CRITICAL BEHAVIOR IN DATA COMPRESSION

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Technical Report # 99-26

Department of Statistics
Purdue University

November 1999

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Abstract

Let $x_1^n = (x_1, x_2, \ldots, x_n)$ be a realization of the independent and identically distributed random variables $(X_1, X_2, \ldots, X_n)$. A compression algorithm operating at distortion level D consists of an encoder that takes strings x_1^n to binary strings of variable length, and a decoder that maps these binary strings to new strings $y_1^n = (y_1, y_2, \ldots, y_n)$, so that the decoded y_1^n is always within distortion D of the encoded x_1^n. Distortion is measured by some single-letter distortion measure such as mean-squared error. The description length $\ell_n(x_1^n)$ is the length of the binary description of x_1^n. For long realizations, the best compression ratio $\ell_n(X_1^n)/n$ that can be achieved by any sequence of algorithms operating at distortion level D is given by Shannon's rate-distortion function $R(D)$.

The following critical phenomenon was recently discovered in [10]. Depending on the distribution of the random variables X_i and the distortion level D, the fastest rate at which $\ell^n(x_1^n)/n$ can converge to $R(D)$ is either of order $\sqrt{n}$ or of order $\log n$. No other possibilities exist. In this paper we show that in most cases the optimal convergence rate is $O(\sqrt{n})$: For any fixed source distribution the rate can be $O(\log n)$ for at most finitely many distortion levels D, unless the X_i are uniformly distributed over a finite set, in which case the rate is $O(\log n)$ for all D.

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[1] I.K.'s research was supported in part by a grant from the Purdue Research Foundation.

1 Introduction

Suppose that data is produced by a stationary memoryless source $\{X_n \; ; \; n \geq 1\}$, so that the X_i are independent and identically distributed (IID) random variables with common distribution P. We will assume throughout that the X_i take values in some subset A of $\mathbb{R}$ called the *source alphabet*, and that the *reproduction alphabet* $\hat{A}$ is a finite subset of $\mathbb{R}$, say $\hat{A} = \{a_1, a_2, \ldots, a_k\}$.

The main objective of data compression is to find efficient approximate representations for realizations $x_1^n = (x_1, x_2, \ldots, x_n)$ from the data source $X_1^n = (X_1, X_2, \ldots, X_n)$. Specifically, we wish to represent each source string x_1^n by a corresponding string $y_1^n = (y_1, y_2, \ldots, y_n)$ taking values in the reproduction alphabet $\hat{A}$, so that the distortion between each x_1^n and its representation lies within some fixed allowable range. For our purposes, distortion is measured by a family of single-letter distortion measures,

$$\rho_n(x_1^n, y_1^n) = \frac{1}{n} \sum_{i=1}^{n} \rho(x_i, y_i) \qquad x_1^n \in A^n, \;\; y_1^n \in \hat{A}^n, \tag{1}$$

where $\rho : A \times \hat{A} \to [0, \infty)$ is a fixed nonnegative function. For example $\rho(x, y)$ may be the Hamming distortion, $\rho(x, y) = 0$ if and only if $x = y$, and $\rho(x, y) = 1$ otherwise; or it may be the squared Euclidean distance $\rho(x, y) = (x - y)^2$, making $\rho_n(x_1^n, y_1^n)$ equal to the mean-squared error between x_1^n and y_1^n.

We consider *variable-length block codes operating at a fixed distortion level*, that is, codes C_n defined by triplets (B_n, ϕ_n, ψ_n) where:

(*a*) B_n is a subset of $\hat{A}^n$ called the *codebook*;

(*b*) $\phi_n : A^n \to B_n$ is a map called the *encoder*;

(*c*) $\psi_n : B_n \to \{0, 1\}^*$ is an invertible representation of the elements of B_n by finite-length binary strings.

In (*c*), $\{0, 1\}^*$ denotes the set of all binary strings of finite length. On top of being invertible, we also assume that the map ψ_n is *prefix-free*, i.e., that in the range of ψ_n no binary string is a prefix of another (this assumption can be made without any essential loss of generality; see Chapter 5 in [5]).

