A PARTIAL INTRODUCTION TO FINANCE

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ABSTRACT. We present an introduction to mathematical Finance Theory for mathematicians. The approach is to start with an abstract setting and then introduce hypotheses as needed to develop the theory. We present the basics of European call and put options, and we show the connection between American put options and backwards stochastic differential equations.

I. Introduction.

We will discuss the modeling of the stock market or more generally "securities", as well as the modeling of interest rates. We will be primarily interested in Financial Derivatives: that is, random variables (representing "contingent claims") that are "derived", or come from, the underlying security price. These claims can be thought of as "portfolio insurance." Indeed, Hans Bühlmann of ETH–Zurich has jokingly characterized actuaries into three types (instead of the usual two):

(1) Life Insurance Actuaries (the first kind);

(2) Property and Casualty Actuaries (the second kind);

(3) Portfolio Insurance Actuaries (Actuaries of the Third Kind).

The three kinds of Actuarial Science use different (albeit overlapping) probabilistic theories. The first kind uses classical probability theory, going back to J. Bernoulli and Ch. Huygens (see [S] for a nice history of classical probability through 1827, the year of the death of Laplace). The second kind involves the Ruin Theory of Cramer–Lundberg and its extensions (this uses martingale theory as well as Large Deviations; see the excellent new book [EKM]). The third kind of insurance — that which interests us here — uses the theory of stochastic integration ("Itô calculus").

II. Introduction to Options and Arbitrage.

Let $X = (X_t)_{0 \leq t \leq T}$ represent the price process of a risky asset (e.g., the price of a stock, a commodity such as "pork bellies," a currency exchange rate, etc.). The present is often thought of as time $t = 0$; one is interested in the price at time T in the future which is unknown, and thus X_T constitutes a "risk". (For example, if an American company contracts at time $t = 0$ to deliver machine parts to Germany at time T, then the unknown price of Euros at time T (in dollars) constitutes a risk for that company.) In order to reduce this risk, one uses " options": one can purchase — at time $t = 0$ — the right to buy Euros at a maximum price at time T. This is one example of an option, called a *call option.*

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A *call option* with strike price K at time T can be represented mathematically as

$$H(\omega) = (X_T(\omega) - K)^+$$

where $x^+ = \max(x, 0)$. Analogously a *put option* with strike price K at time T is

$$H(\omega) = (K - X_T(\omega))^+,$$

and this corresponds to the right to *sell* the security at a minimum price K at time T.

These are two simple examples, often called *European call options* and *European put options.* They are clearly related, and we have

$$X_T - K = (X_T - K)^+ - (K - X_T)^+.$$

We can also use these two simple options as building blocks for more complicated ones. For example if

$$H = \max(K, X_T)$$

then

$$H = X_T + (K - X_T)^+ = K + (X_T - K)^+.$$

More generally if $f: \mathbb{R}_+ \to \mathbb{R}_+$ is convex we can use the well known representation

$$f(x) = f(0) + f'_+(0)x + \int_0^\infty (x - y)^+ \mu(dy) \tag{1}$$

where $f'_+(x)$ is the right continuous version of the derivative of f, and μ is a measure on $\mathbb{R}$ with $\mu = f''$, where the derivative is in the generalized function sense. In this case if

$$H = f(X_T)$$

is our contingent claim, then H is a mixture of European call options, using (1):

$$H = f(0) + f'_+(0)X_T + \int_0^\infty (X_T - K)^+ \mu(dK).$$

Another standard type of option is an *American option.* For the options discussed so far, the contingent claim is a random variable of the form $H = f(X_T)$, that is, a function of the value of X at one fixed and prescribed time T. One can also consider options of the form

$$\begin{aligned} H &= F(X)_T \\ &= F(X_s; 0 \le s \le T) \end{aligned}$$

which are functionals of the paths of X. For example if X has càdlàg paths (càdlàg is a French acronym for "right continuous with left limits") then $F: D \to \mathbb{R}_+$, where D is the space of functions $x: [0, T] \to \mathbb{R}_+$ which are right continuous with left limits. An *American option* is a simple example of such an option. An *American call option* allows the holder to buy the security at a striking price K not only at time T (as is the case for a European call option), but at any time between times $t = 0$ and

time T. Deciding when to exercise such an option is complicated. A strategy for exercising an American option can be represented mathematically by a *stopping rule* τ. (That is, if $(\mathcal{F}_t)_{t\geq 0}$ is the underlying filtration of X then $\{\tau \leq t\} \in \mathcal{F}_t$ for each t, $0 \leq t \leq T$.) For a given τ, the claim is then

$$H(\omega) = (X_{\tau(\omega)}(\omega) - K)^+.$$

We now turn to the *pricing of options.* Let H be a random variable in $\mathcal{F}_T$ representing a contingent claim. Let V_t be its *value* (or price) at time t. What then is V_0?

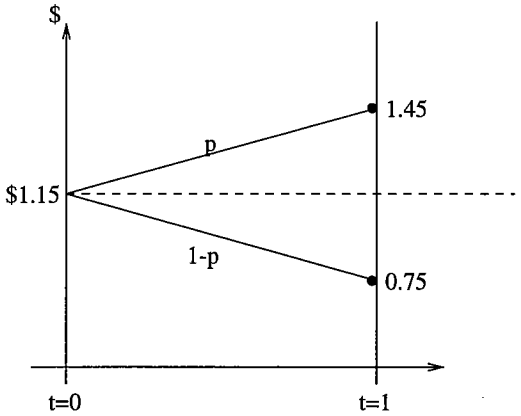
¿From a traditional point of view, classical probability tells us that

$$V_0 = E\{H\}. \tag{2}$$

One could discount for the time value of money (inflation) and assuming a fixed interest rate r one would have

$$V_0 = E\left\{\frac{H}{1+r}\right\} \tag{3}$$

instead of (2). For simplicity we will take $r = 0$ and then show why the obvious price given in (2) does not work (!). For simplicity we consider a binary example. At time $t = 0$, 1 Euro = \$1.15. We assume at time $t = T$ the Euro will be worth either \$0.75 or \$1.45; the probability it goes up to \$1.45 is p and the probability it goes down is $1 - p$.



Let the option have exercise price K =\$1.15, for a European call. That is, $H = (X_T - \$1.15)^+$, where $X = (X_t)_{0\leq t\leq T}$ is the price of one Euro in U.S. dollars. The Huygens–Bernoulli price of H is then

$$E\{H\} = (1.45 - 1.15)p = (0.30)p.$$

For example if $p = 1/2$ we get $V_0 = 0.15$.

The *Black–Scholes* method[1] to calculate the option price however is quite different. We first replace p with a new probability p^* that makes the security price

[1]The "Black-Scholes method" dates back to the fundamental and seminal article [BS] of 1973, where partial differential equations were used; the ideas implicit in that (and subsequent) articles are now referred to as the Black-Scholes methods. M. S. Scholes and R. Merton received the Nobel prize in economics for [BS] and related work (F. Black died prematurely and was not able to share in the prize.)

$X = (X_t)_{t=0,T}$ a martingale. Since this is a two step process, we need only to choose p^* so that X has constant expectation. Since $X_0 = 1.15$, we need

$$E^*\{X_T\} = 1.45p^* + (1 - p^*)0.75 = 1.15.$$

Solving for p^* gives

$$p^* = 4/7.$$

We get now

$$V_0 = E^*\{H\} = (0.30)p^* = \frac{6}{35} \simeq 0.17. \tag{4}$$

The change from p to p^* seems arbitrary. But there is an *economics* argument to justify it; this is where the economics concept of the *absence of arbitrage opportunities* changes the usual intuition dating back to the 16th and 17th centuries.

Suppose for example at time $t = 0$ you sell the option, giving the buyer of the option the right to purchase 1 Euro at time T for \$1.15. He then gives you the price $\pi(H)$ of the option. Again we assume $r = 0$, so there is no cost to borrow money. You can then follow a strategy to prepare for the contingent claim you sold, as follows (calculations are to two decimal places):

Action at time $t = 0$	Result
Sell the option at price $\pi(H)$	$+\pi(H)$
Borrow \$ $\frac{9}{28}$	+\$0.32
Buy $\frac{3}{7}$ Euros at \$1.15	−0.49

The balance at time $t = 0$ is $\pi(H) - 0.17$

At time T there are two possibilities:

(i) The Euro has risen:

Option is exercised	−0.30
Sell $\frac{3}{7}$ Euros at 1.45	+0.62
Pay back loan	−0.32
	0

(ii) The Euro has fallen:

Option is worthless	0
Sell $\frac{3}{7}$ Euros at 0.75	+0.32
Pay back loan	−0.32
	0

Since the balance at time T is zero in both cases, we should have the balance at time 0 also be 0; therefore we must have $\pi(H) = 0.17$. Indeed any price other than $\pi(H) = 0.17$ would allow either the option seller or buyer to make a sure profit *without any risk*: this is called an *arbitrage opportunity* in economics, and it is a standard assumption that such opportunities do not exist. (Of course if they were to exist, market forces would quickly eliminate them.)

Thus we see that — at least in the case of this simple example — that the "no arbitrage price" of the contingent claim H is not $E\{H\}$, but rather must be $E^*\{H\}$, since otherwise there would be an opportunity to make a profit without taking any risk. We emphasize that this is contrary to our standard intuition, since P is the probability measure governing the true laws of chance of the security, while P^* is an artificial construct.

