace median of f is a minimax location of the first vendor. See Small (1990). We show that if f is spherically symmetric and unimodal around some m and g is just spherically symmetric, then m is the first vendor's minimax location even if h is not a constant.

Some strong form of symmetry and unimodality seem to be required due to the increase in the number of dimensions. Indeed, we begin to see the effect of the dimension immediately. We give an example to show that if the market S is a symmetric region, then even if f and g are uniform in S, the first vendor's Bayes solution need not be the point of symmetry of S. But if we make the assumption that f is spherically symmetric and unimodal and g is spherically symmetric, then the point of symmetry m is also the first vendor's Bayes solution whenever $n = 1$. And as we previously saw for the one dimensional case, there again seems to be a movement towards the boundary when n gets large. If S is either a square or a circle, and f, g are uniform in S, then starting at $n = 6$, the Bayes solution moves from the center of S. There is an effect of large n, but the threshold value is 6 in two dimensions although it was 5 in one dimension. We give some heuristic explanation for that. In addition, the threshold value is 6 for both a square and a circle; the shape of S does not appear to have so much of an effect. And the general result on the asymptotic behavior of any sequence of Bayes solutions continues to hold. For very general domains S, an accumulation point of any sequence of Bayes solutions must be a boundary point of S or an interior mode of f/g. These results are illustrated through computation and numerical examples in the rest of Section 4.

The problem we treat is a relevant one and it is quite interesting to see the effect of the number of future rivals and the effect of the dimension on how the first vendor should act. Particularly interesting is the dichotomy of the result as regards n and the associated threshold phenomenon. In the case of one dimension, we were able to prove that the Bayes solutions $\pm x_n$ have the representation $x_n = 1 - \frac{4}{n} + o(\frac{1}{n})$ if g is uniform in $S = [-1, 1]$; f can be general. In the case of two dimensions, we conjecture that the solution (x_n, x_n) in the first quadrant has the representation $x_n = 1 - \frac{c}{\sqrt{n}} + o(\frac{1}{\sqrt{n}})$ for some $c > 0$, when g is uniform in the square $[-1, 1] \times [-1, 1]$. Numerical evidence Suggests that c is about 2.

2 Minimax Solutions in One Dimension

The Hotelling Beach problem studies optimal locations for the first vendor taking into account the possibility of future rivals. The minimax strategy is usually a nice start when we encounter such a competitive situation involving two or more intelligent players. We start with notations and a formulation of the minimax solution.

2.1 Notation

We begin this section by introducing the mathematical model and notations. We denote by S the regular domain in which potential customers and businessmen are located. The one dimensional case where S is either a bounded interval or the whole real line will be considered first in the subsequent sections. Let Z be the possible position of a buyer which we assume to be a random variable with cumulative distribution function F and density f. F is known and has S as its support. The location of our store and the future competitors are represented by x and $y_1, y_2, \ldots, y_n$ respectively, all distinct. Here n is taken as fixed. Assume we are the first store to build a vendor in the region S. As Hotelling (1929) described in his paper, many customers still like to trade with the store nearest to them. Therefore, we assume that a buyer has no preference for any seller and will always visit the closest store. The dollars he will spend in that store is assumed to be a function of the distance between him and the store, denoted by $h(d)$. Note that we assume Z to be absolutely continuous with respect to the Lebesgue measure and all stores are located at different places. Hence, the probability of existence of two or more stores at an equal distance from the customer is 0.

A summary of the notations is given below:

$$\begin{cases} S & : \text{ the region of the market.} \\ Z \sim F \text{ with density } f & : \text{ the position of the customer.} \\ h(d) & : \text{ the amount spent by the customer at a distance } d \text{ from the vendor.} \\ x & : \text{ the location of our vendor.} \\ y_1, \ldots, y_n & : \text{ the locations of our future competitors.} \end{cases}$$

The sales we expect from one customer when we locate ourselves at x and the future competitors are located at $y_1, \ldots y_n$ is

$$D(x; y_1, \cdots, y_n) = E\left[h(|Z-x|)\mathbf{1}_{\{|Z-x|<min_{1\le i\le n}|Z-y_i|\}}\right].$$

The location x is in our control, but the locations $y_1, \ldots, y_n$ are not. Therefore, the minimax optimal strategy for the first vendor is to select x which maximizes $\inf_{y_1,\cdots,y_n\in S} D(x; y_1, \cdots, y_n)$.

2.2 Minimax Solutions with One Competitor

It would be natural to start off with the case when there is only one future competitor. We will denote this sole rival's location by y.

Note that, if this competitor places his vendor to our left, i.e. $y < x$, the customer will visit our store if and only if he is at the right of the middle point $\frac{x+y}{2}$. Therefore, our expected sales will be $E[h(|Z-x|)\mathbf{1}_{Z>(x+y)/2}]$. The domain $\{Z > \frac{x+y}{2}\}$ will decrease to the domain $\{Z \ge x\}$ when y tends to x from the left. Hence, clearly,

$$\inf_{y<x} E\left[h(|Z-x|)\mathbf{1}_{Z>(x+y)/2}\right] = E\left[h(|Z-x|)\mathbf{1}_{Z\ge x}\right].$$

Similarly considering y on the right hand side of x, one has

$$\inf_{y>x} D(x; y) = \min_{y>x} E\left[h(|Z-x|)\mathbf{1}_{Z<(x+y)/2}\right] = E\left[h(|Z-x|)\mathbf{1}_{Z\le x}\right].$$

Combining these we have

$$V(x) \stackrel{def}{=} \inf_y D(x; y) = \min\left(E\left[h(|Z-x|)\mathbf{1}_{Z\ge x}\right], E\left[h(|Z-x|)\mathbf{1}_{Z\le x}\right]\right). \qquad (1)$$

For the case where customers buy the same amount regardless of how far away they are from the seller, Hotelling(1929) showed that the minimax optimal location for the first vendor is the median of F. We state it for completeness.

Proposition 1 (Hotelling) *If $h(\cdot)$ is a constant function, the set of minimax optimals equals the set of medians of F.*

Proof: Without loss of generality, take $h \equiv 1$. Then, from (1), we have $V(x) = \min(P[Z \ge x], P[Z \le x])$. This is maximized when x is any median of F. So the proposition follows. □

The assumption that h is a constant in Proposition 1 may not be always realistic. For example, we may want to consider the possibility that customers too far away do not make the visit to buy anything. In that case, the choice $h(d) = \mathbf{1}_{d \leq d_0}$ may be better. Fortunately, it is possible to prove a general neat result on the minimax optimal location without requiring that $h(\cdot)$ is a constant. The additional assumption needed for this result is on F. The examples following our next proposition would show that without the additional assumption, this general result is false.

Proposition 2 *If f is symmetric and unimodal around some m, then m is a minimax optimal.*

Proof: If we can show $V_1(x) = E[h(|Z - x|)\mathbf{1}_{Z \geq x}]$ is a decreasing function of x for $x \geq m$, $V_2(x) = E[h(|Z - x|)\mathbf{1}_{Z \geq x}]$ is an increasing function for $x \leq m$, and $V_1(m) = V_2(m)$, then it will follow that m is a minimax optimal.

Consider the nonnegative measure u_x defined by

$$u_x([a,b]) = \begin{cases} 0 & \text{if } a \leq b < 0, \\ P(Z \in [x, x+b]) & \text{if } a < 0 \leq b, \\ P(Z \in [x+a, x+b]) & \text{if } 0 \leq a \leq b. \end{cases}$$

Then, we have $V_1(x) = \int h(t) \mathrm{d}u_x(t)$. Let B be any Borel set of $\mathcal{R}$. The assumption that Z is unimodal with mode m implies that $u_x(B)$ is a decreasing function of x for $x \geq m$, and so $V_1(x)$ has the same property.

Similarly one proves that $V_2(x)$ is increasing for $x \leq m$. Furthermore, symmetry of Z easily implies $V_1(m) = V_2(m)$. This therefore completes the proof. $\square$

2.2.1 Discussion and Examples

Note that, in fact we can also make a more general statement: the set of optimals is an interval containing m. This interval property of the minimax solutions is aesthetically nice. However, if f has a unique mode m and h is positive in some neighborhood of $d = 0$, then the minimax solution is unique too, i.e. the interval of minimax solutions has just one point.

Unfortunately, all of these pleasant features disappear when F is not unimodal. Suppose f is bimodal with modes m_1 and m_2 $(m_1 < m_2)$. Then m_1 and m_2 do not have to be the minimax optimals. It is possible to have two or even more optimals in such a case. In addition, these optimals can fall inside or outside of $[m_1, m_2]$. Here are some illustrative examples.

Example 1 Let $F = \frac{1}{2}N(2,1) + \frac{1}{2}N(-2,1)$; then by calculation, the set of minimax optimals for three different choices of $h(\cdot)$ will be $\{0\}$, $\{1.98945, -1.98945\}$, or $\{0, 1.966765, -1.966765\}$ if $h(d) = e^{-d^2/8}$, e^{-d^2}, or $\mathbf{1}_{\{0 \le d \le 1.966765\}}$ respectively.

Example 2 Let $f(z) \propto \begin{cases} e^{-2(|z|-2)^2} & \text{if } |z| \le 2 \\ e^{-(|z|-2)^2/32} & \text{if } |z| \ge 2 \end{cases}$ and $h(d) = \begin{cases} e^{-2} & \text{if } 0 \le d \le 2 \\ e^{-d^2/2} & \text{if } d \ge 2 \end{cases}$.
Then Z is bimodal with modes ± 2, but the minimax optimals can be seen to be ± 3.3.

Example 1 shows that without unimodality of f, we can get different possible numbers of minimax solutions. Example 2 is an example where the optimals are outside $[m_1, m_2]$. The conclusion is that a clean result such as Proposition 2 is not possible in much greater generality.

2.3 Minimax Solutions with many competitors

In contrast to the case when there is just one future rival, if two or more competitors are going to enter the market, the minimax formulation will no longer work. Let us explain what we mean. Suppose that there are two future competitors, say A and B. A sets up his store at y_1 on our left and B at y_2 on our right. Now, if A and B move their stores closer and closer to us, the probability that a customer visits us, i.e. $P[Z \in (\frac{y_1+x}{2}, \frac{y_2+x}{2})]$, will decrease to 0. Hence, no matter where we place our vendor to start with, we always have $\inf_{y_1,y_2} D(x; y_1, y_2) = 0$. Therefore, the minimax approach is not very interesting when $n \ge 2$. This motivates us to look for other reasonable formulations of the problem.