For a fixed distortion level $D \geq 0$, the code $C_n = (B_n, \phi_n, \psi_n)$ is said to *operate at distortion level D* [8] if it encodes each source string with distortion D or less:

$$\rho_n(x_1^n, \phi_n(x_1^n)) \leq D \quad \text{for all } \; x_1^n \in A^n.$$

From the point of view of data compression, the main quantity of interest is the description length of a block code C_n, expressed by its length function $\ell_n : A^n \to \mathbb{N}$, where $\ell_n(x_1^n)$ denotes the description length (in bits) assigned by C_n to the string x_1^n:

$$\ell_n(x_1^n) = \text{ length of } [\psi_n(\phi_n(x_1^n))].$$

Broadly speaking, the smaller the description length, the better the code.

Shannon's 1959 celebrated source coding theorem identified and characterized the best achievable expected compression performance of block codes. It states that, for an arbitrary sequence of block codes $\{C_n = (B_n, \phi_n, \psi_n)\ ;\ n \geq 1\}$ operating at distortion level D, the expected compression ratio $E[\ell_n(X_1^n)]/n$ is asymptotically bounded below by the rate-distortion function,

$$\liminf_{n\to\infty} \frac{E[\ell_n(X_1^n)]}{n} \geq R(D) \quad \text{bits per symbol,}$$

where $R(D) = R(P, D)$ is the rate-distortion function of the memoryless source with distribution P (precise definitions are given in the next section). Moreover, Shannon showed that there exist codes achieving the above lower bound with equality; see Shannon's 1959 paper [11] or Berger's classic text [4].

A stronger version of Shannon's theorem was proved by Kieffer in 1991 [8], where it is shown that the rate-distortion function is a *pointwise* asymptotic lower bound for $\ell_n(X_1^n)$:

$$\liminf_{n\to\infty} \frac{\ell_n(X_1^n)}{n} \geq R(D) \quad \text{with prob. 1.} \tag{2}$$

In [8] it is also demonstrated that the bound in (2) can be achieved with equality.

The following refinement to Kieffer's result was recently given in [10]:

> (POINTWISE REDUNDANCY): For any sequence of block codes $\{C_n\}$ with associated length functions $\{\ell_n\}$, operating at distortion level D,
>
> $$\ell_n(X_1^n) \geq nR(D) + \sum_{i=1}^{n} f(X_i) - 2\log n \quad \text{eventually, with prob. 1,} \tag{3}$$
>
> where $f : A \to \mathbb{R}$ is a bounded function depending on P and D but *not* on the codes $\{C_n\}$, such that $E_P[f(X_1)] = 0$. Moreover, there exist codes $\{C_n, \ell_n\}$ that achieve
>
> $$\ell_n(X_1^n) \leq nR(D) + \sum_{i=1}^{n} f(X_i) + 5\log n \quad \text{eventually, with prob. 1.} \tag{4}$$

[*cf.* Theorems 4 and 5 and eq. (18) in [10]; the function f is defined precisely in Section 3; above and throughout the paper, 'log' denotes the logarithm taken to base 2 and '$\log_e$' denotes the natural logarithm.] This result says that, for any source distribution P and any sequence of codes $\{C_n\}$ operating at distortion level D, the "pointwise redundancy" in the description lengths of the codes C_n, namely, the difference between $\ell_n(X_1^n)$ and the optimum $nR(D)$ bits

$$\text{redundancy} = \ell_n(X_1^n) - nR(D)$$

is essentially bounded below by the sum of the IID, bounded, zero-mean random variables $f(X_i)$. So there are two possibilities:

- Either the random variables $f(X_i)$ are non-constant, in which case the best achievable pointwise redundancy rate is of order $\sqrt{n}$ (by the central limit theorem and the upper and lower bounds in (3) and (4));
- or the random variables $f(X_i)$ are equal to zero with probability one, in which case the best pointwise redundancy is $\leq 5 \log n$ (by (4)).

Our purpose in this paper is to characterize exactly when each one of the above two cases occurs, namely, when the minimal pointwise redundancy is $O(\sqrt{n})$ and when it is $O(\log n)$. In the next section we show that it is almost never the case that $f(X_1) = 0$ with probability one, so the minimal pointwise redundancy is typically of order $\sqrt{n}$. In particular, in the common case when the X_i take values in a finite alphabet $A = \hat{A}$, then (under mild conditions) we show that $f(X_1) = 0$ with probability one if and only if the X_i are uniformly distributed.