This simple binary example can do more than illustrate the idea of using lack of arbitrage to determine to a price. We can also use it to approximate some continuous models. We let the time interval become small (Δt), and we let the binomial model already described become a recombinant tree, which moves up or down to a neighboring node at each time "tick" Δt. By taking Δt small enough, for a time "tick" of interest of length say δ, we can have the price go to 2^n possible values for a given n, by choosing Δt small enough. Thus for example if a continuous time process follows Geometric Brownian motion:

$$dS_t = \sigma S_t dB_t + \mu S_t dt$$

(as is often assumed in practice), then if the security price process S has value $S_t = s$, then it will move up or down at the next tick Δt to

$$\begin{aligned} &s\exp(\mu\Delta t + \sigma\sqrt{\Delta t}) \quad \text{if up} \\ &s\exp(\mu\Delta t - \sigma\sqrt{\Delta t}) \quad \text{if down} \end{aligned}$$

with p being the probability of going up or down (here take $p = \frac{1}{2}$). Thus for a time t, if $n = \frac{t}{\Delta t}$, we get

$$S_t = S_0 \exp\left(\mu t + \sigma\sqrt{t}\left(\frac{2X_n - n}{\sqrt{n}}\right)\right),$$

where X_n counts the number of jumps up. Next using the absence of arbitrage one changes p from $\frac{1}{2}$ to p^*, getting p^* approximately equal to

$$p^* = \frac{1}{2}\left(1 - \sqrt{\Delta t}\left(\frac{\mu + \frac{1}{2}\sigma^2}{\sigma}\right)\right)$$

III. Basic Definitions.

A. The Price Process.

We let $S = (S_t)_{t\geq 0}$ be a semimartingale* which will be the *price process* of a risky security. A *trading strategy* is a predictable process $H = (H_t)_{t\geq 0}$; its economic interpretation is that at time t one holds an amount H_t of the asset. Often one has in concrete situations that H is continuous or at least càdlàg or càglàd (left continuous with right limits). (Indeed, it is difficult to imagine a practical trading

*One *definition of a semimartingale* is a process S that has a decomposition $S = M + A$, with M a local martingale and A an adapted process with càdlàg paths of finite variation on compacts. See [P] for all information regarding semimartingales.

strategy with pathological path irregularities.) In the case H is adapted and càglàd, then

$$\int_0^t H_s dS_s = \lim_{n\to\infty} \sum_{t_i \in \pi^n[0,t]} H_{t_i} \Delta_i S \tag{1}$$

where $\pi^n[0,t]$ is a sequence of partitions of $[0,t]$ with mesh tending to 0 as $n \to \infty$; $\Delta_i S = S_{t_{i+1}} - S_{t_i}$; and with convergence in u.c.p. (uniform in time on compacts and converging in probability). Thus inspired by (1) we let

$$G_t = \int_{0+}^t H_s dS_s$$

and G is called the *(financial) gain process generated by* H.

B. Interest Rates.

Let r be a fixed rate of interest. If one invests D dollars at rate r for one year, at the end of the year one has $D + rD = D(1+r)$. If interest is paid at n evenly spaced times during the year and compounded, then at the end of the year one has $D\left(1 + \frac{r}{n}\right)^n$. This leads us to the notion of an *interest rate r compounded continuously*:

$$\lim_{n\to\infty} D\left(1 + \frac{r}{n}\right)^n = De^r$$

or, for a fraction t of the year, one has \$ De^{rt} after t units of time for an interest rate r compounded continuously. We define

$$R(t) = De^{rt},$$

then R satisfies the ODE (ODE abbreviates Ordinary Differential Equation)

$$dR(t) = rR(t)dt; \quad R(0) = D. \tag{2}$$

Using the ODE(2) as a basis for interest rates, one can treat a variable interest rate $r(t)$ as follows: ($r(t)$ can be random: that is $r(t) = r(t,\omega)$):

$$dR(t) = r(t)R(t)dt; \quad R(0) = D \tag{3}$$

and solving yields $R(t) = D\exp\left(\int_0^t r(s)ds\right)$. We think of the interest rate process $R(t)$ as the *price of a risk–free bond.*

C. Portfolios.

We will assume as given a risky asset with price process S and a risk–free bond with price process R. Let $(a_t)_{t\geq 0}$ and $(b_t)_{t\geq 0}$ be our *trading strategies* for the security and the bond, respectively.

We call our holdings of S and R our *portfolio.*

Definition. The *value at time* t of a portfolio (a, b) is:

$$V_t(a,b) = a_t S_t + b_t R_t. \tag{4}$$

Now we have our first problem. Later we will want to change probabilities so that $V = (V_t(a,b))_{t\geq 0}$ is a martingale. One usually takes the right continuous versions of martingales, so we will want the right side of (4) to be at least càdlàg. Typically this is not a real problem. Even if the process a has no regularity, one can always choose b in such a way that $V_t(a,b)$ is càdlàg.

Let us next define two sigma algebras on the product space $\mathbb{R}_+ \times \Omega$. We recall we are given an underlying probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. We assume $\mathcal{F}_s \subset \mathcal{F}_t$ if $s < t$; $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$; and also that $\bigcap_{s>t} \mathcal{F}_s \equiv \mathcal{F}_{t+} = \mathcal{F}_t$ by hypothesis. This last property is called the *right continuity of the filtration.* (With these hypotheses, one knows that every martingale has a version which is càdlàg, one of the most important consequences of the hypotheses,)

Definition. Let $\mathbb{L}$ denote the space of left continuous processes whose paths have right limits (càglàd), and which are adapted: that is, $H_t \in \mathcal{F}_t$, for $t \geq 0$. The *predictable* σ*-algebra* $\mathcal{P}$ on $\mathbb{R}_+ \times \Omega$ is

$$\mathcal{P} = \sigma\{H : H \in \mathbb{L}\}.$$

That is $\mathcal{P}$ is the smallest σ-algebra that makes all of $\mathbb{L}$ measurable.

Definition. The *optional* σ*-algebra* $\mathcal{O}$ on $\mathbb{R}_+ \times \Omega$ is

$$\mathcal{O} = \sigma\{H : H \text{ is càdlàg and adapted}\}.$$

In general we have $\mathcal{P} \subset \mathcal{O}$; in the case where $B = (B_t)_{t\geq o}$ is a standard Wiener process (or "Brownian motion"), and $\mathcal{F}_t^0 = \sigma(B_s; s \leq t)$ and $\mathcal{F}_t = \mathcal{F}_t^0 \vee \mathcal{N}$ where $\mathcal{N}$ are the P-null sets of $\mathcal{F}$, then we have $\mathcal{O} = \mathcal{P}$. In general $\mathcal{O}$ and $\mathcal{P}$ are not equal. Indeed if they are equal, then every stopping time is predictable: that is, there are no totally inaccessible stopping times.[2] Since the jump times of (reasonable) Markov processes are totally inaccessible, any model which contains a Markov process with jumps (such as a Poisson Process) will have $\mathcal{P} \subset \mathcal{O}$.

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[2] A *totally inaccessible stopping time* is a stopping time that comes with no advance warning: it is a complete surprise. A stopping time T is *totally inaccessible* if whenever there exists a sequence of non-decreasing stopping times $(S_n)_{n\geq 1}$ with $\Lambda = \bigcap_{n=1}^{\infty} \{S_n < T\}$, then

$$P(\{w : \lim S_n = T\} \cap \Lambda) = 0.$$

A stopping time T is *predictable* if there exists a non-decreasing sequence of stopping times $(S_n)_{n\geq 1}$ as above with

$$P(\{w : \lim S_n = T\} \cap \Lambda) = 1.$$

Note that the probabilities above need not be only 0 or 1; thus there are in general stopping times which are neither predictable nor totally inaccessible.

The predictable σ-algebra $\mathcal{P}$ is important because it is the natural σ-field for which stochastic integrals are defined. In the special case of Brownian motion one can use the optional σ-algebra (since they are the same). There is a third σ-algebra which is often used, known as the progressively measurable sets, and denoted π. One has, in general, that $\mathcal{P} \subset \mathcal{O} \subset \pi$; however in practice one gains very little by assuming a process is π-measurable instead of optional, if - as is the case here – one assumes that the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right continuous (that is $\mathcal{F}_{t+} = \mathcal{F}_t$, all $t \geq 0$). The reason is that the primary use of π is to show that adapted, right continuous processes are π-measurable; but they are already optional if $(\mathcal{F}_t)_{t\geq 0}$ is right continuous. Thus there are essentially no "naturally occuring" examples of progressively measurable processes that are not already optional. An example of such a process, however, is the indicator function $1_G(t)$, where G is described as follows: let $\mathbb{Z} = \{(t,\omega) : B_t(\omega) = 0\}$. ($B$ is standard Brownian motion.) Then $\mathbb{Z}$ is a perfect (and closed) set on $\mathbb{R}_+$ for almost all ω. For fixed ω, the complement is an open set and hence a countable union of open intervals. $G(\omega)$ denotes the left end-points of these open intervals. One can then show (using the Markov property of B and P. A. Meyer's section theorems) that G is progressively measurable but not optional. In this case note that $1_G(t)$ is zero except for countably many t for each ω, hence $\int 1_G(s)dB_s \equiv 0$.