The minimax approach is tantamount to saying that we are not willing to make any assumptions about our future competitors. This conservative approach may be rational in some situations; but surely in certain other situations, we may have some prior information about our future rivals. We will now see that fortunately this Bayesian approach remains meaningful for any $n \geq 1$ and unlike the minimax approach, does not lead us to an uninteresting dead end if $n \geq 2$. We move on to the Bayesian approach.

3 Bayes Solutions in one dimension

As we stated in the last section, the Bayesian framework is a reasonable one to adopt. Practically, the distribution imposed on the location of the ith rival may depend on the locations of the already existing stores, $x, y_1, \ldots, y_{i-1}$. This will unfortunately complicate the problem very much. To avoid this, we will assume that the future rivals decide their locations independently according to some distribution G which has S as its support and a density g; symbolically, $Y_1, \ldots, Y_n \overset{i.i.d.}{\sim} G$. Then $D_n(x)$, our expected sales when we build our store at x and believe there will be n future rivals, can be expressed as

$$D_n(x) = E_{Z\sim F, Y_1, \cdots, Y_n \overset{i.i.d.}{\sim} G}\left[h(|Z-x|)\mathbf{1}_{\{|Z-x|<\min_{1\leq i\leq n}|Z-Y_i|\}}\right]. \tag{2}$$

The Bayes solution x_n is the value of x which maximizes the utility function $D_n(x)$. Usually, there will be no problem regarding the existence of x_n; e.g. if S is closed and h is continuous, x_n will exist. Nevertheless, as we will see, uniqueness of the Bayes solution is an exception rather than a rule. This is especially so when n is moderately large. In fact, a very interesting threshold phenomenon holds.

In the remainder of this paper, we will also assume that $h(\cdot) \equiv 1$ since no clear results seem possible if $h(\cdot)$ is not a constant. Therefore (2) simplifies to

$$D_n(x) = P_{Z\sim F, Y_1, \cdots, Y_n \overset{i.i.d.}{\sim} G}\left(|Z-x| < \min_{1\leq i\leq n}|Z-Y_i|\right). \tag{3}$$

3.1 Bayes Solutions with One Competitor

As before, let us start with the simpler case : that of one competitor. In this situation, formula (3) for the utility function becomes

$$D_1(x) = P_{Z\sim F, Y\sim G}\left(Y < x, Z > \frac{Y+x}{2}\right) + P_{Z\sim F, Y\sim G}\left(Y > x, Z < \frac{Y+x}{2}\right). \tag{4}$$

The following question seems to be a natural one:

Question: Hotelling (1929) showed that if h is a constant, then any median of F is a minimax solution. Is the median of F a Bayes solution as well under some general conditions on F and G?

Theorem 1 below addresses this question.

Theorem 1 *Let m be a median of F.*

(a) (Sufficient condition). *If for every $x \neq m$, we have*

$$E\left[\left\{F(\frac{Y+x}{2}) - F(\frac{Y+m}{2})\right\}\mathbf{1}_{\{Y<m\}}\right] \geq E\left[\left\{F(\frac{Y+x}{2}) - F(\frac{Y+m}{2})\right\}\mathbf{1}_{\{Y>m\}}\right], \quad (5)$$

then m will be a Bayes solution for the first vendor. Moreover, if F and G are both strictly increasing in some neighborhood of m, then m is the unique Bayes solution.

(b) (Necessary condition). *Suppose that S is an interval and g is continuous in S. Then*

$$\int_{-\infty}^{m} f(\frac{y+m}{2})g(y)\mathrm{d}y = \int_{m}^{\infty} f(\frac{y+m}{2})g(y)\mathrm{d}y \quad (6)$$

is a necessary condition for m to be a Bayes solution for the first vendor.

It is true that verifying condition (a) of Theorem 1 takes as much numerical effort as directly verifying that m is a Bayes solution. But in fact the sufficient condition of part (a) has a positive redeeming feature, for the following corollary of Theorem 1 shows analytically that the median is always a Bayes solution if we impose some reasonable conditions on F and G.

Corollary 1 *Suppose that the measures F and G are symmetric around m and f is unimodal with mode m. Then m is a Bayes solution. Moreover, m is the unique Bayes solution for the first vendor if g is positive in some neighborhood of m.*

Proof of Corollary 1: Since G and F are symmetric around m, we have

$$Y \overset{\mathcal{D}}{\equiv} 2m - Y \quad \text{and} \quad (7)$$

$$F(x) = 1 - F(2m - x) \quad \text{for all } x. \quad (8)$$

Therefore,

$$\begin{aligned}
\text{the left hand side of (5)} &= E\left[\left\{F(m + \frac{-Y+x}{2}) - F(m + \frac{-Y+m}{2})\right\}\mathbf{1}_{Y>m}\right] \quad \text{(by (7))} \\
&= E\left[\left\{F(m - \frac{-Y+m}{2}) - F(m - \frac{-Y+x}{2})\right\}\mathbf{1}_{Y>m}\right] \quad \text{(by (8))} \\
&= E\left[\left\{F(\frac{Y+m}{2}) - F(\frac{Y+m}{2} - \frac{x-m}{2})\right\}\mathbf{1}_{Y>m}\right]. \quad (9)
\end{aligned}$$

On the other hand, obviously,

$$\text{the right hand side of (5)} = E\left[\left\{(F(\frac{Y+m}{2}+\frac{x-m}{2})-F(\frac{Y+m}{2})\right\}\mathbf{1}_{Y>m}\right]. \quad (10)$$

Recall that f is symmetric and unimodal with mode m, which implies $F(z)-F(z-\delta) \geq F(z+\delta)-F(z)$ for all $z \geq m$ and all δ. Therefore (9) $\geq$ (10) for all x. Consequently, inequality (5) holds. Furthermore, m is obviously a median of F, and hence the Bayes solution by part (a) of Theorem 1.

It is easy to see that F and G will be both strictly increasing in some neighborhood of m if f and g are positive in some neighborhood of m. Then again by part (a) of Theorem 1, the uniqueness follows immediately. □

Proof of Theorem 1, (a) Part I: (m is a Bayes solution). We will prove it by showing that inequality (5) implies $D_1(m) \geq D_1(x)$ for all $x \neq m$.

First, let us consider $x > m$; by using the expression in (4) one can easily get

$$D_1(m) - D_1(x) = P(B) + P(C) - P(D) - P(E) \text{ where}$$

$$B = \left\{Y < m, \frac{Y+m}{2} < Z < \frac{Y+x}{2}\right\}, \quad C = \left\{m < Y < x, Z < \frac{Y+m}{2}\right\},$$
$$D = \left\{m < Y < x, Z > \frac{Y+x}{2}\right\}, \quad E = \left\{Y > x, \frac{Y+m}{2} < Z < \frac{Y+x}{2}\right\}.$$

We will show that $P(C) \geq P(D)$ and $P(B) \geq P(E)$.

First, let us show that $P(C) \geq P(D)$. This follows from

$$\begin{aligned} P(C) &= \int_{(m,x)} P(Z < \frac{y+m}{2})\mathrm{d}G(y) \geq \int_{(m,x)} P(Z \leq m)\mathrm{d}G(y) \\ &= \int_{(m,x)} P(Z \geq m)\mathrm{d}G(y) \geq \int_{(m,x)} P(Z \geq \frac{y+x}{2})\mathrm{d}G(y) = P(D) \end{aligned}$$

Now let us prove that $P(B) \geq P(E)$. First observe that, the left hand side of (5) equals $P(Y < m, \frac{Y+m}{2} < Z < \frac{Y+x}{2})$, which is $P(B)$ exactly. On the other hand, the right hand side of (5) equals $P(Y > m, \frac{Y+m}{2} < Z < \frac{Y+x}{2})$, which is greater than or equal to $P(E)$ because we have assumed that $x > m$. Therefore if (5) holds, then $P(B) \geq P(E)$.

Thus we have showed that for all $x > m$, $D_1(m) \geq D_1(x)$ if (5) holds. Exactly similar arguments can be given for the case where $x < m$. Hence, under the condition

(5), m is indeed a Bayes solution.

(a) Part II: (uniqueness). To prove that m is the unique Bayes solution, we have to show that $D_1(m) > D_1(x)$ for all $x \neq m$.

First, let us consider the case $x > m$. The proof is exactly like the one we gave in Part I, except that we can show furthermore that $P(C) > P(D)$ (strictly larger) if F and G are assumed to be strictly increasing in some neighborhood of m. Towards this end, we first observe that there will exist some w_0 in the interval (m, x) such that $P(m < Z < w), P(m < Y < w)$ are strictly positive for all w in (m, w_0). Consequently,

$$\begin{aligned} P(C) &= \int_{(m,x)} P(Z < \frac{y+m}{2})\mathrm{d}G(y) > \int_{(m,x)} P(Z \leq m)\mathrm{d}G(y) \\ &= \int_{(m,x)} P(Z \geq m)\mathrm{d}G(y) > \int_{(m,x)} P(Z \geq \frac{y+x}{2})\mathrm{d}G(y) = P(D) \end{aligned}$$

So $D_1(m) > D_1(x)$ for $x > m$ if inequality (5) holds and if F and G are both strictly increasing in some neighborhood of m. Similar arguments can be given for the case $x < m$. These imply that m is the unique Bayes solution.

(b): Since G has the density g, the expression for $D_1(x)$ in (4) is equivalent to

$$D_1(x) = \int_{-\infty}^{x} (1 - F(\frac{y+x}{2}))g(y)\mathrm{d}y + \int_{x}^{\infty} F(\frac{y+x}{2})g(y)\mathrm{d}y. \tag{11}$$

Note now that we have assumed g to be continuous; and clearly F is continuous. Therefore, from (11), it follows that $D_1(x)$ is differentiable at every interior point x of S, and indeed,

$$D_1'(x) = g(x)\,(1 - 2F(x)) + \frac{1}{2}\left(-\int_{-\infty}^{x} f(\frac{y+x}{2})g(y)\mathrm{d}y + \int_{x}^{\infty} f(\frac{y+x}{2})g(y)\mathrm{d}y\right). \tag{12}$$

So if the median m is a Bayes solution, $D_1(x)$ must have a local maximum at m, and hence from (12), $D_1'(x)|_{x=m} = \int_m^\infty f(\frac{y+m}{2})g(y)\mathrm{d}y - \int_{-\infty}^m f(\frac{y+m}{2})g(y)\mathrm{d}y = 0$. This proves that equation (6) is indeed a necessary condition for m to be a Bayes solution, as was claimed in the statement of Theorem 1. □

3.1.1 Examples

Again we give a few illustrative examples.