Before stating our main results (Theorems 1, 2 and 3 in the next section) in detail, we recall the following representative examples from [9] and [10].

Example 1. (Lossless Compression). In the case of lossless compression the objective is to find efficient, uniquely decodable representations for the source strings x_1^n. For a source $\{X_n\}$ with distribution P on the finite alphabet A, a lossless code C_n is a prefix-free map $\psi_n : A^n \to \{0,1\}^*$. [Or, to be pedantic, in our setting a lossless code is a code operating at distortion level $D = 0$ with respect to Hamming distortion.] In this case the function f has the simple form

$$f(x) = -\log P(x) - H(P) \tag{5}$$

where $H(P) = E_P[-\log P(X_1)]$ is the entropy of P, and the lower bound (3) is simply

$$\begin{aligned} \ell_n(X_1^n) &\geq nH(P) + \sum_{i=1}^{n} f(X_i) - 2\log n \\ &= -\log P(X_1^n) - 2\log n \quad \text{eventually, with prob. 1.} \end{aligned} \tag{6}$$

The lower bound (6) is a well-known information-theoretic fact called Barron's lemma (see [2][3] and the discussion in [10]). It says that the description lengths $\ell_n(X_1^n)$ of an arbitrary sequence of codes are (eventually eventually with probability 1) bounded below by the idealized Shannon code lengths $-\log P(X_1^n)$, up to terms of order $\log n$. From (5) it is obvious that $f(X_1) = 0$ with probability one if and only if P is the uniform distribution on A.

Example 2. (Binary Source, Hamming Distortion). This is the simplest non-trivial lossy example. Suppose $\{X_n\}$ is a binary source with Bernoulli(p) distribution for some $p \in (0, 1/2]$. Let $A = \hat{A} = \{0,1\}$ and take ρ to be the Hamming distortion measure, $\rho(x,y) = 0$ when $x = y$, and equal to 1 otherwise. For each fixed $D \in (0,p)$ it is shown in [10] that

$$f(x) = -\log\left(\frac{P(x)}{1-D}\right) - E_P\left[-\log\left(\frac{P(X_1)}{1-D}\right)\right],$$

from which it is again obvious that $f(X_1) = 0$ with probability one if and only if $p = 1/2$, i.e., if and only if P is the uniform distribution on $A = \{0,1\}$.

In a third example presented in [10] it is also found that $f(X_1) = 0$ with probability one if and only if P is the uniform distribution, and the natural question is raised as to whether this pattern persists in general. In the next section we answer this question by showing (in Theorem 1 and Corollary 1) that for a fixed source distribution P on a finite alphabet A, $f(X_1)$ can be equal to zero with probability one for at most finitely many distortion levels D, *unless* P is the uniform distribution and ρ is a "permutation" distortion measure. In Theorems 2 and 3 and in Corollary 2 the continuous case is considered, and it is shown that when P is a continuous distribution it essentially never happens that $f(X_1) = 0$ with probability one. Section 3 contains the proofs of Theorems 1, 2 and 3 and Corollaries 1 and 2.

2 Results

Suppose that the source alphabet A is an arbitrary (Borel) subset of $\mathbb{R}$, and let P be a (Borel) probability measure on $\mathbb{R}$, supported on A (the special cases when P is purely discrete or purely continuous are considered separately below). Let $\hat{A} = \{a_1, a_2, \ldots, a_k\}$ be the finite reproduction alphabet of size k. Given an arbitrary, bounded, nonnegative function $\rho : A \times \hat{A} \to [0, M]$ (for some finite constant M), define a sequence of single-letter distortion measures $\rho_n : A^n \times \hat{A}^n \to [0, M]$ as in (1). Throughout the paper, we make the usual assumption:

$$\sup_{x\in A} \min_{y\in\hat{A}} \rho(x,y) = 0. \tag{7}$$

[If this is not satisfied, for example when A is an interval of real numbers, $\hat{A}$ is a finite set, and $\rho(x,y) = (x-y)^2$, we may consider the distortion measure $\rho'(x,y) = \rho(x,y) - \min_{z\in\hat{A}} \rho(x,z)$ instead.] For $D \geq 0$, the *rate-distortion function* of the memoryless source with distribution P is defined by