Finally we note that if $a = (a_s)_{s\geq 0}$ is progressively measurable, then $\int_0^t a_s dB_s = \int_0^t \dot{a}_s dB_s$, where $\dot{a}$ is the predictable projection of a.[3]

Let us now recall a few details of stochastic integration. Let S be a (càdlàg) semimartingale, and let H be càdlàg and adapted, or alternatively $H \in \mathbb{L}$. Let $H_- = (H_{s-})_{s\geq 0}$ denote the left continuous version of H. (If $H \in \mathbb{L}$, then of course $H = H_-$.) Let $\pi^n[0,t]$ be a sequence of finite partitions of $[0,t]$ with $\lim_{n\to\infty} \text{mesh}(\pi^n) = 0$. We then have:

Theorem. *H càdlàg, adapted or $H \in \mathbb{L}$. Then*

$$\lim_{n\to\infty} \sum_{t_i \in \pi^n[0,t]} H_{t_i}(S_{t_{i+1}} - S_{t_i}) = \int_0^t H_{s-} dS_s,$$

with convergence uniform in s on $[0,t]$ in probability.

We remark that it is crucial that we sample H at the left endpoint of the interval $[t_i, t_{i+1}]$. Were we to sample at, say, the right endpoint or the midpoint, then the sums would not converge in general (they converge for example if the quadratic

[3]Let H be a bounded, measurable process. (H need not be adapted.) The *predictable projection* of H is the unique predictable process $\dot{H}$ such that

$$\dot{H}_T = E\{H|\mathcal{F}_{T-}\} \quad \text{a.s. on } \{T < \infty\}$$

for all predictable stopping times T. Here $\mathcal{F}_{T-} = \sigma\{A \cap \{t < T\}; A \in \mathcal{F}_t\} \vee \mathcal{F}_0$. For a proof of the existence and uniqueness of $\dot{H}$ see [P, p.119].

covariation process [H,S] exists); in cases where they do converge, the limit is in general different. Thus while the above theorem gives a pleasing "limit as Riemann sums" interpretation to a stochastic integral, it is not at all a perfect analogy.

The basic idea of Theorem 1 can be extended to bounded predictable processes in a method analogous to the definition of the Lebesgue integral for real-valued functions. Note that

$$\sum_{t_i \in \pi^n[0,t]} H_{t_i}(S_{t_{i+1}} - S_{t_i}) = \int_{0+}^{t} H_s^n dS_s,$$

where $H_t^n = \sum H_{t_i} 1_{(t_i . t_{i+1}]}$ which is in $\mathbb{L}$; thus these "simple" processes are the building blocks, and since $\sigma(\mathbb{L}) = \mathcal{P}$, it is unreasonable to expect to go beyond $\mathcal{P}$ when defining the stochastic integral.

There is, of course, a maximal space of integrable processes; without describing it, we define:

Definition. For a semimartingale S we let $L(S)$ denote the space of predictable processes a, where a is integrable with respect to S.

We are now ready for a key definition.

Definition. A strategy (a, b) is called *self-financing* if $a \in L(S)$, b is optional and $b \in L(R)$, and

$$a_t S_t + b_t R_t = a_0 S_0 + b_0 R_0 + \int_0^t a_s dS_s + \int_0^t b_s dR_s \tag{1}$$

for all $t \geq 0$,

Note that the equality (1) above implies that $a_t S_t + b_t R_t$ is càdlàg. We also remark that it is reasonable that a be predictable: a is the trader's holdings at time t, and this is based on information obtained at times strictly before t, but not t itself.

We remark that we are assuming we have only one risky asset, for simplicity.

The next concept is of fundamental importance. An *arbitrage opportunity* is the chance to make a profit *without risk.* One way to model that mathematically is as follows:

Definition. A model is *arbitrage free* if there does not exist a self–financing strategy (a, b) such that $V_0(a, b) = 0$, $V_T(a, b) \geq 0$, and $P(V_T(a, b) > 0) > 0$.

D. Equivalent Martingale Measures.

Let $S = (S_t)_{0 \leq t \leq T}$ be our risky asset price process, which we are assuming is a *semimartingale.* Let

$$S_t = S_0 + M_t + A_t$$

be the semimartingale decomposition of S; M is a local martingale and A is an adapted càdlàg process of finite variation on compacts. We are working on a fixed and given filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$.

Definition. A model is *good* if there exists an equivalent** probability measure Q such that S is a Q–local martingale.

We remark that a price process S can easily not be "good". Indeed, if $Z = \frac{dQ}{dP}$ and $Z_t = E_P\{Z|\mathcal{F}_t\}$, then the Meyer–Girsanov theorem gives the Q decomposition of S by:

$$S_t = (M_t - \int_0^t \frac{1}{Z_s}\, d[Z,M]_s) + (A_t + \int_0^t \frac{1}{Z_s}\, d[Z,M]_s).$$

In order for S to be a Q–local martingale we need* to have $A_t = -\int_0^t \frac{1}{Z_s} d[Z,M]_s$. The Kunita–Watanabe inequality implies that $d[Z,M] << d[M,M]$, hence a *necessary* condition for a model to be good is that

$$dA_t << d[M,M]_t \quad \text{a.s.}$$

Note that this implies in particular in the Brownian case that if $M_t = \int_0^t \theta_s dB_s$, then A must of necessity be of the form $A_t = \int_0^t \gamma_s \theta_s^2 ds$ for some process γ.

E. The Fundamental Theorem of Asset Pricing.

In Section II we saw that with the "No Arbitrage" assumption, at least in the case of a very simple example, we needed to change from the "true" underlying probability measure P, to an equivalent one P^*, and the price of a contingent claim H was not $E\{H\}$ as one might expect, but rather $E^*\{H\}$. The idea that led to this price was to find a probability P^* that gave the price process X a constant expectation.

In continuous time a sufficient condition for the price process $S = (S_t)_{t\geq 0}$ to have constant expectation is that it be a martingale. That is, if S is a martingale then the function $t \to E\{S_t\}$ is constant. Actually this property is not far from characterizing martingales. A classic theorem from martingale theory is the following (cf, eg, [P]):

Theorem. *Let $S = (S_t)_{t\geq 0}$ be càdlàg and suppose $E\{S_\tau\} = E\{S_0\}$ for any bounded stopping time τ (and of course $E\{|S_\tau|\} < \infty$). Then S is a martingale.*

That is, if we require constant expectation at stopping times (instead of only at fixed times), then S is a martingale. Thus the general idea is the following.

**Q is equivalent to P if Q and P have the same sets of probability zero.

*At least in the case of continuous paths

Folk Theorem. *Let S be a price process on a given space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. Then there is an absence of arbitrage opportunities if and only if there exists a probability P^*, equivalent to P, such that S is a martingale under P^*.*

Before stating a more rigorous theorem (our version is due to Delbaen and Schachermeyer [DS]), let us examine a needed hypothesis. We need to avoid problems that arise from the classical doubling strategy. Here a player bets \$1 at a fair bet. If he wins, he stops. If he loses he next bets \$2. Whenever he wins, he stops, and his profit is \$1. If he continues to lose, he continues to play, each time doubling his bet. This strategy leads to a certain gain of \$1 without risk. However the player needs to be able to tolerate arbitrarily large losses before he might gain his certain profit. Of course no one has such infinite resources to play such a game. Mathematically one can eliminate this type of problem by requiring trading strategies to give martingales that are bounded below by a constant. This leads to the next definition.

Definition. Let $\alpha > 0$, and let S be a semimartingale. A predictable trading strategy θ is *α-admissible* if $\theta_0 = 0$, $\int_0^t \theta_s dS_s \geq -\alpha$, all $t \geq 0$. θ is called *admissible* if there exists $\alpha > 0$ such that θ is α-admissible.

Before we make more definitions, let us recall the basic idea. Suppose θ is admissible, self-financing, with $\theta_0 S_0 = 0$ and $\theta_T S_T \geq 0$. Assume we can neglect the bond or "numeraire" process, so that self-financing reduces to

$$\theta_T S_T = \theta_0 S_0 + \int_0^T \theta_s dS_s.$$

Then if P^* exists such that $\int \theta_s dS_s$ is a martingale, we have

$$E^*\{\theta_T S_T\} = 0 + E^*\{\int_0^T \theta_s dS_s\}.$$

In general $\int_0^t \theta_s dS_s$ is only a *local* martingale; if we know that it is a true martingale then $E^*\{\int_0^T \theta_s dS_s\} = 0$, whence $E^*\{\theta_T S_T\} = 0$, and since $\theta_T S_T \geq 0$ we deduce $\theta_T S_T = 0$, P^* a.s., and since P^* is equivalent to P, we have $\theta_T S_T = 0$ a. s. (dP) as well. This implies no arbitrage exists. The technical part of this argument is to show $\int_0^t \theta_s dS_s$ is a P^* true martingale, and not just a local martingale. The converse is typically harder: that is, that no arbitrage implies P^* exists. The converse is proved using a version of the Hahn-Banach theorem.