Example 3 Consider the case when F and G are both uniform on a bounded interval $[a, b]$. Practically, this is a natural case to consider because it corresponds to lack of information about customers and the future competitor. In this case, by Corollary 1, $x = \frac{a+b}{2}$ is the unique Bayes solution for the first vendor. So the first vendor should build his store right in the middle if he has no information about the customers and his competitor. This might seem to be trivially obvious; but, interestingly, we shall see that this result is false if the number of future competitors is large.

The next example explains the mutual relation among the sufficient condition and the necessary condition in Theorem 1 and the unimodality assumption on f in Corollary 1.

Example 4 Let $Z \sim \frac{1}{2}N(\mu, 1) + \frac{1}{2}N(-\mu, 1)$, where $\mu \geq 0$, and let $Y \sim N(0, 1)$. Then 0 is the median of Z and by some calculations, one sees:
(a) The necessary conditions (6) holds for all $\mu \geq 0$.
(b) The sufficient condition (5) holds if $\mu \leq 1.07543$.
(c) $x = 0$ is the (unique) Bayes solution if $\mu \leq 3.14876$.
(d) Z is unimodal if and only if $\mu \leq 1$.

This Example therefore shows that neither (5) nor (6) is a necessary and sufficient condition for the median of Z to be a Bayes solution. It also shows that the median of Z can be the unique Bayes solution even if the conditions in Corollary 1 fail. In other words, Hotelling's (1929) result that the median of Z is the first vendor's optimal solution according to the minimax criterion is often valid according to the Bayes criterion as well under reasonable conditions on F and G. Example 4 indicates that sometimes unimodality of F may not be that important. This is instructive in the sense that unanimity of solutions according to different criteria is generally a desirable outcome.

3.2 Bayes Solution with Many Competitors

Let us now move on to the cases where there exist two or more future competitors. In the remainder of this section, we will take S to be the bounded interval $[a, b]$; without loss of generality, we may choose $[a, b]$ to be $[-1, 1]$. Let also G be the uniform distribution on $[-1, 1]$. However, the case of a general G would be commented on. In Section 3.1, we saw that the median of F is a Bayesian solution for the first vendor under some suitable assumptions when there is only one future competitor. But we come to a completely different conclusion when there are many future competitors. The main difference is that the Bayes solution may move towards the boundary when n gets large. There is a threshold result. A preview is given in Section 3.2.1, and Section 3.2.2 and 3.2.3 provide the results and the proofs.

Generally, one observes that a customer located at z on our left, i.e. $z < x$, will visit us if and only if there are no competitors located between $2z-x$ and x and similarly if the customer is on our right. Consequently, the general formula for the utility function in (3) becomes

$$D_n(x) = \int_{-\infty}^{x} (G(2z-x) + 1 - G(x))^n f(z)\mathrm{d}z + \int_{x}^{\infty} (G(x) + 1 - G(2z-x))^n f(z)\mathrm{d}z. \tag{13}$$

For $S = [-1, 1]$ and $G = U[-1, 1]$, (13) itself will simplify to:

$$\begin{aligned} D_n(x) &= (\frac{1-x}{2})^n F(\frac{x-1}{2}) + \int_0^{\frac{1+x}{2}} (1-t)^n f(x-t)\mathrm{d}t \\ &\quad + \int_0^{\frac{1-x}{2}} (1-t)^n f(x+t)\mathrm{d}t + (\frac{1+x}{2})^n (1 - F(\frac{1+x}{2})) \qquad (14) \\ &= (I)_1 + (I)_2 + (I)_3 + (I)_4 \qquad (15) \end{aligned}$$

One can observe from (14) that $D_n(x)$ is continuous and bounded for $x \in [-1, 1]$. Consequently, a Bayes solution exists.

The actual analysis will get rather technical. So it would be helpful to have a preview of the principal phenomena that arise from the results. The preview would be useful to appreciate how the case of many competitors differs from the case of one competitor and to provide an explanation for why these differences arise. We now provide a brief preview.

3.2.1 Preview

The key difference between the case of small and large n is that for large n (for some F's), the first vendor's optimal strategy is to move away from the center and choose a location near the boundary. For example, if F is uniform too, then the first vendor should set up his or her store at 0 only if $n \leq 4$ and move to locations $\pm x_n$ near the boundary if $n \geq 5$. There is a clear threshold. x_n is a root of a complicated equation, but there is a nice approximation, namely, $x_n = 1 - \frac{4}{n} + o(\frac{1}{n})$. Heuristically, one can guess that the Bayes solution will move from zero starting at $n = 5$ by noting that the second derivative of the utility function satisfies $D_n''(0) = \frac{n(n-4)}{2^{n+1}}$ which is > 0 for $n \geq 5$ and $D_n'(0) = 0$ for any n from symmetry. Figure 1 illustrates the shape of $D_n(x)$.

For general F, the final results are similar, although at first they seem counter to intuition. For instance, intuition might suggest that if F has a unique mode m in the interior of S, then the first vendor ought to locate at that unique mode. For large n, this is not exactly correct. If the unique mode m is a very pronounced one, then indeed the first vendor will stay there. But if the density f at a boundary point is only a bit less than the density at that interior mode, then for large n, the first vendor should move close to that boundary point. We make precise what is the meaning of only a bit less. For example, if f is symmetric with a mode at 0, and $f(\pm 1) \geq .9366 f(0)$, then for large n, the first vendor should abandon the center location and seek a location $\pm x_n$ near the boundary. And fortunately, the approximation $x_n = 1 - \frac{4}{n} + o(\frac{1}{n})$ is valid for general F, not just when F is uniform.

What is the intuition? The intuition is that if n is large, then the first vendor would be sandwiched between competitors on his left as well as on his right if he were to select a central location. By moving close to the boundary, he can essentially eliminate competition from one side, say his right. If there are still a good number of customers on his right, they are all necessarily his clients and in this way he is better off moving towards the boundary than staying at the central location.

Theorem 4 describes precisely this verbal exposition; if $\{x_n\}_{n=1}^{\infty}$ is any sequence of optimal locations of the first vendor, then every accumulation point of $\{x_n\}$ must be ± 1 or an interior mode of f. So asymptotically, the first vendor's task is simple: consider only those interior locations (if any) that are modes of f, and also consider the two points $1 - \frac{4}{n}$ and $-1 + \frac{4}{n}$. If he limits his search to just these locations, he would be approximately correct. One final point: the main result guiding Theorem 4 is Lemma

1. Lemma 1 (parts (a) and (b)) says that for all interior locations x, the utility function $D_n(x)$ (see equation (15)) is well approximated by $\frac{2}{n}f(x)$. So at any interior location x at which f(x) is large, the utility function $D_n(x)$ is also large. But the boundary points ± 1 are not covered by this. So these two boundary points must also be considered in addition to the interior modes of f as possible accumulation points. And that is what Theorem 4 says.

We will present the special case $F =$ uniform first. There are two reasons for doing so; first, it is an important special case and second, there is a very explicit threshold result for the uniform case. We should remind the reader at this point that we have already assumed G to be uniform on $[-1, 1]$ for the entire Section 3.2.

3.2.2 The case of Uniform F

Theorem 2 *Suppose that F and G are both uniform on $[-1, 1]$. Then,*
(a) *When $n \leq 4$, there is a unique Bayes solution $x_n = 0$.*
(b) *When $n \geq 5$, there are exactly two Bayes solutions ,$-x_n$ and x_n, for some $0 < x_n <$ 1.*
(c) $x_n = 1 - \frac{4}{n} + o(\frac{1}{n})$.

Some plots of the utility functions for the case that F and G are both uniform on $[-1, 1]$ are shown in Figure 1.

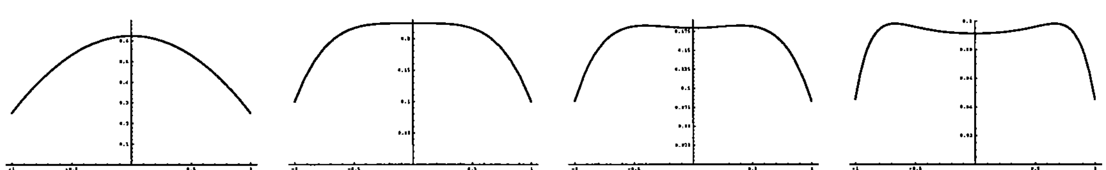


Figure 1: Plots of $D_n(x)$ when F and G are both uniform distributions on $[-1, 1]$.

Proof: First, note that the utility function $D_n(x)$ can be written in a simple closed form when $F = U[-1, 1]$. Simply plugging $f(z) = \frac{1}{2}\mathbf{1}_{-1\leq z\leq 1}$ and $F(z) = \frac{z+1}{2}\mathbf{1}_{-1\leq z\leq 1} + \mathbf{1}_{1<z}$

into (14) and simplifying it, we get

$$D_n(x) = \frac{1}{2(n+1)}\left[2-(n+2)\left((\frac{1+x}{2})^{n+1}+(\frac{1-x}{2})^{n+1}\right)+(n+1)\left((\frac{1+x}{2})^{n}+(\frac{1-x}{2})^{n}\right)\right]. \tag{16}$$

One can observe that $D_n(x)$ is a symmetric polynomial of degree $n+1$. Therefore, if we find one nonzero Bayes solution, then there will automatically be two.

Step 1: It is easy to see from (16) that

$$D_n'(x) = \frac{1}{4}\left[-(n+2)\left((\frac{1+x}{2})^{n}-(\frac{1-x}{2})^{n}\right)+n\left((\frac{1+x}{2})^{n-1}-(\frac{1-x}{2})^{n-1}\right)\right]. \tag{17}$$

Since $D_n(x)$ is symmetric, we only need to check the sign of $D_n'(x)$ for x in $(0,1)$ to find the Bayes solutions.

Define $a_n(x) = (\frac{1+x}{2})^n - (\frac{1-x}{2})^n$ for $n \geq 1$ and $b_n(x) = a_n(x)/a_{n-1}(x)$ for $n \geq 2$. Both $a_n(x)$ and $b_n(x)$ are well defined, continuous, and positive in $(0,1)$. Note that

$$D_n'(x) >,\ =\ \text{or}\ <\ 0 \quad \text{if and only if} \quad b_n(x) <, = \ \text{or}\ > \frac{n}{n+2}, \quad \text{respectively.} \tag{18}$$

Step2: (recurrence equations for a_n and b_n). For any two numbers a and b, $(a^n - b^n) = (a+b)(a^{n-1}-b^{n-1}) - ab(a^{n-2}-b^{n-2})$. Plugging $a = \frac{1+x}{2}$ and $b = \frac{1-x}{2}$ in, we immediately have the recurrence equation for a_n:

$$a_n(x) = a_{n-1}(x) - \frac{1-x^2}{4}a_{n-2}(x) \quad \text{for all } n \geq 3,$$

and hence we obtain the recurrence equation for $b_n(x)$ which is

$$b_n(x) = 1 - \frac{1-x^2}{4}\frac{1}{b_{n-1}(x)}, \quad \text{for all } n \geq 3. \tag{19}$$

Step 3: By a straight application of L'Hospital's rule, we get $\lim_{x\to 0} b_n(x) = \frac{n}{2(n-1)}$ and $\lim_{x\to 1} b_n(x) = 1$.