$$R(D) = \inf_{(X,Y)} I(X;Y) \tag{8}$$

where the infimum is over all jointly distributed random variables (X,Y) with values in $A\times\hat{A}$ such that X has distribution P and $E[\rho(X,Y)] \leq D$; $I(X;Y)$ denotes the mutual information (in bits) between two random variables X and Y (see [4] for more detailed definitions and various properties of $R(D)$). Under our assumptions, the rate-distortion function $R(D)$ is a convex, nonincreasing function of $D \geq 0$, and it is finite for all D.

For a fixed distribution P on A, let

$$D_{\max} \;=\; D_{\max}(P) \;=\; \min_{y\in\hat{A}}\; E_P[\rho(X,y)]$$

and recall that $R(D) = 0$ for $D \geq D_{\max}$ (see, e.g., Proposition 1 in Section 3). In order to avoid the trivial case when $R(D)$ is identically zero we assume that $D_{\max} > 0$, and from now on we restrict our attention to the interesting range of distortion levels $D \in (0, D_{\max})$.

2.1 The Discrete Case: $A = \hat{A}$

We first consider the most common case where the source $\{X_n\}$ takes values in a finite alphabet. Suppose that the random variables X_i are IID with common distribution P on $A = \hat{A} = \{a_1, a_2, \ldots, a_k\}$, and assume, without loss of generality, that $P_i = P(a_i) > 0$ for all $i = 1, \ldots, k$. Given a distortion measure ρ, write ρ_{ij} for $\rho(a_i, a_j)$. We assume throughout this section that ρ is symmetric,

$$\rho_{ij} = \rho_{ji} \quad \text{for all } 1 \leq i, j \leq k,$$

and also that $\rho_{ij} = 0$ if and only if $i = j$. We call ρ a *permutation distortion measure*, if all rows of the matrix $(\rho_{ij})_{i,j=1,\ldots,k}$ are permutations of one another (which, by symmetry, is equivalent to saying that all columns are permutations of one another).

Recall that the minimal pointwise redundancy is $O(\log n)$ if and only if $f(X_1) = 0$ with probability one; otherwise it is $O(\sqrt{n})$. Our first result says that the rate cannot be $O(\log n)$ for many distortion levels D, unless the distribution P is uniform in which case the rate is $O(\log n)$ for all distortion levels D.

Theorem 1.

(a) If P is the uniform distribution on A and ρ is a permutation distortion measure, then $f(X_1) = 0$ with probability one for all $D \in (0, D_{\max})$.

(b) If $f(X_1) = 0$ with probability one for a sequence of distortion values $D_n \in (0, D_{\max})$ such that $D_n \downarrow 0$, then P is the uniform distribution and ρ is a permutation distortion measure, and therefore $f(X_1) = 0$ with probability one for *all* $D \in (0, D_{\max})$.

As we mentioned above, the rate-distortion function $R(D)$ is convex for $D \in (0, D_{\max})$. If it is *strictly* convex (as it is usually the case – see the discussion in [4, Chapter 2]), then Theorem 1 can be strengthened to the following.

Corollary 1. Suppose the rate-distortion function $R(D)$ is strictly convex over $(0, D_{\max})$. If $f(X_1) = 0$ with probability one for infinitely many $D \in (0, D_{\max})$ then P is the uniform distribution and ρ is a permutation distortion measure, and therefore $f(X_1) = 0$ with probability one for *all* $D \in (0, D_{\max})$.

Remark. In the examples presented in the previous section, it turned out that either $f(X_1) = 0$ with probability one for *all* D, or it was never the case. But it may happen that $f(X_1) = 0$ with probability one only for a few isolated values of D, while P is *not* the uniform distribution. Such an example is given after Lemma 3 in Section 3.2.