Following Delbaen and Schachermeyer, we make a sequence of definitions:

$$K_0 = \left\{ \int_0^\infty \theta_s dS_s | \quad \theta \text{ is admissible and } \lim_{t\to\infty} \int_0^t \theta_s dS_s \text{ exists a.s.} \right\}$$

$$C_0 = \{\text{all functions dominated by elements of } K_0\}$$

$$= K_0 - L^0_+, \text{ where } L^0_+ \text{ are positive, finite random variables.}$$

$$K = K_0 \cap L^\infty$$

$$C = C_0 \cap L^\infty$$

$$\overline{C} = \text{the closure of } C \text{ under } L^\infty.$$

Definition. A semimartingale price process S satisfies

(i) the *No Arbitrage* condition if $C \cap L^\infty_+ = \{0\}$ (this corresponds to no chance of making a profit with risk);

(ii) the *No Free Lunch with Vanishing Risk* condition ($NFLVR$) if $\overline{C} \cap L^\infty_+ = \{0\}$.

Clearly condition (ii) implies condition (i). Condition (i) is slightly too restrictive to imply the existence of an equivalent martingale measure P^*. (One can construct a trading strategy of $H_t(\omega) = 1_{\{[0,1]\setminus\mathbb{Q}\times\Omega\}}(t,\omega)$, which means one sells before each rational time and buys back immediately after it; combining H with a specially constructed càdlàg semimartingale shows that (i) does not imply the existence of P^* - see [DS, p.511].)

Let us examine then condition (ii). If NFLVR is not satisfied then there exists an $f_0 \in L^\infty_+$, $f_0 \not\equiv 0$, and also a sequence $f_n \in C$ such that $\lim_{n\to\infty} f_n = f_0$ a. s, such that for each n, $f_n \geq f_0 - \frac{1}{n}$. In particular $f_n \geq -\frac{1}{n}$. This is almost the same as an absence of arbitrage, as the risk of the trading strategies becomes arbitrary small.

Fundamental Theorem. *Let S be a bounded semimartingale. There exists an equivalent martingale measure P^* for S if and only if S satisfies NFLVR.*

Proof. Let us assume we have NFLVR. Since S satisfies the no arbitrage property we have $C \cap L^\infty_+ = \{0\}$. However one can use the property NFLVR to show C is weak closed in L^∞ (that is, it is closed in $\sigma(L^1, L^\infty)$), and hence there will exist a probability P^* equivalent to P with $E^*\{f\} \leq 0$, all f in C. (This is the Kreps-Yan separation theorem - essentially the Hahn-Banach theorem). For each $s < t$, $B \in \mathcal{F}_s$, $\alpha \in \mathbb{R}$, we deduce $\alpha(S_t - S_s)1_B \in C$, since S is bounded. Therefore $E^*\{(S_t - S_s)1_B\} = 0$, and S is a martingale under P^*.

For the converse, note that NFLVR remains unchanged with an equivalent probability, so without loss of generality we may assume S is a Martingale under P itself. If θ is admissible, then $\left(\int_0^t \theta_s dS_s\right)_{t\geq 0}$ is a local martingale, hence it is a supermartingale. Since $E\{\theta_0 S_0\} = 0$, we have as well $E\{\int_0^\infty \theta_s dS_s\} \leq E\{\theta_s S_0\} = 0$.

This implies that for any $f \in C$, we have $E\{f\} \leq 0$. Therefore it is true as well for $f \in \overline{C}$, the closure of C in L^∞. Thus we conclude $\overline{C} \cap L^\infty_+ = \{0\}$. □

Corollary. *Let S be a locally bounded semimartingale. There is an equivalent local martingale measure P^* for S if and only if S satisfies NFLVR.*

We refer to [DS, p.479] for the proof of the corollary. Examples show that in general P^* can make S only a local martingale, not a martingale. We also note that any semimartingale with continuous paths is locally bounded.

F. Normalizing the Bond Price.

Our Portfolio as described in III.C consists of

$$V_t(a,b) = a_t S_t + b_t R_t$$

where (a,b) are trading strategies, S is the risky security price, and $R_t = D\ \exp(\int_0^t r_s ds)$ is the price of a risk-free bond. The process R is often called a *numeraire*. One often takes $D = 1$ and then R_t represents the time value of money. One can then deflate future monetary values by multiplying by $\frac{1}{R_t} = \exp\left(-\int_0^t r_s ds\right)$. Let us write $Y_t = \frac{1}{R_t}$ and we shall refer to the process Y_t as a *deflator*. By multiplying S and R by $Y = \frac{1}{R}$, we can effectively reduce the situation to the case where the price of a risk free bond is constant and equal to one. The next theorem allows us to do that.

Theorem (Numeraire Invariance). *Let (a,b) be a strategy for (S,R). Let $Y = \frac{1}{R}$. Then (a,b) is self-financing for (S,R) if and only if (a,b) is self-financing for $(YS,1)$.*

Proof. Let $Z = \int_0^t a_s dS_s + \int_0^t b_s dR_s$. Then using integration by parts we have (since Y is continuous and of finite variation)

$$\begin{aligned} d(Y_t Z_t) &= Y_t dZ_t + Z_t dY_t \\ &= Y_t a_t dS_t + Y_t b_t dR_t + (\int_0^t a_s dS_s + \int_0^t b_s dR_s) dY_t \\ &= a_t(Y_t dS_t + S_t dY_t) + b_t(Y_t dR_t + R_t dY_t) \\ &= a_t d(YS)_t + b_t d(YR)_t \end{aligned}$$

and since $YR = \frac{1}{R} R = 1$, this is

$$= a_t d(YS)_t$$

since $dYR = 0$ because YR is constant. Therefore

$$a_t S_t + b_t R_t = a_0 S_0 + b_0 + \int_0^t a_s dS_s + \int_0^t b_s dR_s$$

if and only if

$$a_t \frac{1}{R_t} S_t + b_t = a_0 S_0 + b_0 + \int_0^t a_s d(\frac{1}{R} S)_s.$$

The Numeraire Invariance Theorem allows us to assume $R \equiv 1$ without loss of generality. Note that one can check as well that there is no arbitrage for (a, b) with (S, R) if and only if there is no arbitrage for (a, b) with $\left(\frac{1}{R} S, 1\right)$. By renormalizing, we no longer write $\left(\frac{1}{R} S, 1\right)$, but simply S.

G. Redundant Claims.

Let us assume given a security price process S, and as we have seen in Section F we take $R_t \equiv 1$ without loss of generality. Let $\mathcal{F}_t^0 = \sigma(S_r; r \leq t)$ and let $\mathcal{F}_t^{\sim} = \mathcal{F}_t^0 \vee \mathcal{N}$ where $\mathcal{N}$ are the null sets of $\mathcal{F}$ and $\mathcal{F} = \bigvee_t \mathcal{F}_t^0$, under P, defined on $(\Omega, \mathcal{F}, P)$. Finally we take $\mathcal{F}_t = \bigcap_{u>t} \mathcal{F}_u^{\sim}$. A *contingent claim* on S is then a random variable $H \in \mathcal{F}_T$, for some fixed time T. One goal of Finance Theory is to show there exists a trading strategy (a, b) that one can use either to obtain H at time T, or to come as close as possible - in an appropriate sense - to obtaining H.

Definition. Let S be the price process of a risky security and let R be the price process of a risk free bond (numeraire). A contingent claim $H \in \mathcal{F}_T$ is said to be *redundant* if there exists a strategy (a, b) such that

$$H = a_0 S_0 + b_0 R_0 + \int_0^T a_s dS_s + \int_0^T b_s dR_s.$$

Let us normalize S by writing $M = \frac{1}{R} S$, then H will still be redundant under M and hence we have (taking $R_0 = 1$):

$$H = a_0 M_0 + b_0 + \int_0^T a_s dM_s.$$

Next note that if P^* is any equivalent martingale measure making M a martingale, we then have

$$E^*\{H\} = E^*\{a_0 M_0 + b_0\} + E^*\{\int_0^T a_s dM_s\}$$

provided all expectations exist,

$$= E^*\{a_0 M_0 + b_0\} + 0.$$

Theorem. *Let H be a redundant contingent claim such that there exists an equivalent martingale measure P^* with $H \in \mathcal{L}^*(M)$. (See the second definition following for a definition of $\mathcal{L}^*(M)$). Then the no arbitrage price of H must be $E^*\{H\}$.*

Proof. First we note that the quantity $E^*\{H\}$ *is the same* for every equivalent martingale measure. Indeed if Q_1 and Q_2 are both equivalent martingale measures, then

$$E_{Q_i}\{H\} = E_{Q_i}\{a_0 M_0 + b_0\} + E_{Q_i}\{\int_0^T a_s dM_s\}.$$

But $E_{Q_i}\{\int_0^T a_s dM_s\} = 0$, and $E_{Q_i}\{a_0M_0 + b_0\} = a_0M_0 + b_0$, since we assume a_0, M_0, and b_0 are known at time 0 and thus without loss of generality are taken to be constants.

Next suppose one offers a price $\pi > E^*\{H\} = a_0M_0 + b_0$. Then one follows the strategy $a = (a_s)_{s\geq 0}$ and (we are ignoring transaction costs) at time T one has H to present to the purchaser of the option. One thus has a sure profit (that is, risk free) of $\pi - (a_0M_0 + b_0) > 0$. This is an arbitrage opportunity. On the other hand if one can buy the claim H at a price $\pi < a_0M_0 + b_0$, analogously at time T one will have achieved a risk-free profit of $(a_0M_0 + b_0) - \pi$. □

Definition. If H is a redundant claim, then there exists (a, b) such that

$$H = a_0M_0 + b_0 + \int_0^T a_s dM_s;$$

the strategy a is said to *replicate* the claim H.