Step 4: We claim for each fixed $x \in (0,1)$, $\{b_n(x)\}_{n=2}^{\infty}$ is a strictly decreasing sequence.

Let $N = \{n > 1 : b_n(x) > b_{n+1}(x) \text{ for all } x \text{ in } (0,1)\}$. First, note that $b_2(x) = 1 > 1 - \frac{1-x^2}{4} = b_3(x)$ for all $x \in (0,1)$. Hence $2 \in N$. Now, let $k \geq 2$ belong to N; namely, $b_k(x) > b_{k+1}(x)$ for all $x \in (0,1)$. By (19), one gets $b_{k+1}(x) = 1 - \frac{1-x^2}{4}\frac{1}{b_k(x)} > 1 - \frac{1-x^2}{4}\frac{1}{b_{k+1}(x)} = b_{k+2}(x)$ for all $x \in (0,1)$. Therefore $k+1$ also belongs to N. This proves the claim.

Step 5: We claim for each fixed $n \geq 3$, $b_n(x)$ is a strictly increasing function of x in $(0,1)$. Again, this follows by induction by using (19).

Step6: (Proof of part (a) of Theorem). In Example 3, it has been shown that the statement of part (a) is true for $n = 1$. Thus, only the cases $n = 2, 3, 4$ are left.

To show that 0 is the unique Bayes solution, it is enough to show that $D_n'(x) < 0$ for all $x \in (0,1)$. By (18), this is equivalent to $b_n(x) > \frac{n}{n+2}$ for all $x \in (0,1)$. When $n = 2$, this is obvious. When $n = 3$, or 4, by Step 3 and Step 5, for every $x \in (0,1)$, we have $b_n(x) > \lim_{x\to 0} b_n(x) = \frac{n}{2(n-1)}$. But $\frac{n}{2(n-1)}$ is greater than or equal to $\frac{n}{n+2}$ when $n = 3, 4$. Hence the proof for part (a) is complete.

Step 7: (Proof of part (b)). If we can show that there exists $0 < x_n < 1$ such that

$$b_n(x_n) = \frac{n}{n+2}, \; b_n(x) < \frac{n}{n+2} \text{ for } 0 < x < x_n, \text{ and } b_n(x) > \frac{n}{n+2} \text{ for } x_n < x < 1, \tag{20}$$

it will follow from (18) that $D_n(x)$ strictly increases in the interval $(0, x_n)$ and strictly decreases in the interval $(x_n, 1)$. This implies x_n and $-x_n$ are the only two Bayes solutions. But the facts in Step 3 and Step 4 imply that $b_n(x)$ increases strictly from $\frac{n}{2(n-1)}$ to 1 as x goes from 0 to 1. Furthermore, when $n \geq 5$, one has $\frac{n}{2(n-1)} < \frac{n}{n+2}$. Therefore, there exists a unique x_n between 0 and 1 satisfying (20).

Step 8: (Proof of part (c)). For $n \geq 5$, plugging $x = x_n$ into (19), we have

$$b_n(x_n) = \frac{n}{n+2} = 1 - \frac{1-x_n^2}{4}\frac{1}{b_{n-1}(x_n)} \iff (n+2)(1-x_n) = \frac{8b_{n-1}(x_n)}{1+x_n} \tag{21}$$

Furthermore, for any $n \geq 6$, Steps 3, 4, and 5 imply $\frac{n}{n+2} = b_n(x_n) < b_{n-1}(x_n) \leq b_5(x_n) < \lim_{x\to 1} b_5(x) = 1$. From this, immediately, we see that $\lim_{n\to\infty} b_{n-1}(x_n) = 1$. Hence, from (21),

$$(n+2)(1-x_n^2) = 8 + o(1) \;\Rightarrow\; x_n^2 = 1 - \frac{8}{n+2} + o(\frac{1}{n}) \;\Rightarrow\; x_n = 1 - \frac{4}{n} + o(\frac{1}{n}),$$

as was claimed, and this completes the proof of part (c). □

In reality, the number of the future competitors is usually unknown. Therefore, we may want to treat n as a parameter and impose a prior on it.

Let us assume $n \sim Poisson(\lambda)$, i.e. $P(n = k) = e^{-\lambda}\frac{\lambda^k}{k!}, k = 0, 1, \ldots$. Note that under this model it is possible that there will be no future competitors. When there exists no competitor, the customer will certainly visit us, and hence $D_0(x) \equiv 1$. One can check that formula (16) is also valid for $n = 0$. So the utility function for the first vendor becomes

$$\begin{aligned} D_\lambda(x) &= \sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} D_n(x) \\ &= \frac{1}{\lambda} + \frac{1}{2}\left(\frac{1-x}{2} e^{-\frac{1-x}{2}\lambda} + \frac{1+x}{2} e^{-\frac{1+x}{2}\lambda}\right) - \frac{1}{2\lambda}\left(e^{-\frac{1-x}{2}\lambda} + e^{-\frac{1+x}{2}\lambda}\right) \qquad (22) \end{aligned}$$

Notice, again, $D_\lambda(x)$ is an even function of x. Therefore, if x_λ is a Bayes solution for the first vendor, so is $-x_\lambda$. Interestingly, even under this model, there is a threshold result:

Theorem 3 *Suppose that F and G are both uniform on $[-1, 1]$, and n is Poisson distributed with parameter λ. Then we have*
(a) *When $\lambda \le 6$, the Bayes solution is unique and it is* 0.
(b) *When $\lambda > 6$, there are exactly two Bayes solutions, x_λ and $-x_\lambda$, where $0 < x_\lambda < 1$.*
(c) $x_\lambda = 1 - \frac{4}{\lambda} + o(\frac{1}{\lambda})$.

Remark: It is interesting that the threshold value for λ is 6, a whole number.
Proof: The proof is very similar to that of Theorem 2. From expression (22), one can easily derive

$$D'_\lambda(x) = \frac{1}{4}\left\{\lambda\left(\frac{1-x}{2} e^{-\frac{1-x}{2}\lambda} - \frac{1+x}{2} e^{-\frac{1+x}{2}\lambda}\right) - 2\left(e^{-\frac{1-x}{2}\lambda} - e^{-\frac{1+x}{2}\lambda}\right)\right\}$$

Since $D_\lambda(x)$ is symmetric around 0, it is enough to check the sign of $D'_\lambda(x)$ for $x \in (0, 1)$ in order to find where $D_\lambda(x)$ has a maximum.

First, let us define, for all $x \in (0, 1)$ and all $\lambda > 0$,

$$b_\lambda(x) = \frac{\frac{1-x}{2} e^{-\frac{1-x}{2}\lambda} - \frac{1+x}{2} e^{-\frac{1+x}{2}\lambda}}{e^{-\frac{1-x}{2}\lambda} - e^{-\frac{1+x}{2}\lambda}} = \frac{\frac{1-x}{2} - \frac{1+x}{2} e^{-\lambda x}}{1 - e^{-\lambda x}}.$$

The following facts will be used in the proof:

(P1) $D'_\lambda >, =$, or < 0 if and only if $b_\lambda(x) >, =$, or $< \frac{2}{\lambda}$, respectively.

(P2) $\lim_{x \to 0} b_\lambda(x) = \frac{\lambda - 2}{2\lambda}$ and $\lim_{x \to 1} b_\lambda(x) = -\frac{e^{-\lambda}}{1 - e^{-\lambda}}$.

(P3) $\frac{\lambda - 2}{2\lambda} \le \frac{2}{\lambda} \Leftrightarrow \lambda \le 6$.

(P4) For any fixed $x \in (0,1)$, $b_\lambda(x)$ is a strictly increasing function of λ and converges to $\frac{1-x}{2}$ as λ tends to ∞.

(P5) For any fixed $\lambda > 0$, $b_\lambda(x)$ is a strictly decreasing function of x in $(0,1)$.

Proof of part (a): By (P2), (P3), and (P5), when $\lambda \le 6$, we have, for all $x \in (0,1)$, $b_\lambda(x) < \lim_{x\to 0} b_\lambda(x) = \frac{\lambda-2}{2\lambda} \le \frac{2}{\lambda}$. This and (P1) imply $D_\lambda'(x) < 0$ for all $x \in (0,1)$. It follows that 0 is the unique Bayes solution when $\lambda \le 6$.

Proof of part (b): If we show that there exists $0 < x_\lambda < 1$ such that $D_\lambda'(x_\lambda) = 0$, $D_\lambda'(x) > 0$ if $0 < x < x_\lambda$, and $D_\lambda'(x) < 0$ if $x_\lambda < x < 1$, the statement of (b) will follow immediately. But by (P1), it means we need to find $0 < x_\lambda < 1$ satisfying $b_\lambda(x_\lambda) = \frac{2}{\lambda}$. From (P5), it will automatically follow that $b_\lambda(x) > \frac{2}{\lambda}$ for $x < x_\lambda$ and $b_\lambda(x) < \frac{2}{\lambda}$ for $x > x_\lambda$.

Now note that, (P2) and (P3) imply that $\lim_{x\to 0} b_\lambda(x) = \frac{\lambda-2}{2\lambda} > \frac{2}{\lambda} > 0 > -\frac{e^{-\lambda}}{1-e^{-\lambda}} = \lim_{x\to 1} b_\lambda(x)$ when $\lambda > 6$. Therefore, by (P5), such an x_λ exists, and $\pm x_\lambda$ are the Bayes solutions.

Proof of part (c): From the proof of part (b), we have x_λ satisfying

$$\frac{2}{\lambda} = b_\lambda(x_\lambda) = \frac{\frac{1-x_\lambda}{2} - \frac{1+x_\lambda}{2}e^{-\lambda x_\lambda}}{1-e^{-\lambda x_\lambda}} \iff \frac{1}{2} - \frac{2}{\lambda} = \frac{1}{2}\frac{x_\lambda(1+e^{-\lambda x_\lambda})}{1-e^{-\lambda x_\lambda}} \text{ for all } \lambda > 6. \quad (23)$$

It is easily proved that x_λ itself is increasing in λ, and therefore $x^* = \lim_{\lambda\to\infty} x_\lambda$ exists and cannot be zero. This immediately implies that $e^{-\lambda x_\lambda} = o(\frac{1}{\lambda})$. So from equation (23) above,

$$(1-\frac{4}{\lambda})(1+o(\frac{1}{\lambda})) = x_\lambda(1+o(\frac{1}{\lambda})) \Rightarrow x_\lambda = 1 - \frac{4}{\lambda} + o(\frac{1}{\lambda}),$$

as was claimed and this completes the proof of Theorem 3 and finishes our discussion for the case when F is uniform. □

3.2.3 The Case of a General F

For theoretical purposes as well as for applications, the case of a general F is very important. We will now look at this case.