2.2 The Continuous Case: $A = \mathbb{R}$

Here we take $A = \mathbb{R}$ and we assume that the distribution P of the source has a positive density g (with respect to Lebesgue measure), or, more generally, that there exists a (nonempty) open interval $I \subset \mathbb{R}$ on which P has an absolutely continuous component with density g such that $g(x) > 0$ for $x \in I$. Since the reproduction alphabet $\hat{A} = \{a_1, a_2, \ldots, a_k\}$ is finite, given a distortion measure ρ we can write

$$\rho_j(x) = \rho(x, a_j) \quad \text{for all } 1 \leq j \leq k,\ x \in A.$$

We assume that for all j the functions ρ_j are continuous on I. For convenience we also define, for $j = 0$, $\rho_j(x) \equiv 0$ on I.

Our next result gives a sufficient condition on the distortion measures ρ_j, under which the best redundancy rate in (2) can *never* be $O(\log n)$.

Theorem 2. If for every $\lambda < 0$ the functions $\{e^{\lambda\rho_j(\cdot)}\ ;\ 0 \leq j \leq k\}$ are linearly independent on I, then $f(X_1)$ cannot be equal to zero with probability one for any distortion level $D \in (0, D_{\max})$.

Next we provide a somewhat simpler set of conditions, under which we get a weaker conclusion. Theorem 3 says that the best redundancy rate in (2) cannot be $O(\log n)$ for many distortion levels D (as long as the distortion measure satisfies one of two mild conditions).

Theorem 3. Under either one of the following two conditions, $f(X_1)$ cannot be equal to zero with probability one for distortion levels $D > 0$ arbitrarily close to zero.

(a) There exist (distinct) points $\{x_0, x_1, \ldots, x_k\}$ in I such that, for all $0 \leq i \neq j \leq k$, with $j \neq 0$, we have $\rho_j(x_j) > \rho_j(x_i)$.

(b) There exist (distinct) points $\{x_0, x_1, \ldots, x_k\}$ in I such that, for every permutation π of the indices $\{0, 1, \ldots, k\}$ with π not equal to the identity, we have

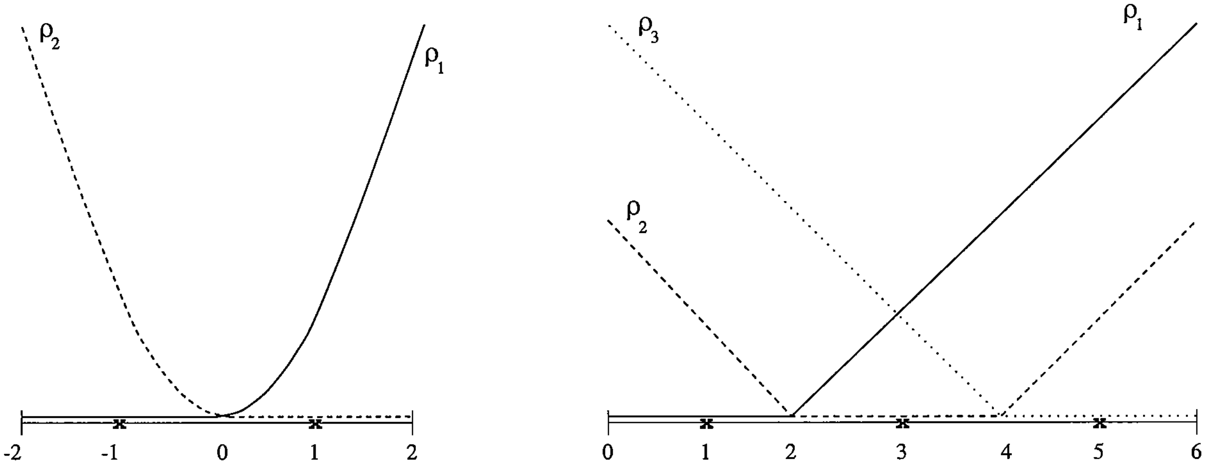
$$\sum_{j=0}^{k} \rho_j(x_j) \neq \sum_{j=0}^{k} \rho_j(x_{\pi(j)}).$$

Although the conditions of Theorems 2 and 3 may seem unusual, they are natural and generally easy to verify. To illustrate this, we present below two simple examples.