Corollary. *If H is a redundant claim, then one can replicate H in a self-financing manner with initial capital equal to $E^*\{H\}$, where P^* is any equivalent martingale measure for the normalized price process M.*

It is tempting to consider markets where all contingent claims are redundant. Unfortunately this is too large a space of random variables; we wish to restrict ourselves to claims that have good integrability properties.

Let us fix an equivalent martingale P^*, so that M is a martingale (or even a local martingale) under P^*.

We need to recall some details of stochastic integration. If X is a semimartingale, let $\pi^n[0,t]$ denote a sequence of partitions of $[0,t]$ with $\lim_{n\to\infty} \text{mesh}(\pi^n) = 0$. We then have

$$(*) \qquad \lim_{n\to\infty} \sum_{t_i\in\pi^n[0,t]} (X_{t_{i+1}} - X_{t_i})^2 = [X,X]_t$$

with convergence in u. c. p. (uniform in s on compact time intervals, and in probability). The process $t \to [X,X]_t(w)$ is càdlàg, adapted, and nondecreasing. If X, Y are two semimartingales, then

$$[X,Y]_t = \frac{1}{2}\{[X+Y, X+Y]_t - [X,X]_t - [Y,Y]_t\}$$

which is a type of polarization identity. Moreover

$$\lim_{n\to\infty} \sum_{t_i\in\pi^n[0,t]} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i}) = [X,Y]_t$$

where convergence is again in u. c. p. The process $[X,Y]$ is càdlàg, adapted, and has paths of finite variation on compacts. Since X is in general only càdlàg, $[X,X]$ will

also be only càdlàg, but since it is non-decreasing it has a path by path Lebesgue decomposition into a continuous part and a jump part:

$$(**) \qquad [X,X]_t = [X,X]^c_t + \sum_{s\le t} \Delta[X,X]_s$$

where $\Delta[X,X]_s = [X,X]_s - [X,X]_{s-}$, the jump at time s. We observe that it follows from (*) that

$$\Delta[X,X]_s = (\Delta X_s)^2,$$

hence (**) can be written

$$[X,X]^c_t = [X,X]_t - \sum_{0\le s\le t} (\Delta X_s)^2.$$

The process $[X,X]^c$ arises in Itô's formula, which for a càdlàg semimartingale X has the form:

Theorem. *Let X be a semimartingale and let f be C^2. Then*

$$f(X_t) = f(X_0) + \int_0^t f'(X_{s-})dX_s + \frac{1}{2}\int_0^t f''(X_{s-})d[X,X]^c_s + \sum_{0<s\le t} \{f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s\}.$$

We now return to Finance. Recall P^* is an equivalent local martingale measure for M. We consider all self-financing strategies (a,b) such that the process $\left(\int_0^t a_s^2 d[M,M]_s\right)^{1/2}$ is locally integrable: that means that there exists a sequence of stopping times $(T_n)_{n\ge 1}$ which can be taken $T_n \le T_{n+1}$, a. s. , such that $\lim_{n\to\infty} T_n \ge T$ a. s. and

$E^*\left\{(\int_0^{T_n} a_s^2 d[M,M]_s)^{1/2}\right\} < \infty$, each T_n. Let $\mathcal{L}^*(M)$ denote the class of such strategies, under P^*.

Definition. A market model $(M, \mathcal{L}^*(M), P^*)$ is *complete* if every claim $H \in L^1(\mathcal{F}_T, dP^*)$ is redundant for $\mathcal{L}^*(M)$. That is for any $H \in L^1(\mathcal{F}_T, dP^*)$, there exists a strategy (a,b) with $a \in \mathcal{L}^*(M)$ such that

$$H = a_0 M_0 + b_0 + \int_0^T a_s dM_s.$$

In essence, then, a complete market is one for which every claim is redundant.

We note that in Probability Theory a martingale M is said to have the *predictable representation property* if for any $H \in L^2(\mathcal{F}_T)$ one has

$$H = E\{H\} + \int_0^T a_s dM_s$$

for some predictable $a \in \mathcal{L}(M)$. This is of course *exactly* the property of market completeness. Martingales with predictable representation are well studied and this theory can usefully be applied to Finance. Unfortunately having the predictable representation property is rather delicate, and few martingales process this property. Examples include Brownian motion, the Compensated Poisson process (but not mixtures of the two nor even the difference of two Poisson processes), and the Azéma martingales. (One can consult [DP] for more on the Azéma martingales.)

Most models are therefore *not* complete, and most practitioners believe the actual financial world being modeled is not complete. We have the following result:

Theorem. *There is a unique P^* such that M is a local martingale only if the market is complete.*

This theorem is a trivial consequence of Dellacherie's approach to Martingale Representation: if there is a unique probability making a process M a local martingale, then M must have the martingale representation property. The theory has been completely resolved in the work of Jacod and Yor. To give an example of what can happen, let $\mathcal{M}^2$ be the set of equivalent probabilities making M an L^2-martingale. Then M has the predictable representation property (and hence market completeness) for every extremal element of $\mathcal{M}^2$. If $\mathcal{M}^2 = \{P^*\}$, only one element, then of course P^* is extremal. (See [P, p. 152]). Indeed P^* is in fact unique in the proto-typical example of Brownian motion; since many diffusions can be constructed as pathwise functionals of Brownian motion they inherit the completeness of the Brownian model.

Note that if H is a redundant claim, then the no arbitrage price of H is $E^*\{H\}$, for any equivalent martingale measure P^*. (If H is redundant then we have seen the quantity $E^*\{H\}$ is the same under every P^*.) However, if the market is not complete, then

(i) there will arise non-redundant claims

(ii) there will be more than one equivalent martingale measure P^*.

We now have the conundrum: if H is non-redundant, what is the no arbitrage price of H? We can no longer argue that it is $E^*\{H\}$, because there are many such values!

The absence of this conundrum is a large part of the appeal of complete markets.

Finally let us note that when H is redundant there is always a replication strategy a. However, when H is non-redundant it cannot be replicated; in this event we do the best we can in some appropriate sense (for example expected squared error loss), and we call the strategy we follow a *hedging strategy.* See for example [FS] and [JMP] for results about hedging strategies.

H. Finding a Replication Strategy.

It is rare that we can actually "explicitly" compute a replication strategy, and rarer still that we can explicitly compute a hedging strategy. However, there are simple cases where miracles happen; and when there are no miracles, then we can often approximate hedging strategies accurately using numerical techniques.

A standard, and relatively simple, type of contingent claim is one which has the form

$$H = f(S_T)$$

where S is the price of the risky security. The two most important examples (already discussed in Section II) are

(i) *The European call option*: Here $f(x) = (x - K)^+$ for a constant K, so the contingent claim is $H = (S_T - K)^+$. K is referred to as the *strike price* and T is the expiration time. In words, the European call option gives the holder the right to *buy* one unit of the security at the price K at time T. Thus the (random) value of the option at time T is $(S_T - K)^+$.

(ii) *The European put option*: Here $f(x) = (K - x)^+$. This option gives the holder the right to *sell* one unit of the security at time T at price K. Hence the (random) value of the option at time T is $(K - S_T)^+$.

The European call and put options are clearly related. Indeed we have

$$(S_T - K)^+ - (K - S_T)^+ = S_T - K.$$

An important difference between the two is that $(K - S_T)^+$ is a bounded random variable with values in $[0, K]$, while $(S_T - K)^+$ is in general an unbounded random variable.

To illustrate the ideas involved, let us take $R_t \equiv 1$ by a change of the numeraire, and let us suppose that $H = f(S_T)$ is a redundant claim. The *value* of the claim is, we recall,

$$V_t = E^*\{f(S_T)|\mathcal{F}_t\} = a_0 S_0 + b_0 + \int_0^t a_s dS_s.$$

We now make a series of hypotheses in order to obtain an easier analysis:

Hypothesis 1. S is a Markov process under P^*.

Under hypothesis 1 we have:

$$V_t = E^*\{f(S_T)|\mathcal{F}_t\} = E^*\{f(S_T)|S_t\}.$$

But measure theory tells us that there exists a function $\varphi(t, \cdot)$, for each t, such that

$$E^*\{f(S_T)|S_t\} = \varphi(t, S_t).$$

Hypothesis 2. $\varphi(t, x)$ is $\mathcal{C}^1$ in t and $\mathcal{C}^2$ in x.