As we discussed in our preview in Section 3.2.1, asymptotically the first vendor's optimal location(s) will be close to either an interior mode of f or to ± 1. Theorem

4 below gives a precise statement. The statement itself is a bit awkward; Remark 1 following the statement may be helpful to understand the statement more easily.

Theorem 4 *Suppose that f is continuous in $S = [-1,1]$ (here $f(-1)$ and $f(1)$ are allowed to be infinite). Let S_m be the set of all modes of f and let $M_m = \sup_{z\in[-1,1]} f(z)$. Let also $S_{-1} = \{-1\}, S_1 = \{1\}, M_{-1} = \frac{2+e^{-2}}{2} f(-1)$, and $M_1 = \frac{2+e^{-2}}{2} f(1)$.*

Denote $M^* = \max(M_m, M_{-1}, M_1)$. Construct a set S^* by including S_i into S^* if and only if $M^* = M_i$; here $i = m, -1, or\ 1$. Let $\{x_n\}_{n=1}^{\infty}$ be any sequence of Bayes solutions for the first vendor. Then the accumulation points of $\{x_n\}_{n=1}^{\infty}$ are contained in S^*.

Remark 1 Suppose f has a unique mode m in the interior and suppose that $f(1) > f(-1)$ but 1 is not a mode of f. Then Theorem 4 tells us that if $f(1) < \frac{2}{2+e^{-2}} f(m)$, asymptotically x_n will be close to m. But if $f(1) > \frac{2}{2+e^{-2}} f(m)$, x_n will be close to 1 rather than the mode m. In other words, if the density of the customers around 1 is higher than a certain proportion (which is $\frac{2}{2+e^{-2}} \approx 0.9366$) of the density of the customers at the mode m, the first vendor will prefer to locate close to 1 rather than close to m when the number of the future competitors is large. Usually $n = 10$ is large enough. Similar interpretations hold when $f(-1) > f(1)$ or when $f(-1) = f(1)$.

Remark 2 A similar result holds for more general G; i.e., G does not have to be uniform. Suppose that G has a density g which is uniformly bounded away from 0 and both f and g satisfy some Hölder conditions. Then the accumulation points of $\{x_n\}_{n=1}^{\infty}$ will always be contained in the union of $\{-1,1\}$ and the modes of $\frac{f}{g}$.

Remark 3 We will see from the proof of Theorem 4 that if x_n converges to 1, then we still have $x_n = 1 - \frac{4}{n} + o(\frac{1}{n})$, as we had seen for the special case of F=uniform before (Theorem 2). Likewise, if x_n converges to -1, then $x_n = -1 + \frac{4}{n} + o(\frac{1}{n})$.

Theorem 4 rests on a key lemma. The lemma describes the asymptotic behavior of $D_n(x)$; more precisely, it describes the interplay between $D_n(x)$ and $f(x)$ as $n \to \infty$.

Lemma 1 *Suppose that f is continuous in $[-1,1]$.*

(a) Let C be any closed subset of $(-1,1)$. Then, $nD_n(x)$ is uniformly convergent to $2f(x)$ in C.

(b) Let b_n be a sequence with limit b in the interior, i.e., $b \in (-1,1)$. Then

$$\lim_{n\to\infty} nD_n(b_n) = 2f(b).$$

(c) Let a_n be any sequence such that $\lim_{n\to\infty} a_n = \infty$ and such that $a = \lim_{n\to\infty} \frac{n}{a_n}$ exists (here 'a' can be 0 or ∞). Let $b_n = -1 + \frac{2}{a_n}$. Then

$$\lim_{n\to\infty} nD_n(b_n) = (2+(a-1)e^{-a})f(-1) \quad \textit{and} \quad \lim_{n\to\infty} nD_n(-b_n) = (2+(a-1)e^{-a})f(1).$$

We will first provide the proof of Theorem 4, assuming Lemma 1. Then we will give a prove of the lemma itself.

Proof of Theorem 4: For simplicity of presentation, we will provide the proof for the case when f has a unique mode m. This m could be ± 1 or in the interior $(-1,1)$. The proof for the case of multiple modes is very similar.

Step1 : We claim $\liminf_{n\to\infty} nD_n(x_n) \geq 2M^*$ (M^* is as in the statement of Theorem 4).

First consider the case $m \in (-1,1)$. Let $a_n = m + \frac{1}{n}$ and $b_n = -1 + \frac{4}{n}$. By Lemma 1, $\lim_{n\to\infty} nD_n(a_n) = 2f(m)$, $\lim_{n\to\infty} nD_n(b_n) = (2+e^{-2})f(-1)$, and $\lim_{n\to\infty} nD_n(-b_n) = (2+e^{-2})f(1)$. Let x_n be a Bayes solution for a given n. Then, obviously, $D_n(x_n) \geq D_n(a_n)$, $D_n(b_n)$, and $D_n(-b_n)$ separately, and hence,

$$\begin{aligned}
\liminf_{n\to\infty} nD_n(x_n) &\geq \max(\liminf_{n\to\infty} nD_n(a_n), \liminf_{n\to\infty} nD_n(b_n), \liminf_{n\to\infty} nD_n(-b_n)) \\
&= \max(2f(m), (2+e^{-2})f(-1), (2+e^{-2})f(1)) = 2M^*,
\end{aligned}$$

as we claimed.

Exactly similar arguments prove that $\liminf_{n\to\infty} nD_n(x_n) \geq 2M^*$ also when $m = \pm 1$.

Step 2: We claim m, -1, and 1 are the only candidates for accumulation points of $\{x_n\}_{n=1}^{\infty}$ (here m may be -1 or 1).

So let x^* be any accumulation point of $\{x_n\}_{n=1}^{\infty}$; then there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ with limit x^*. Suppose $x^* \neq m, -1$, or 1. By Lemma 1, we have $\lim_{k\to\infty} n_k D_{n_k}(x_{n_k}) =$

$2f(x^*)$. Hence $\liminf_{n\to\infty} nD_n(x_n) \le 2f(x^*) < 2f(m) \le 2M^*$. This contradicts what we just proved in Step 1. Therefore, necessarily, $x^* = m, -1$, or 1.

Step 3: Now we characterize x^* more precisely, i.e., we pin down x^* as stated in Theorem 4.

First, consider the case $-1 < m < 1$ and suppose $M_m < M^*$. We want to show that then x^* cannot be equal to m. If x^* is m, then there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ with limit m. By applying Lemma 1, we have $\lim_{k\to\infty} n_k D_{n_k}(x_{n_k}) = 2f(m) < 2M^*$ which is again a contradiction to the conclusion made in Step 1. So if $M_m < M^*$, then x^* cannot be equal to m.

Next, let us look at the case $M_{-1} < M^*$. We want to show that then x^* cannot be equal to -1. For if $x^* = -1$, then there is a subsequence $\{x_{n_k}\}$ which converges to -1, and if we take $a_{n_k} = \frac{2}{1+x_{n_k}}$, $\lim_{k\to\infty} a_{n_k} = \infty$ and $a = \lim_{k\to\infty} \frac{n}{a_{n_k}}$ exists. Therefore by part (c) of Lemma 1,

$$\lim_{k\to\infty} n_k D_{n_k}(x_{n_k}) = (2 + (a-1)e^{-a})f(-1) \le (2 + e^{-2})f(-1) < 2M^*,$$

which again is a contradiction to the conclusion in Step 1. So if $M_{-1} < M^*$, then x^* cannot be equal to -1.

Similarly one proves that if $M_1 < M^*$, then x^* cannot be equal to 1.

Step4: In Step 3, we had assumed that $-1 < m < 1$. But the proof is exactly similar if m is ± 1. So we will omit that proof.

This completes the proof of Theorem 4. □

It remains to give a proof of Lemma 1, which is provided below.

Proof of Lemma 1:

Proof of (a): For any given closed subset of $(-1,1)$, we can always find a closed interval in $(-1,1)$ containing it. Therefore, we only need to show that the statement is true for the case $C = [a,b]$, where $-1 < a < b < 1$. Furthermore, by the decomposition $D_n(x) = (I)_1 + (I)_2 + (I)_3 + (I)_4$ in (15), it is enough to show that $\lim_{n\to\infty} n(I)_1 = \lim_{n\to\infty} n(I)_4 = 0$, and $\lim_{n\to\infty} n(I)_2 = \lim_{n\to\infty} n(I)_3 = f(x)$, and that these convergences are uniform in C.

First, let us show that $n(I)_1$ is convergent to 0 uniformly for $x \in C$. Evidently, F is bounded by 1. Thus for all $x \in C$, we have $|n(I)_1| \le n(\frac{1-x}{2})^n \le n(\frac{1-a}{2})^n$. Furthermore, the assumption that $a > -1$ implies $\frac{1-a}{2} < 1$ and hence $n(\frac{1-a}{2})^n$ goes to zero as n goes to infinity. We conclude that $n(I)_1$ converges to zero uniformly in C. Similarly, $n(I)_4$ converges to zero uniformly in C.