Example 1. (Mean-Squared Error). Suppose P has a positive density on the interval $I = [-2, 2]$, let $\hat{A}$ consist of the two reproduction points ± 1, and let ρ be the mean-squared error distortion measure. Recall that, to satisfy (7), $\rho(x, y)$ is actually defined by $\rho(x, y) = (x - y)^2 - \min\{(x-1)^2, (x+1)^2\}$; the corresponding distortion functions $\rho_1(x) = \rho(x, -1)$ and $\rho_2(x) = \rho(x, +1)$ are shown in Figure 1 (a). Here, condition (a) of Theorem 3 is easily seen to hold with $x_0 = 0$, $x_1 = 2$ and $x_2 = -2$.

Example 2. (L^1 Distance). Suppose P has a positive density on the interval $I = [0, 6]$, let $\hat{A} = \{1, 3, 5\}$, and take ρ to be the normalized L^1 distance $|x - y|$ adjusted so that (7) is satisfied; the resulting functions $\rho_j(\cdot)$ are shown in Figure 1 (b). Here it is easy to verify that the condition of Theorem 2 is satisfied, i.e., that the functions $\{e^{\lambda\rho_j(\cdot)}\ ;\ 0 \leq j \leq 3\}$ are linearly independent on I. For this it suffices to observe that $e^{\lambda\rho_1}$ and $e^{\lambda\rho_3}$ are linearly independent on $[2, 4]$ (essentially because the functions $e^{\lambda x}$ and $e^{-\lambda x}$ are linearly independent on $[0, 2]$), and that $e^{\lambda\rho_2}$ is not constant outside $[2, 4]$.

Like in the discrete case, under some additional assumptions on the rate-distortion function $R(D)$, it is possible to get a stronger version of Theorem 3:



(a)

(b)

Figure 1: Distortion measures in Examples 1 and 2. Reproduction points are shown as **x**'s.

Corollary 2. Suppose the rate-distortion function $R(D)$ is differentiable and strictly convex over $(0, D_{\max})$. Under either one of the assumptions (a) and (b) in Theorem 3, there can be at most finitely many $D \in (0, D_{\max})$ such that $f(X_1) = 0$ with probability one.

Remark. Under somewhat more restrictive assumptions on the distortion measure ρ, it is possible to prove that, for any P with a continuous component as above, there can be at most $k(k+1)/2$ distortion levels D for which $f(X_1) = 0$ with probability one. Since the proof of this slightly stronger result relies on an argument different from the ones used to prove Theorems 2 and 3, we omit it here.

3 Proofs

3.1 Preliminaries

Before giving the proofs of Theorems 1, 2 and 3, we recall some definitions and notation from [10] and give the precise form of the function f (see equation (12) below).

Let P be a source distribution on A, and let Q be an arbitrary probability mass function on $\hat{A}$. Write X for a random variable with distribution P on A, and Y for an independent random variable with distribution Q on $\hat{A}$. Let $S = \{a \in \hat{A} \; : \; Q(a) > 0\}$ be the support of Q and define

$$\begin{aligned} D_{\min}^{P,Q} &= E_P \left[\min_{a \in S} \rho(X, a) \right] \\ D_{\max}^{P,Q} &= E_{P \times Q} \left[\rho(X, Y) \right]. \end{aligned}$$

For $\lambda \leq 0$, let

$$\Lambda_{P,Q}(\lambda) \;=\; E_P\left[\log_e E_Q\left(e^{\lambda\rho(X,Y)}\right)\right],$$

and for $D \geq 0$ write $\Lambda^*_{P,Q}$ for the Fenchel-Legendre transform of $\Lambda_{P,Q}$,

$$\Lambda^*_{P,Q}(D) = \sup_{\lambda \leq 0}[\lambda D - \Lambda_{P,Q}(\lambda)].$$

We also define

$$R(P,Q,D) \;=\; \inf_{(X,Z)}[I(X;Z) + H(Q_Z\|Q)]$$

where $H(R\|Q) = \sum_{a\in\hat{A}} R(a)\log[R(a)/Q(a)]$ denotes the relative entropy (in bits) between R and Q, Q_Z denotes the distribution of Z, and the infimum is over all jointly distributed random variables (X,Z) with values in $A\times\hat{A}$ such that X has distribution P and $E[\rho(X,Z)] \leq D$. In view of (8), we clearly have

$$R(D) = \inf_{\text{all } Q} R(P,Q,D). \tag{9}$$

In Lemma 1 and Proposition 1 below we summarize some useful properties of $\Lambda_{P,Q}$, $\Lambda^*_{P,Q}$ and $R(P,Q,D)$ (see Lemma 1 and Propositions 1 and 2 in [10]).