We now use Itô's formula:

$$\begin{aligned} V_t &= E^*\{f(S_T)|\mathcal{F}_t\} = \varphi(t, S_t) \\ &= \varphi(0, S_0) + \int_0^t \varphi'_x(s, S_{s-}) dS_s \\ &\quad + \int_0^t \varphi'_s(s, S_{s-}) ds + \frac{1}{2} \int_0^t \varphi''_{xx}(s, S_{s-}) d[S, S]^c_s \\ &\quad + \sum_{0<s\leq t} \{\varphi(s, S_s) + \varphi(s, S_{s-}) - \varphi'_x(s, S_{s-}) \Delta S_s\}. \end{aligned}$$

Hypothesis 3. S has continuous paths. With hypothesis 3 Itô's formula simplifies:

$$V_t = \varphi(t, S_t) = \varphi(0, S_0) + \int_0^t \varphi'_x(s, S_s)dS_s + \int_0^t \varphi'_s(s, S_s)ds + \frac{1}{2}\int_0^t \varphi''_{xx}(s, S_s)d[S, S]_s. \tag{1}$$

Since V is a P^* martingale, the right side of (1) must also be a P^* martingale. This is true if

$$\int_0^t \varphi'_s(s, S_s)ds + \frac{1}{2}\int_0^t \varphi''_{xx}(s, S_s)d[S, S]_s = 0. \tag{2}$$

There is not much chance of this happening without the next hypothesis:

Hypothesis 4. $[S, S]_t = \int_0^t h(s, S_s)^2 ds$.

We then get that (2) certainly holds if φ is the solution of the partial differential equation:

$$\frac{1}{2}h(s, x)^2 \frac{\partial^2 \varphi}{\partial x^2}(s, x) + \frac{\partial \varphi}{\partial s}(s, x) = 0$$

with boundary condition $\varphi(T, x) = f(x)$. Note that if we combine Hypotheses 1–4, we have a continuous Markov process with quadratic variation $\int_0^t h(s, S_s)^2 ds$. An obvious candidate for such a process is the solution of a stochastic differential equation

$$dS_t = h(s, S_s)dB_t + b(s; S_r; r \leq s)ds,$$

where B is a standard Wiener process (Brownian motion) under P. S is a continuous Markov process under P^*, with quadratic variation $[S, S]_t = \int_0^t h(s, S_s)^2 ds$ as desired. The quadratic variation is a path property and is unchanged by changing to an equivalent probability measure P^* (see [P] for example). But what about the Markov property? Why is S a Markov process under P^* when b can be path dependent?

Here we digress a bit. Let us analyze in more detail P^*. Since P^* is equivalent to P, we can let $Z = \frac{dP^*}{dP}$ and $Z > 0$ a.s. (dP). Let $Z_t = E\{Z|\mathcal{F}_t\}$, which is clearly a martingale. By Girsanov's theorem (see, eg, [P]) we have that

$$\int_0^t h(s, S_s)dB_s - \int_0^t \frac{1}{Z_s} d[Z, \int_0^{\cdot} h(r, S_r)dB_r]_s \tag{3}$$

is a P^* martingale.

Let us suppose that $Z_t = \int_0^t H_s Z_s dB_s$, which is reasonable since we have martingale representation for B and Z is a martingale. We then have that (3) becomes

$$\int_0^t h(s, S_s)dB_s - \int_0^t \frac{1}{Z_s} Z_s H_s h(s, S_s)ds = \int_0^t h(s, S_s)dB_s - \int_0^t H_s h(s, S_s)ds.$$

If we choose $H_s = \dfrac{b(s; S_r; r \leq s)}{h(s, S_s)}$, then we have

$$S_t = \int_0^t h(s, S_s) dB_s + \int_0^t b(s; S_r; r \leq s) ds$$

is a martingale under P^*; moreover we have $M_t = B_t + \displaystyle\int_0^t \frac{b(s; S_r; r \leq s)}{h(s, S_s)} ds$ is a P^* martingale; since $[M, M]_t = [B, B]_t = t$, by Lévy's theorem it is a P^*-Brownian motion (see, e. g. , [P]), and we have

$$dS_t = h(t, S_t) dM_t$$

and thus S *is a Markov process under* P^* . The last step in this digression is to show it is possible to construct such a P^*! Recall that the *stochastic exponential* of a semimartingale X is the solution of the "exponential equation"

$$dY_t = Y_t dX_t; \quad Y_0 = 1.$$

The solution is known in closed form and is given by

$$Y_t = \exp(X_t - \frac{1}{2}[X, X]_t^c) \prod_{s \leq t} (1 + \Delta X_s) e^{-\Delta X_s}.$$

If X is continuous then

$$Y_t = \exp(X_t - \frac{1}{2}[X, X]_t),$$

and it is denoted $Y_t = \mathcal{E}(X)_t$. Recall we wanted $dZ_t = H_t Z_t dB_t$; we let $N_t = \int_0^t H_s dB_s$, and we have $Z_t = \mathcal{E}(N)_t$. Then we set $H_t = \dfrac{-b(t; S_r, r \leq t)}{h(t, S_t)}$ as planned and let $dP^* = Z_T dP$, and we have achieved our goal. Since $Z_T > 0$ a.s. (dP), we have that P and P^* are equivalent.

Let us now summarize the foregoing. We assume we have a price process given by

$$dS_t = h(t, S_t) dB_t + b(t; S_r, r \leq t) dt.$$

We form P^* by $dP^* = Z dP$, where $Z_T = \mathcal{E}(N)_T$ and $N_t = \displaystyle\int_0^t \frac{-b(s; S_r, r \leq s)}{h(s, S_s)} dB_s$. We let φ be the (unique) solution of the boundary value problem.

$$\frac{1}{2} h(t, x)^2 \frac{\partial^2 \varphi}{\partial x^2}(t, x) + \frac{\partial}{\partial s} \varphi(t, x) = 0 \tag{4}$$

and $\varphi(T, x) = f(x)$, where φ is $\mathcal{C}^2$ in x and $\mathcal{C}^1$ in t. Then

$$V_t = \varphi(t, S_t) = \varphi(0, S_0) + \int_0^t \frac{\partial \varphi}{\partial x}(s, S_s) dS_s.$$

Thus, under these four rather restrictive hypotheses, we have found our replication strategy! It is $a_s = \dfrac{\partial \varphi}{\partial x}(s, S_s)$. We have also of course found our value process

$V_t = \varphi(t, S_t)$, provided we can solve the partial differential equation (4). However even if we cannot solve it in closed form, we can always approximate φ numerically.

Conclusion: It is a convenient hypothesis to assume that the price process S of our risky asset follows a stochastic differential equation driven by Brownian motion.

Important Comment: Although our price process is assumed to follow the SDE

$$dS_t = h(t, S_t)dB_t + b(t; S_r, r \leq t)dt,$$

we see that the PDE (4) does not involve the "drift" coefficient b at all! Thus the price and the replication strategy do not involve b either. The economic explanation of this is two fold: first, the drift term b is already reflected in the market price: it is based on the "fundamentals" of the security; second, what is important is the degree of risk involved, and this is reflected in the term h.

Remark. Hypothesis (2) is not a benign hypothesis. Since φ turns out to be the solution of a partial differential equation (given in (4)), we are asking for regularity of the solution. This is typically true when f is smooth (which of course the canonical example $f(x) = (K - x)^+$ is not!). Nevertheless this analysis essentially works for the cases of European calls and puts as we describe in Section I that follows.

I. A special Case.

In Section H we saw how it is convenient to assume S verifies a stochastic differential equation. Let us now assume S follows a linear SDE (= Stochastic Differential Equation) with constant coefficients:

$$dS_t = \sigma S_t dB_t + \mu S_t dt; \quad S_0 = 1. \tag{1}$$

Let $X_t = \sigma B_t + \mu t$ and we have

$$dS_t = S_t dX_t; \quad S_0 = 1$$

so that

$$S_t = \mathcal{E}(X)_t = e^{\sigma B_t + (\mu - \frac{1}{2}\sigma^2)t}.$$

The process S of (1) is known as *geometric Brownian motion* and has been used to study stock prices since at least the 1950's and the work of P. Samuelson. In this simple case the solution of the PDE (4) of Section H *can be found explicitly*, and it is given by

$$\varphi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(xe^{\sigma u\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)}) e^{-\frac{u^2}{2}} du. \tag{2}$$

In the case of a European *call option* we have $f(x) = (x - K)^+$ and in this case we get

$$\varphi(x,t) = x\Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(\log\frac{x}{K} + \frac{1}{2}\sigma^2(T-t)\right)\right) - K\Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(\log\frac{x}{K} - \frac{1}{2}\sigma^2(T-t)\right)\right).$$

Here $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{u^2}{2}} du$. In the case of the call option we can also compute the replication strategy:

$$a_t = \Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(\log\frac{S_t}{K} + \frac{1}{2}\sigma^2(T-t)\right)\right). \tag{3}$$

Third we can compute as well the *price* of the European call option (here we assume $S_0 = s$):

$$\begin{aligned} V_0 = \varphi(x,0) = x\Phi\left(\frac{1}{\sigma\sqrt{T}}\left(\log\frac{x}{K} + \frac{1}{2}\sigma^2 T\right)\right) \\ - K\Phi\left(\frac{1}{\sigma\sqrt{T}}\left(\log\frac{x}{K} - \frac{1}{2}\sigma^2 T\right)\right). \end{aligned} \tag{4}$$

These formulas, (3) and (4) are the celebrated *Black-Scholes option formulas*, with $R_t \equiv 1$.

These relatively simple, explicit, and easily computable formulas make working with European call and put options very simple. It is perhaps because of the beautiful simplicity of this model that security prices are often assumed to follow geometric Brownian motions even when there is significant evidence that such a structure poorly models the real markets. Finally note that - as we observed earlier - the drift coefficient μ does not enter into the Black-Scholes formulas.