Now let us show $n(I)_2$ converges uniformly to $f(x)$. But

$$|n(I)_2 - f(x)| \le |n\int_0^{\frac{1}{\sqrt{n}}}(1-t)^n f(x-t)\mathrm{d}t - f(x)| + n\int_{\frac{1}{\sqrt{n}}}^{\frac{1+x}{2}}(1-t)^n f(x-t)\mathrm{d}t, \quad (24)$$

of which the second term is

$$\le n(1-\frac{1}{\sqrt{n}})^n \int_{\frac{1}{\sqrt{n}}}^{\frac{1+x}{2}} f(x-t)\mathrm{d}t \le n(1-\frac{1}{\sqrt{n}})^n \to 0 \text{ as } n \to \infty.$$

As to the first term in (24), let $\delta_n = \max_{a\le x\le b}\max_{x-\frac{1}{\sqrt{n}}\le z\le x}|f(z)-f(x)|$. Then, the first term is

$$\begin{aligned} &\le \left|n\textstyle\int_0^{\frac{1}{\sqrt{n}}}(1-t)^n\{f(x-t)-f(x)\}\mathrm{d}t\right| + \left|n\textstyle\int_0^{\frac{1}{\sqrt{n}}}(1-t)^n f(x)\mathrm{d}t - f(x)\right| \\ &\le \delta_n\left(\tfrac{n}{n+1}\right)\left(1-(1-\tfrac{1}{\sqrt{n}})^{n+1}\right) + f(x)\left\{1-\tfrac{n}{n+1}\left(1-(1-\tfrac{1}{\sqrt{n}})^{n+1}\right)\right\}. \end{aligned} \quad (25)$$

Since f is uniformly continuous in any closed interval of $(-1,1)$, we have $\lim_{n\to\infty}\delta_n = 0$. Moreover, $\lim_{n\to\infty}(\frac{n}{n+1})(1-(1-\frac{1}{\sqrt{n}})^{n+1}) = 1$. Consequently, (25) goes to 0 as n goes to ∞. It now follows from (24) that $n(I)_2$ is uniformly convergent to $f(x)$ for x in C.

Similarly, $n(I)_3$ is uniformly convergent to $f(x)$ in C. Therefore, part (a) of Lemma 1 is now proved.

Proof of (b): Let C_n be the smallest closed interval which contains b and each b_i for $i \ge n$. By the assumption that $\lim_{n\to\infty} b_n = b$, C_n converges to the singleton $\{b\}$.

To prove $\lim_{n\to\infty} nD_n(b_n) = 2f(b)$, just observe that

$$\begin{aligned} |nD_n(b_n) - 2f(b)| &\le |nD_n(b_n) - 2f(b_n)| + 2|f(b_n) - f(b)| \\ &\le \sup_{x\in C_n}|nD_n(x) - 2f(x)| + 2\sup_{x\in C_n}|f(x) - f(b)| \end{aligned} \quad (26)$$

By applying part (a) of this lemma, we have the first term of (26) convergent to zero as n goes to infinity, and the continuity of f at b implies the second term of (26) goes to zero too. And this proves part (b) of Lemma 1.

Proof of (c): We will only establish the limit for $nD_n(b_n)$ as the proof for $nD_n(-b_n)$ is virtually identical. Again, the key is the decomposition of $D_n(x)$ into $(I)_1, \ldots, (I)_4$ given in equation (15).

Step 1: $(\lim_{n\to\infty} n(I)_1 = ae^{-a}f(-1))$.

From the definition of $(I)_1$,

$$n(I)_1 = n(1-\frac{1}{a_n})^n F(\frac{1}{a_n}-1) = \frac{n}{a_n}\left((\frac{1}{a_n}-1)^{a_n}\right)^{\frac{n}{a_n}}\left(\frac{1}{(\frac{1}{a_n})}\int_{-1}^{-1+\frac{1}{a_n}} f(z)\mathrm{d}z\right).$$

Since $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty}\frac{n}{a_n} = a$ by assumption, and f is continuous at -1, one gets $\lim_{n\to\infty} n(I)_1 = ae^{-a}f(-1)$.

Step 2: $(\lim_{n\to\infty} n(I)_2 = (1-e^{-a})f(-1))$.

Let $M_n = \max_{-1\leq z\leq b_n} f(z)$ and $m_n = \min_{-1\leq z\leq b_n} f(z)$. Then, from the definition of $(I)_2$,

$$m_n n\int_0^{\frac{1}{a_n}}(1-t)^n\mathrm{d}t \leq n(I)_2 \leq M_n n\int_0^{\frac{1}{a_n}}(1-t)^n\mathrm{d}t.$$

By the continuity of f, $\lim_{n\to\infty} M_n = \lim_{n\to\infty} m_n = f(-1)$. Also, $n\int_0^{\frac{1}{a_n}}(1-t)^n\mathrm{d}t = \frac{n}{n+1}(1-(1-\frac{1}{a_n})^{n+1}) \to 1-e^{-a}$ as $n\to\infty$. Thus, $\lim_{n\to\infty} n(I)_2 = (1-e^{-a})f(-1)$.

Step 3: $(\lim_{n\to\infty} n(I)_3 = f(-1))$.

Again, from the definition of $(I)_3$,

$$n(I)_3 = n\int_0^{\frac{1}{\sqrt{n}}}(1-t)^n f(b_n+t)\mathrm{d}t + n\int_{\frac{1}{\sqrt{n}}}^{1-\frac{1}{a_n}}(1-t)^n f(b_n+t)\mathrm{d}t. \tag{27}$$

By an argument analogous to the one in Step 2, the first term converges to $f(-1)$, and the second term is easily seen to converge to 0. Thus $\lim_{n\to\infty} n(I)_3 = f(-1)$.

Step 4: $(\lim_{n\to\infty} n(I)_4 = 0)$.

This is trivial.

Step 5: Combining the conclusions in Step 1, 2, 3, and 4, part (c) of Lemma 1 is now proved.

4 Two Dimensional Markets

Instead of a linear market, it may be more realistic to consider a two dimensional market. We will continue to use the notations introduced in Section 2.1 for one dimensional markets. Also, we still assume that a buyer visits the closest store. As before, we would like to investigate these two dimensional markets from two perspectives: minimax and Bayesian. We will discuss them in Section 4.1 and 4.2 respectively.

4.1 Minimax Solutions in Two Dimensions

As we did in the case of a linear market, first let us consider the case of only one future rival. Let us denote the future rival's location by $\underset{\sim}{y}$. The expected sale for the first vendor then becomes

$$D(\underset{\sim}{x}, \underset{\sim}{y}) = E(h(\|\underset{\sim}{Z} - \underset{\sim}{x}\|)\mathbf{1}_{\|\underset{\sim}{Z}-\underset{\sim}{x}\|^2 < \|\underset{\sim}{Z}-\underset{\sim}{y}\|^2}) = E(h(\|\underset{\sim}{Z} - \underset{\sim}{x}\|)\mathbf{1}_{\underset{\sim}{Z} \in H((\underset{\sim}{x}+\underset{\sim}{y})/2, \underset{\sim}{x}-\underset{\sim}{y})}),$$

where $H(\underset{\sim}{p}, \underset{\sim}{n})$ is the open half space containing the point $\underset{\sim}{p} + \underset{\sim}{n}$ with its boundary line passing through the point $\underset{\sim}{p}$ and perpendicular to the vector $\underset{\sim}{n}$; i.e., $H(\underset{\sim}{p}, \underset{\sim}{n}) = \{\underset{\sim}{z} : \underset{\sim}{n} \cdot (\underset{\sim}{z} - \underset{\sim}{p}) > 0\}$. Let $e_\theta = (\cos\theta, \sin\theta)$. By simple arguments, one can see that the minimax solutions are those points which maximize

$$V(\underset{\sim}{x}) \stackrel{def}{=} \min_{\underset{\sim}{y} \neq \underset{\sim}{x}} D(\underset{\sim}{x}, \underset{\sim}{y}) = \min_\theta E(h(\|\underset{\sim}{Z} - \underset{\sim}{x}\|)\mathbf{1}_{\underset{\sim}{Z} \in H(\underset{\sim}{x}, e_\theta)}) = \min_\theta V_\theta(\underset{\sim}{x}), \tag{28}$$

$$\text{where} \quad V_\theta(\underset{\sim}{x}) \stackrel{def}{=} E(h(\|\underset{\sim}{Z} - \underset{\sim}{x}\|)\mathbf{1}_{\underset{\sim}{Z} \in H(\underset{\sim}{x}, e_\theta)}). \tag{29}$$

When $h(\cdot) \equiv$ a constant, say 1, from (28), the minimax solutions are those points which maximize $V(\underset{\sim}{x}) = \min_\theta P(Z \in H(\underset{\sim}{x}, e_\theta))$. But points which maximize $\min_\theta P_{\underset{\sim}{Z} \sim F}(\underset{\sim}{Z} \in H(\underset{\sim}{x}, e_\theta))$ are in fact defined as the halfspace medians for a bivariate distribution F. Indeed, the halfspace median, a generalization of the univariate median to higher dimensions due to Tukey (1975) and Donoho (1982), is motivated by this minimaxity result. See Small (1990) for details. Therefore, we conclude a result similar to Proposition 1: if $h(\cdot)$ is a constant function, the set of minimax optimals for the first vendor equals the set of halfspace medians of F.

If we intend to discard the assumption that $h(\cdot) \equiv$ a constant and still want to get a clean result, more conditions on F will be needed. The following is a general result analogous to Proposition 2.

Proposition 3 *If f is spherically symmetric and unimodal around some $\underset{\sim}{m}$, then $\underset{\sim}{m}$ is a minimax optimal.*

Proof: Without loss of generality, we can take $\underset{\sim}{m}$ to be the origin.

Step 1: We claim $V_\theta(re_\theta)$ is a decreasing function of r in $[0,\infty)$ for all θ; recall that the formula $V_\theta(\cdot)$ is as defined in (29).

It is obvious that

$$V_\theta(re_\theta) = \int_{\underset{\sim}{z}\in H(re_\theta,e_\theta)} h(\|\underset{\sim}{z} - re_\theta\|) f(\underset{\sim}{z}) \mathrm{d}\underset{\sim}{z} = \int_{\underset{\sim}{t}\in H(0,e_\theta)} h(\|\underset{\sim}{t}\|) f(\underset{\sim}{t} + re_\theta) \mathrm{d}\underset{\sim}{t}. \tag{30}$$

Now observe $\|\underset{\sim}{t} + re_\theta\|^2 = \|\underset{\sim}{t}\|^2 + r^2 + 2r\underset{\sim}{t}\cdot e_\theta$, which is an increasing function of r for $\underset{\sim}{t} \in H(\underset{\sim}{0}, e_\theta)$. Therefore, the assumption that f is spherically symmetric and unimodal around the origin implies that for any fixed θ, $f(\underset{\sim}{t} + re_\theta)$ is a decreasing function of r for $\underset{\sim}{t} \in H(\underset{\sim}{0}, e_\theta)$. Applying this to (30), we prove this claim.

Step 2: We claim there exists a constant c such that $V_\theta(\underset{\sim}{0}) = c$ for all θ.

This is obvious; since f is spherically symmetric to the origin, it is evident that $V_\theta(\underset{\sim}{0})$ does not depend on θ.