J. Other options in the Brownian paradigm: a general view.

In Sections H and I we studied contingent claims of the form $H = f(S_T)$, that depend only on the final value of the price process. There we showed that the computation of the price and also the hedging strategy can be obtained by solving a partial differential equation, provided the price process S is assumed to be Markov under P^*.

Other contingent claims can depend on the values of S between 0 and T. A *look-back option* depends on the entire path of S from 0 to T. To give an illustration of how to treat this phenomenon (in terms of calculating both the price and replication strategy of a look-back option), let us return to the very simple model of Geometric Brownian motion:

$$dS_t = \sigma S_t dB_t + \mu S_t dt.$$

Proceeding as in Section H we change to an equivalent probability measure P^* such that $B_t^* = B_t + \frac{\mu}{\sigma} t$ is a standard Brownian motion under P^*, and now S is a martingale satisfying:

$$dS_t = \sigma S_t dB_t^*. \tag{1}$$

Let F be a functional defined on $C[0,t]$, the continuous functions with domain $[0,T]$. Then $F(u) \in \mathbb{R}$, where $u \in C[0,T]$, and let us suppose that F is *Fréchet* differentiable; let DF denote its Fréchet derivative. Under some technical conditions on F, if $H = F(B^*)$, then one can show

$$H = E^*\{H\} + \int_0^T {}^p(DF(B^*;(t,T]))dB_t^* \tag{2}$$

where ${}^p(X)$ denotes the predictable projection of X. (This is often written $E^*\{X|\mathcal{F}_t\}$ in the literature by an abuse of notation. The reason it is an abuse is that for the process $X = (X_t)_{0\le t\le T}$, $E^*\{X_t|\mathcal{F}_t\}$ is defined for each t a. s. The null set N_t depends on t. Thus $E^*\{X_t|\mathcal{F}_t\}$ does not uniquely define a process, since if $N = \bigcup_{0\le t\le T} N_t$, then $P(N_t) = 0$ for each t, but $P(N)$ need not be zero. The theory of predictable projections avoids this problem.) Using (1) we then have a formula for the hedging strategy:

$$a_t = \frac{1}{\sigma S_t}{}^p(DF(\cdot,(t,T])).$$

If we have $H(w) = \sup_{0\le t\le T} S_t(w) = S_T^* = F(B^*)$, then we can let $\tau(B^*)$ denote the random time where the trajectory of S attains its maximum on $[0,t]$. Such an operation is Fréchet differentiable and

$$DF(B^*,\cdot) = \sigma F(B^*)\delta_{\tau(B^*)},$$

where δ_α denotes the Dirac measure at α.

Let

$$M_{s,t} = \max_{s\le u\le t}\left(B_u^* - \frac{1}{2}\sigma u\right)$$

with $M_t = M_{0,t}$. Then the Markov property gives

$$\begin{aligned} E^*\{DF(B^*,(t,T])|\mathcal{F}_t\}(B^*) &= E^*\{\sigma F(B^*)1_{\{M_{t,T}>M_t\}}\big|\mathcal{F}_t\}(B^*) \\ &= \sigma S_t E^*\{\exp(\sigma M_{T-t}); M_{T-t} > M_t(B^*)\}. \end{aligned}$$

For a given fixed value of B^*, this last expectation depends only on the distribution of the maximum of a Brownian motion with constant drift. But this distribution is explicitly known. Thus we obtain an explicit hedging strategy for this look-back option:

$$\begin{aligned} a_t(\omega) = &\left(-\log\frac{M_t}{S_t}(\omega) + \frac{\sigma^2(T-t)}{2} + 2\right)\Phi\left(\frac{-\log\frac{M_t}{X_t}(\omega) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) \\ &+ \sigma\sqrt{T-t}\varphi\left(\frac{-\log\frac{M_t}{S_t}(\omega) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) \end{aligned}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-u^2/2}du$ and $\varphi(x) = \Phi'(x)$.

The value of this look-back option is then:

$$V_0 = E^*\{H\} = S_0\left(\frac{\sigma^2 T}{2} + 2\right)\Phi\left(\frac{1}{2}\sigma\sqrt{T}\right) + \sigma\sqrt{T}S_0\varphi\left(\frac{1}{2}\sigma\sqrt{T}\right).$$

Requiring that the claim be of the form $H = F(B^*)$ where F is Fréchet differentiable is very restrictive. One can weaken this hypothesis substantially by requiring that F be only Malliavin differentiable. If we let D denote now the Malliavin derivative of F, then equation (2) is still valid. Nevertheless explicit strategies and prices can be computed only in a few very special cases, and usually only when the price process S is Geometric Brownian motion.

IV. American Options.

A. The General View.

We begin with an abstract definition, in the *case of a unique equivalent martingale measure.*

Definition. We consider given an adapted process U and an expiration time T. An *American Security* is a claim to the payoff U_τ at a stopping time $\tau \leq T$; the stopping time τ is chosen by the holder of the security and is called the *exercise policy.*

We let $V_t =$ the price of the security at time t. One wants to find $(V_t)_{0 \leq t \leq T}$ and especially V_0. Let $V_t(\tau)$ denote the value of the security at time t if the holder uses exercise policy τ. Let us further assume (only for simplicity) that $R_t \equiv 1$. Then

$$V_t(\tau) = E^*\{U_\tau | \mathcal{F}_t\} \tag{1}$$

where of course E^* denotes expectation with respect to the equivalent martingale measure P^*.

Let $\mathcal{T}(t) = \{\text{all stopping times with values in } [t, T]\}$.

Definition. A *rational exercise policy* is a solution to the optimal stopping problem

$$V_0^* = \sup_{\tau \in \mathcal{T}(0)} V_0(\tau). \tag{2}$$

We want to establish a price for an American security. That is, how much should one charge to give a buyer the right to purchase U in between $[0, T]$ at a stopping rule of his choice?

Suppose first that the supremum in (A2) is achieved. That is, let us assume there exists a rule τ^* such that $V_0^* = V_0(\tau^*)$, where V_0^* is defined in (A2).

Lemma 1. *V_0^* is a lower bound for the no arbitrage price of our security.*

Proof. Suppose it is not. Let $V_0 < V_0^*$ be another price. Then one should buy the security at V_0 and use stopping rule τ^* to purchase U at time τ^*. One then spends $-U_{\tau^*}$, which gives an initial payoff of $V_0^* = E^*\{U_{\tau^*} | \mathcal{F}_0\}$; one's initial profit is $V_0^* - V_0 > 0$. This is an arbitrage opportunity. □

To prove V_0^* is also an upper bound for the no arbitrage price (and thus finally equal to the price!), is more difficult.

Definition. A super-replicating trading strategy θ is a self-financing trading strategy θ such that $\theta_t S_t \geq U_t$, all t, $0 \leq t \leq T$, where S is the price of the underlying risky security on which the American security is based. (We are again assuming $R_t \equiv 1$.)

Lemma 2. *Suppose a super replicating strategy θ exists, with $\theta_0 S_0 = V_0^*$. Then V_0^* is an upper bound for the no arbitrage price of the American security U.*

Proof. If $V_0 > V_0^*$, then one can sell the American security and adapt a super-replicating trading strategy θ with $\theta S_0 = V_0^*$. One then has an initial profit of $V_0 - V_0^* > 0$, while we are also able to cover the payment U_τ asked by the holder of the security at his exercise time τ, since $\theta_\tau S_\tau \geq U_\tau$. Thus we have an arbitrage opportunity. □

The existence of super-replicating trading strategies can be established using *Snell Envelopes.*

Recall that a collection of random variables $\mathcal{X}$ is uniformly integrable if

$$\lim_{c\to\infty} \sup_{X\in\mathcal{X}} E\{|X|1_{\{|X|>c\}}\} = 0.$$

A stochastic process Y is of "class D" if the collection $\mathcal{H} = \{Y_\tau : \tau \text{ a stopping time}\}$ is uniformly integrable.

Theorem. *Let Y be a càdlàg, adapted process, $Y > 0$ a.s., and of "Class D". Then there exists a positive càdlàg supermartingale Z such that*

(i) $Z \geq Y$, and for every other positive supermartingale Z' with $Z' \geq Y$, also $Z' \geq Z$;

(ii) Z is unique and also belongs to Class D;

(iii) For any stopping time τ

$$Z_\tau = \operatorname*{ess\,sup}_{\nu\geq\tau} E\{Y_\nu|\mathcal{F}_\tau\}$$

(ν also a stopping time).

For a proof consult [DM]. Z is called the *Snell Envelope of Y.*

One then needs to make some regularity hypotheses on the American security U. For example if one assumes U is a continuous semimartingale and $E^*\{[U,U]_T\} < \infty$, it is more than enough. One then uses the existence of Snell envelopes to prove:

Theorem. *Under regularity assumptions there exists a super-replicating trading strategy θ with $\theta_t S_t \geq k$ for all t for some constant k and such that $\theta_0 S_0 = V_0^*$. A rational exercise policy is*

$$\tau^* = \inf\{t > 0 : Z_t = U_t\},$$

where Z is the Snell Envelope of U under P^*.

B. The American Call Option.

Let us here assume that for a price process $(S_t)_{0\leq t\leq T}$ and a bond process $R_t \equiv 1$, there exists a unique equivalent martingale measure P^* which means that there is No Arbitrage and the market is complete.