Step 3: Now we shall show $V(\underset{\sim}{x}) \leq V(\underset{\sim}{0})$ for all $\underset{\sim}{x}$ in R^2. But for any $\underset{\sim}{x}$ in R^2, there exist $r_0 > 0$ and $0 \leq \theta_0 < 2\pi$ such that $\underset{\sim}{x} = r_0 e_{\theta_0}$. Therefore one has $V(\underset{\sim}{x}) = \min_\theta V_\theta(r_0 e_{\theta_0}) \leq V_{\theta_0}(r_0 e_{\theta_0})$, which is smaller than $V_{\theta_0}(\underset{\sim}{0})$ by Step 1. On the other hand, Step 2 implies $V(\underset{\sim}{0}) = \min_\theta V_\theta(\underset{\sim}{0}) = c = V_{\theta_0}(\underset{\sim}{0})$ for all θ_0. Therefore, we have $V(\underset{\sim}{x}) \leq V(\underset{\sim}{0})$ for all $\underset{\sim}{x}$. This completes the proof of Proposition 3. □

For reasons identical to the case of one dimension, the minimax formulation is uninteresting if the number of future rivals is more than 1. So again, we will now proceed to consideration of the Bayesian approach.

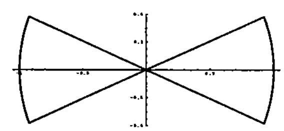
4.2 Bayes Solutions in Two Dimensions

Through out the rest of this section, we will take $h(\cdot) \equiv 1$. We discuss first the case of one future competitor.

4.2.1 Bayes Solutions with One Competitor

In a linear market, from Corollary 1, we know that 0 is a Bayes solution if f and g are both symmetric around 0 and f is unimodal. To make a similar conclusion in a two dimensional market, simply assumptions of symmetries about $(0,0)$ of f and g are not enough. In fact, even if F and G are each uniform on a symmetric set S, the Bayes solution need not be the origin. We need to put more assumptions on f and g. Example 5 below is a counterexample. This says that there is an effect of the number of dimensions and the effect already shows up in two dimensions itself.

Example 5 Let $S = \{(r\cos\theta, r\sin\theta) : -1 \leq r \leq 1, -\frac{\pi}{8} \leq \theta \leq \frac{\pi}{8}\}$, union of two opposite "fans"; a plot of the set S is provided below. Suppose F and G are both uniform distributions on S. Then, $(0,0)$ is not a Bayes solution for the first vendor. For example, one can compute and see that $(0.1, 0)$ is a better location than $(0,0)$ for the first vendor.



Example 5 suggests that more structure is needed in higher dimensions if we want to have a general result analogous to Corollary 1 for the linear market. Spherical symmetry automatically suggests itself. The next result says that this is enough structure.

Theorem 5 *Suppose that f and g are both spherically symmetric around $(0,0)$ and in addition f is unimodal. Then $(0,0)$ is a Bayes solution for the first vendor.*

Proof: It is well known that any random vector $\underset{\sim}{Z}$ with a spherically symmetric unimodal distribution may be written as $\underset{\sim}{Z} = s\underset{\sim}{U}$ where $\underset{\sim}{U}$ has a uniform distribution in the unit sphere and $s > 0$ is independent of $\underset{\sim}{U}$. Likewise, any random vector $\underset{\sim}{Y}$ with a spherically symmetric distribution may be written as $\underset{\sim}{Y} = t\underset{\sim}{V}$, where $\underset{\sim}{V}$ has a uniform distribution on the boundary of the unit sphere and $t > 0$ is independent of $\underset{\sim}{V}$. By following the same notations as in the previous sections, the utility function (i.e., the expected sales of the first vendor) is

$$D(\underset{\sim}{x}) = P(||\underset{\sim}{Z} - \underset{\sim}{x}|| \leq ||\underset{\sim}{Z} - \underset{\sim}{Y}||),$$

where $\underset{\sim}{x} = (x_1, x_2)$ is a general location for the first vendor. The utility function $D(\underset{\sim}{x})$ itself is now a function of $||\underset{\sim}{x}||$ alone and so if we can prove that for $\underset{\sim}{x}$ of the form $\underset{\sim}{x} = (x, 0)$, $D(\underset{\sim}{x})$ is maximized at $x = 0$, the theorem would follow. For notational convenience, in the rest of this proof, we will denote the utility function as $D(x)$ with a scalar x. Furthermore, it will be enough to prove that for every fixed $s > 0$,

$$D(x) = D(x,s) = P(\|s\underset{\sim}{U} - (x,0)\|^2 \leq \|s\underset{\sim}{U} - \underset{\sim}{V}\|^2) \tag{31}$$

is maximized at $x = 0$; in the above, $\underset{\sim}{U} = (U_1, U_2)$, $V = (V_1, V_2)$ are independent and distributed uniformly in the unit sphere and its boundary, respectively.

Define $T = ((V_1 - x)U_1 + V_2U_2)/\sqrt{1 + x^2 - 2V_1x}$. The following are easy to verify by a direct calculation:

(i) T and V_1 are independent;

(ii) T is distributed as $Beta(\frac{3}{2}, \frac{3}{2})$ on $(-1, 1)$ (which is a symmetric distribution);

(iii) V_1 has the density $\frac{1}{\pi\sqrt{1-v_1^2}}$, $-1 \leq v_1 \leq 1$.

By simple algebra from (31),

$$D(x) = P\left(T < \frac{1 - x^2}{2s\sqrt{1 + x^2 - 2V_1x}}\right). \tag{32}$$

This implies:

For $x > 1$, $D(x) \leq P(T \leq 0) = \frac{1}{2} < P(T < \frac{1}{2s}) = D(0)$, and so the maxima of $D(x)$ is necessarily within the interval $[0, 1]$. This is a very useful reduction for the rest of the proof.

Let therefore $0 \leq x \leq 1$. Note that T and V_1 both have distributions symmetric around 0. Therefore from only the symmetries,

$$\begin{aligned} D(x) &= P(T < \frac{1 - x^2}{2s\sqrt{1 + x^2 - 2V_1x}}, V_1 > 0) + P(T > -\frac{1 - x^2}{2s\sqrt{1 + x^2 + 2V_1x}}, V_1 > 0) \\ &= \frac{1}{2} + P(-t(x, -V_1) < T < t(x, V_1), V_1 > 0), \end{aligned} \tag{33}$$

where $t(x, v) = \frac{1-x^2}{2s\sqrt{1+x^2-2vx}}$. Now, when $x = 0$, the interval $-t(x, -v) < T < t(x, v)$ is exactly symmetric around zero for all $v > 0$. If we take $0 < x \leq 1$, two things happen: the lower limit $-t(x, -v)$ of this interval moves closer to zero and at the same time, the length $t(x, v) + t(x, -v)$ of this interval decreases. Since $T|V_1$ has a $Beta(\frac{3}{2}, \frac{3}{2})$ distribution which is symmetric and unimodal around 0, it follows that the expression in (33) is maximized when $x = 0$. The two properties of the function $t(x, v)$ stated above are proved by straightforward calculations; it is in this calculus argument that $x \leq 1$ is used. That is why the first reduction of x to $[0, 1]$ was necessary.

This proves Theorem 5. □

4.2.2 Bayes Solutions with Many Competitors

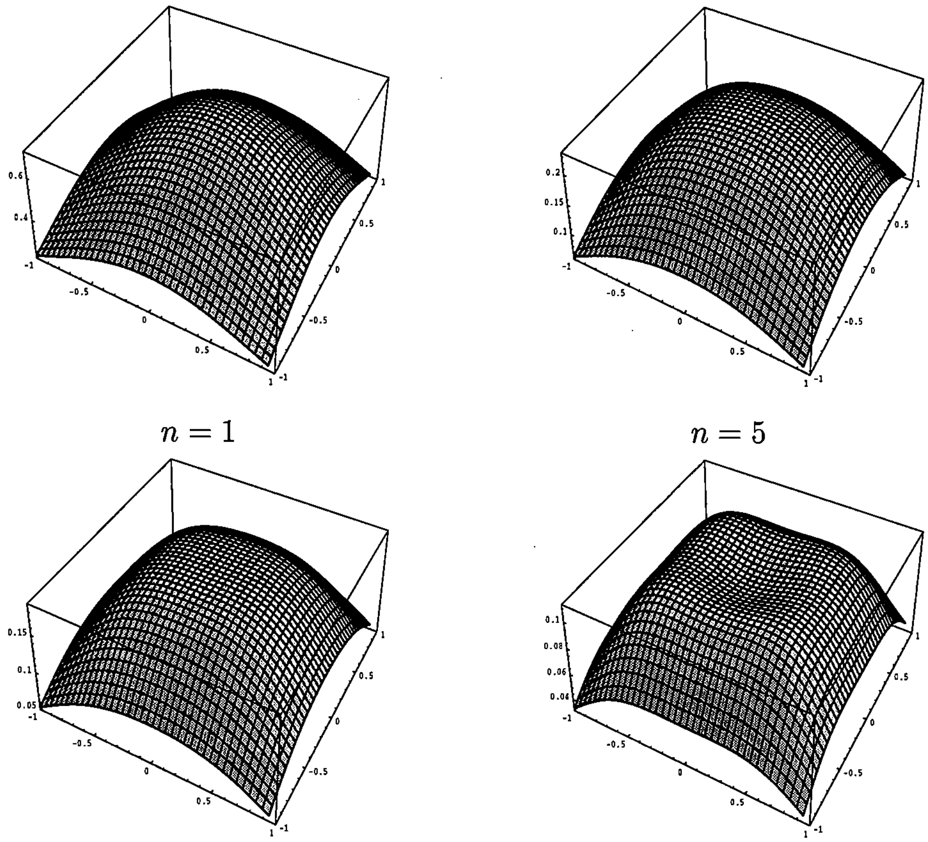
Analogous to Theorem 4, the Bayes solutions for the first vendor in a planar market can also possibly move towards the boundary of the market rather than staying around the interior modes of f when the number of future competitors n is large. We will give some illustrations first and then state a general result.

Example 6 Let $S = [-1,1] \times [-1,1]$. Take both of F and G to be uniform distributions in S. Figure 2 illustrates the shape of the utility functions for some selected values of n. Some Bayes optimals $\underset{\sim}{x}_n$ are also calculated and they are $\underset{\sim}{x}_1 = \ldots = \underset{\sim}{x}_5 = (0,0), \underset{\sim}{x}_6 = (\pm.11, \pm.11), \underset{\sim}{x}_7 = (\pm.27, \pm.27), \underset{\sim}{x}_{10} = (\pm.41, \pm.41), \underset{\sim}{x}_{20} = (\pm.58, \pm.58)$, and $\underset{\sim}{x}_{30} = (\pm.65, \pm.65)$ respectively. From these observations, we believe that when $n \leq 5$, the origin will be the unique Bayes solution, and when $n \geq 6$, there will be four Bayes solutions, one in each quadrant, and they will move towards the four corner points of S as n goes to infinity. If this statement is true, we again get a threshold result for a two dimensional market. And the threshold value will be 6, which is larger than the threshold value for the one dimensional market which was 5. One possible explanation for this larger threshold value is that due to an increase in the number of dimensions, the competitors themselves will spread out, making the central location relatively safer for the first vendor.

The next example is similar to example 6 except that the shape of the market now is round.

Example 7 Let S be the unit sphere in R^2. Suppose again that both of F and G are uniform distributions in S. Then, by some calculations, one has $\underset{\sim}{x}_1 = \ldots = \underset{\sim}{x}_5 = (0,0)$, $\underset{\sim}{x}_6 = .20(\cos\theta, \sin\theta)$, $\underset{\sim}{x}_{10} = .48(\cos\theta, \sin\theta)$, $\underset{\sim}{x}_{20} = .64(\cos\theta, \sin\theta)$, and $\underset{\sim}{x}_{30} = .71(\cos\theta, \sin\theta)$, where θ is arbitrary. We can see that the threshold value is 6, the same as the threshold value for a rectangular market. It seems that the threshold value does not depend on the shape of S very much.

Now, let us look at an example where the customers prefer to stay near the center and yet the first vendor's Bayes solutions move towards the boundary for large n.



$n = 1$ $n = 5$ $n = 6$ $n = 10.$

Figure 2: Plots of $D_n(x)$ when F and G are both uniform distributions on $[-1,1]\times[-1,1]$.

Example 8 Take S to be the unit sphere in R^2. Suppose the position of a customer has the density $f(\underset{\sim}{z}) = \frac{10}{9\pi}(1 - \frac{\|\underset{\sim}{z}\|^2}{5})$ and the future rivals are uniformly located in S, i.e., G is still uniform. Then we have, for example, $\underset{\sim}{x}_1 = \ldots = \underset{\sim}{x}_6 = (0,0), \underset{\sim}{x}_7 = .23(\cos\theta, \sin\theta)$, and $\underset{\sim}{x}_{10} = .41(\cos\theta, \sin\theta)$, where θ is arbitrary. We now see that the threshold value seems to be $n = 7$.

The next general result says that an accumulation point of any sequence of Bayes solutions must be either an interior mode of f or a boundary point of S, as was the case for one dimension.

Theorem 6 *Let S be a bounded region in R^2 such that for each point $\underset{\sim}{x}$ in S, there exists a sequence of interior points of S convergent to $\underset{\sim}{x}$. Suppose that f is continuous and G is the uniform distribution in S. Let $\{\underset{\sim}{x}_n\}_{n=1}^{\infty}$ be any sequence of Bayes solutions* *for the first vendor. Then any accumulation point of* $\{\underset{\sim}{x}_n\}_{n=1}^{\infty}$ *is either an interior mode of* f *or a boundary point of* S.

Proof: The proof is similar to the proof of Theorem 4. Let s denote the area of S. We will first show that $nD_n(\underset{\sim}{x})$ converges to $sf(\underset{\sim}{x})$ uniformly in any compact set C contained in S°, the interior of S. This corresponds to part (a) of Lemma 1. Then we will show that $\lim_{n\to\infty} nD_n(\underset{\sim}{b}_n) = sf(\underset{\sim}{b})$ for any sequence satisfying $\underset{\sim}{b} = \lim_{n\to\infty} \underset{\sim}{b}_n$, if b belongs to S°. This is parallel to part (b) of Lemma 1. Unfortunately, we are unable to give a finer result corresponding to part (c) of Lemma 1.

Step 1: We claim $nD_n(\underset{\sim}{x})$ converges to $sf(\underset{\sim}{x})$ uniformly in any compact set $C \subset S^\circ$. Let $C(\underset{\sim}{x}, r)$ denote the disk centered at $\underset{\sim}{x}$ with radius r. It is enough to verify this claim for $C = C(\underset{\sim}{x}_0, \epsilon_0)$, where $\underset{\sim}{x}_0$ is an interior point of S and ϵ_0 satisfies $C(\underset{\sim}{x}_0, 2\epsilon_0) \subset S^\circ$. Without loss of generality, we may also take $s = 1$.

Since G is the uniform distribution in S, the utility function with n future rivals can be expressed as

$$D_n(\underset{\sim}{x}) = \int_{\underset{\sim}{z}\in S} \{1 - \text{Area}(S \cap C(\underset{\sim}{z}, \|\underset{\sim}{x} - \underset{\sim}{z}\|))\}^n f(\underset{\sim}{z}) \mathrm{d}\underset{\sim}{z}. \tag{34}$$

Let $M = \max_{\underset{\sim}{x}\in C(\underset{\sim}{x}_0, 2\epsilon_0)} f(\underset{\sim}{x})$ and $\delta_n = \max_{\underset{\sim}{x}\in C(\underset{\sim}{x}_0,\epsilon_0)} \max_{\underset{\sim}{z}:\|\underset{\sim}{z}-\underset{\sim}{x}\|\le 1/n^{\frac{1}{4}}} |f(\underset{\sim}{z}) - f(\underset{\sim}{x})|$. By the continuity of f, one has $M < \infty$ and $\lim_{n\to\infty} \delta_n = 0$. Due to our assumption that $C(\underset{\sim}{x}_0, 2\epsilon_0) \subset S^\circ$, for all large n we have $C(\underset{\sim}{x}, 1/n^{\frac{1}{4}}) \subset S^\circ$ for all$\underset{\sim}{x} \in C(\underset{\sim}{x}_0, \epsilon_0)$. Thus,

$$\begin{aligned}
&|nD_n(\underset{\sim}{x}) - f(\underset{\sim}{x})| \\
&\le \int_{\underset{\sim}{z}\in S\setminus C(\underset{\sim}{x},1/n^{\frac{1}{4}})} n\{1 - \text{Area}(S \cap C(\underset{\sim}{z}, \|\underset{\sim}{x} - \underset{\sim}{z}\|))\}^n f(\underset{\sim}{z}) \mathrm{d}\underset{\sim}{z} \\
&\quad + \left| \int_{\underset{\sim}{z}\in C(\underset{\sim}{x},1/n^{\frac{1}{4}})} n(1 - \pi\|\underset{\sim}{z} - \underset{\sim}{x}\|^2)^n f(\underset{\sim}{z}) \mathrm{d}\underset{\sim}{z} - f(\underset{\sim}{x}) \right| \\
&\le n(1 - \frac{\pi}{\sqrt{n}})^n + \int_{0\le\theta\le 2\pi} \int_{0\le r\le 1/n^{\frac{1}{4}}} n(1 - \pi r^2)^n |f((r\cos\theta, r\sin\theta)) - f(\underset{\sim}{x})| r \mathrm{d}r \mathrm{d}\theta \\
&\quad + M \left| \int_{0\le\theta\le 2\pi} \int_{0\le r\le 1/n^{\frac{1}{4}}} n\left(1 - \pi r^2\right)^n r \mathrm{d}r \mathrm{d}\theta - 1 \right| \\
&\le n(1 - \frac{\pi}{\sqrt{n}})^n + \delta_n \frac{n}{n+1}(1 - (1 - \frac{\pi}{\sqrt{n}})^{n+1}) + M\frac{n}{n+1}(1 - \frac{\pi}{\sqrt{n}})^{n+1},
\end{aligned} \tag{35}$$

which does not depend on $\underset{\sim}{x}$ and tends to zero as $n \to \infty$. This verifies the stated claim.

Step 2: Let $\{\underset{\sim}{b}_n\}_{n=1}^{\infty}$ be a sequence convergent to some $\underset{\sim}{b} \in S^\circ$. We claim $\lim_{n\to\infty} nD_n(\underset{\sim}{b}_n) = sf(\underset{\sim}{b})$. Again, without loss of generality, we can assume $s = 1$.

Let $C_n = C(\underset{\sim}{b}, \max_{i \geq n} \|\underset{\sim}{b}_i - \underset{\sim}{b}\|)$. It is obvious that C_n is compact and contained in S° when n is large. By Step 1 and the continuity of f, we have

$$|nD_n(\underset{\sim}{b}_n) - f(\underset{\sim}{b})| \leq \sup_{\underset{\sim}{x} \in C_n} |nD_n(\underset{\sim}{x}) - f(\underset{\sim}{x})| + \sup_{\underset{\sim}{x} \in C_n} |f(\underset{\sim}{x}) - f(\underset{\sim}{b})| \to 0 \text{ as } n \to \infty,$$

and so this proves the stated claim.

Step 3: Assume $\underset{\sim}{x}^* = \lim_{k \to \infty} \underset{\sim}{x}_{n_k}$ for some subsequence $\{\underset{\sim}{x}_{n_k}\}_{k=1}^\infty$ of $\{\underset{\sim}{x}_n\}_{n=1}^\infty$. We claim that $\underset{\sim}{x}^*$ is either an interior mode of f or in the boundary of S.

By the assumptions that $S \subset \bar{S}^\circ$, the closure of S°, and that f is continuous in S, there always exists an interior point of S with its f value arbitrarily close to the maxima of f. Therefore, if $\underset{\sim}{x}^*$ is not an interior mode of f, there exists $\underset{\sim}{x}_0 \in S^\circ$ such that $f(\underset{\sim}{x}_0) > f(\underset{\sim}{x}^*)$. If $\underset{\sim}{x}^*$ is neither on the boundary of S, by Step 2, $\lim_{k \to \infty} n_k D_{n_k}(\underset{\sim}{x}_0) = sf(\underset{\sim}{x}_0) > sf(\underset{\sim}{x}^*) = \lim_{k \to \infty} n_k D_{n_k}(\underset{\sim}{x}_{n_k})$, which is a contradiction to the assumption that $\underset{\sim}{x}_n$'s are Bayes solutions. This completes the proof of Theorem 6. □

5 Summary

We have studied the Hotelling beach model for spatial competition in one and two dimensional markets in the spirit of a statistical decision problem. We ask how the first chooser should act keeping future rivals in mind. We show that some general results are possible under reasonable assumptions, typically symmetry and unimodality.

The number of future rivals is seen to play a very decisive role. In particular, we show that even if customers prefer to stay near a central location, the first chooser may prefer to locate near the boundary if there are a large number of future rivals. For one dimensional markets, we can also analytically approximate the first chooser's optimum location as the number of future rivals tends to infinity. For two dimensional markets, a similar affinity for the boundary continues to hold, and there is some evidence that the shape of the market does not very much affect the threshold value of the number of rivals at which the first chooser should start to move towards the boundary. This is in contrast to classic literature on equilibrium in competition, where equilibrium heavily depends on the shape of a two dimensional market.

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