Definition. An *American call option* with terminal time T and strike price K gives the holder the right to buy the security S at any time τ, at price K, between 0 and T.

It is of course reasonable to consider the random time τ where the option is exercised to be a stopping time, and it is standard to assume that it is then $(S_\tau - K)^+$, corresponding to which rule τ the holder uses.

We note first of all that since the holder of the option is free to choose the rule $\tau \equiv T$, he or she is always in a better position than the holder of a European call option, whose worth is $(S_T - K)^+$. Thus the price of an American call option should be bounded below by the price of the corresponding European call option.

Following Section IV.A we let

$$V_t(\tau) = E^*\{U_\tau | \mathcal{F}_t\} = E^*\{(S_\tau - K)^+ | \mathcal{F}_t\}$$

denote the value of our American call option at time t assuming τ is the exercise rule. We then have that the price is

$$V_0^* = \sup_{\substack{\tau \\ 0 \leq \tau \leq T}} E^*\{(S_\tau - K)^+\}. \tag{1}$$

We note however that $S = (S_t)_{0 \leq t \leq T}$ is a martingale under P^*, and since $f(x) = (x - K)^+$ is a convex function we have $(S_t - K)^+$ is a submartingale under P^*; hence from (1) we have

$$V_0^* = E^*\{(S_T - K)^+\}$$

since $t \to E^*\{(S_t - K)^+\}$ is an increasing function, and the sup - even for stopping times - of the expectation of a submartingale is achieved at the terminal time (this can be easily seen as a trivial consequence of the Doob-Meyer decomposition theorem). This leads to a surprising result.

Theorem. *In a complete market (with no arbitrage) the price of an American call option with terminal time T and strike price K is the same as the price for a European call option with the same terminal time and strike price.*

Corollary. *If the price process S_t follows the SDE*

$$dS_t = \sigma S_t dB_t + \mu S_t dt; S_0 = 1$$

then the price of an American call option with strike price K and terminal time T is the same as that of the corresponding European call option and is given by the formula (III.I.4) of Black and Scholes.

We note that while we have seen that the prices of the European and American call options are the same, we have said nothing about the replication strategies.

C. Backwards Stochastic Differential Equations and the American Put Option.

Let ξ be in L^2 and suppose $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is Lipschitz in space. Then a simple backwards ordinary differential equation (ω by ω) is

$$Y_t(\omega) = \xi(\omega) + \int_t^T f(s, Y_s(\omega))ds.$$

However if $\xi \in L^2(\mathcal{F}_T, dP)$ and one requires that a solution $Y = (Y_t)_{0 \le t \le T}$ be *adapted* (that is, $Y_t \in \mathcal{F}_t$), then the equation is no longer simple. For example if $Y_t \in \mathcal{F}_t$ for every t, $0 \le t \le T$, then one has

$$Y_t = E\{\xi + \int_t^T f(s, Y_s)ds | \mathcal{F}_t\}. \tag{1}$$

An equation such as (1) is called a *Backwards Stochastic Differential Equation.* Next we write

$$\begin{aligned} Y_t &= E\{\xi + \int_0^T f(s, Y_s)ds | \mathcal{F}_t\} - \int_0^t f(s, Y_s)ds \\ &= M_t - \int_0^t f(s, Y_s)ds \end{aligned}$$

where M is the martingale $E\{\xi + \int_0^T f(s, Y_s)ds | \mathcal{F}_t\}$. Let us suppose we are solving (1) on the canonical space for Brownian motion. Then we have that the martingale representation property holds, and hence there exists a predictable $Z \in \mathcal{L}(B)$ such that

$$M_t = M_0 + \int_0^t Z_s dB_s$$

where B is Brownian motion. We then have

$$\begin{aligned} Y_T - Y_t &= M_T - M_t - \left(\int_0^T f(s, Y_s)ds - \int_0^t f(s, Y_s)ds \right) \\ \xi - Y_t &= \int_t^T Z_s dB_s - \int_t^T f(s, Y_s)ds \end{aligned}$$

or, the equivalent equation:

$$Y_t = \xi + \int_t^T f(s, Y_s)ds - \int_t^T Z_s dB_s. \tag{2}$$

Thus to find an adapted Y that solves (1) is equivalent to find a pair (Y, Z) with Y adapted and Z predictable that solve (2).

Now that one has introduced Z, one can consider a more general version of (2) in the form

$$Y_t = \xi + \int_0^t f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s. \tag{3}$$

We next wish to consider a more general equation than (3), however: Backward Stochastic Differential Equations where the solution Y is forced to stay above an obstacle. This can be formulated as follows (here we follow [EKPPQ]):

$$(4)\quad \begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dB_s \\ Y_t \geq U_t \quad (U \text{ is optional}) \\ K \text{ is continuous, increasing, adapted, } K_0 = 0, \text{ and } \int_0^T (Y_t - U_t)dK_t = 0. \end{cases}$$

The obstacle process U is given, as are the random variables ξ and the function f, and the unknowns to find are (Y, Z, K). Once again it is Z that makes both Y and K adapted.

Theorem ([EKPPQ]). *Let f be Lipschitz in (y, z) and assume $E\{\sup_{0\leq t\leq T}(U_t^+)^2\} < \infty$. Then there exists a unique solution (Y, Z, K) to equation (4).*

Two proofs are given in [EKKPQ]: one uses the Skorohod problem, a priori estimates and Picard iteration; the other uses a penalization method.

Now let us return to American options. Let S be the price process of a risky security and let us take $R_t \equiv 1$. An American put option then takes the form $(K - S_\tau)^+$ where K is a striking price and the exercise rule τ is a stopping time with $0 \leq \tau \leq T$. Thus we should let $U_t = (K - S_t)^+$, and if X is the Snell envelope of U, we see from IV.A that a rational exercise policy is

$$\tau^* = \inf\{t > 0 : X_t = U_t\}$$

and that the price is $V_0^* = V_0(\tau^*) = E^*\{U_{\tau^*}|\mathcal{F}_0\} = E^*\{(K - S_{\tau^*})^+\}$. Therefore finding the price of an American put option is related to finding the Snell envelope of U. Recall that the Snell envelope is a supermartingale such that

$$X_\tau = \operatorname*{ess\ sup}_{\nu \geq \tau} E\{U_\nu|\mathcal{F}_\tau\}$$

where ν is also a stopping time.

We consider the situation where $U_t = (K - S_t)^+$ and $\xi = (K - S_T)^+$. We then have

Theorem ([EKKPQ]). *Let (Y, K, Z) be the solution of (4). Then*

$$Y_t = \operatorname*{ess\ sup}_{\substack{t\leq\nu\leq T \\ \nu \text{ a stopping time}}} E\left\{\int_t^\nu f(s, Y_s, Z_s)ds + U_\nu|\mathcal{F}_t\right\}.$$

Proof (Sketch). In this case

$$Y_t = U_T + \int_t^T f(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dB_s,$$

hence

$$Y_\nu - Y_t = -\int_t^\nu f(s, Y_s, Z_s)ds + (K_t - K_\nu) + \int_t^\nu Z_s dB_s$$

and since $Y_t \in \mathcal{F}_t$ we have

$$Y_t = E\left\{\int_t^\nu f(s, Y_s, Z_s)ds + Y_\nu + (K_\nu - K_t)|\mathcal{F}_t\right\} \geq E\left\{\int_t^\nu f(s, Y_s, Z_s)ds + U_\nu|\mathcal{F}_t\}\right\}.$$

Next let $\gamma_t = \inf\{t \leq u \leq T : Y_u = U_u\}$, with $\gamma_t = T$ if $Y_u > U_u$, $t \leq u \leq T$. Then

$$Y_t = E\left\{\int_t^{\gamma_t} f(s, Y_s, Z_s)ds + Y_{\gamma_t} + K_{\gamma_t} - K_t|\mathcal{F}_t\right\}.$$

However on $[t, \gamma_t)$ we have $Y > U$, and thus $\int_t^{\gamma_t}(Y_s - U_s)dK_s = 0$ implies that $K_{\gamma_t -} - K_t = 0$; however K is continuous by assumption, hence $K_{\gamma_t} - K_t = 0$. Thus (using $Y_{\gamma_t} = U_{\gamma_t}$):

$$Y_t = E\left\{\int_t^{\gamma_t} f(s, Y_s, Z_s)ds + U_{\gamma_t}|\mathcal{F}_t\right\}$$

and we have the other implication. □

The next corollary shows that we obtain the price of an American put option via reflected backwards stochastic differential equations.

Corollary. *The American put option has the price Y_0, where (Y, K, Z) solves the reflected obstacle backwards SDE with obstacle $U_t = (K - S_t)^+$ and where $f = 0$.*

Proof. In this case the previous theorem becomes

$$Y_0 = \operatorname*{ess\,sup}_{\substack{0 \leq \nu \leq T \\ \nu \text{ a stopping time}}} E\{U_\nu|\mathcal{F}_t\},$$

and $U_\nu = (K - S_\nu)^+$. □

This relationship between the American put option and backwards SDEs can be exploited to price numerically an American put option.

We note that one can generalize these results to American Game Options, using Forward-Backward Reflected Stochastic Differential Equations, see, eg, [MC].

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