Lemma 1.

(*i*) $\Lambda_{P,Q}$ is infinitely differentiable on $(-\infty, 0)$, and $\Lambda''_{P,Q}(\lambda) \geq 0$ for all $\lambda \leq 0$.

(*ii*) If $D \in (D^{P,Q}_{\text{min}}, D^{P,Q}_{\text{max}})$ then there exists a unique $\lambda < 0$ such that $\Lambda'_{P,Q}(\lambda) = D$ and $\Lambda^*_{P,Q}(D) = \lambda D - \Lambda_{P,Q}(\lambda)$.

Proposition 1.

(*i*) For all $D \geq 0$, $R(P,Q,D) = \inf_W E_P[H(W(\cdot|X)\|Q(\cdot))]$, where the infimum is over all probability measures W on $A \times \hat{A}$ such that the A-marginal of W equals P and $E_W[\rho(X,Y)] \leq D$.

(*ii*) For all $D \geq 0$, $R(P,Q,D) = (\log e)\Lambda^*_{P,Q}(D)$.

(*iii*) For $0 < D < D_{\text{max}}$ we have $0 < R(D) < \infty$, whereas for $D \geq D_{\text{max}}$, $R(D) = 0$.

(*iv*) For every $D \in (0, D_{\text{max}})$ there exists a $Q = Q^*$ on $\hat{A}$ achieving the infimum in (9), and $D \in (D^{P,Q^*}_{\text{min}}, D^{P,Q^*}_{\text{max}})$.

For any distribution P on A and any distortion level $D \in (0, D_{\text{max}}(P))$, by Proposition 1 we can pick a Q^* achieving the infimum in (9) so that $R(D) = R(P,Q^*,D)$ and also $D \in (D^{P,Q^*}_{\text{min}}, D^{P,Q^*}_{\text{max}})$, so by Lemma 1 we can pick a $\lambda^* < 0$ with

$$\lambda^* D - \Lambda_{P,Q^*}(\lambda^*) \;=\; \Lambda^*_{P,Q^*}(D) \;=\; (\log_e 2)R(P,Q^*,D) \;=\; (\log_e 2)R(D). \tag{10}$$

Note also that

$$\lambda^* \to -\infty \quad \text{as} \quad D \to 0 \tag{11}$$

(see the Appendix for a short proof). Finally we can define the function f:

$$f(x) \triangleq (\log e)\left[\lambda^* D - \log_e E_{Q^*}\left(e^{\lambda^* \rho(x,Y)}\right)\right] - R(D), \qquad x \in A. \tag{12}$$

Since $E_P[f(X_1)] = 0$, $f(X_1) = 0$ with probability one if and only if

$$\sum_{j=1}^{k} Q^*(a_j) e^{\lambda^* \rho(x,a_j)} = \text{Constant} \quad \text{for } P - \text{almost all } x. \tag{13}$$

Next we give an useful interpretation for the constant λ^* in the representation of $R(D)$ in (10): Lemma 2 says that if the rate-distortion function is differentiable at D, then λ^* is proportional to its slope at D; it is proved in the Appendix.

Lemma 2. For any $D \in (0, D_{\max})$:

(*i*) $(\log_e 2)R(D) = \sup_{\lambda \leq 0}[\lambda D - \Gamma(\lambda)]$, where $\Gamma(\lambda) = \sup_Q \Lambda_{P,Q}(\lambda)$.

(*ii*) Let λ^* be chosen as in (10). If $R(\cdot)$ is differentiable at D, then $\lambda^* = (\log_e 2)R'(D)$.

3.2 Proofs in the Discrete Case

For the proof of Theorem 1 we will need the following lemma. It easily follows from Theorem 3.7 in Chapter 2 of [6] (see the Appendix). Recall the notation $P_i = P(a_i)$ and $\rho_{ij} = \rho(a_i, a_j)$.

Lemma 3. A probability mass function Q^* on A achieves the infimum in (9) if and only if there exists a $\lambda^* < 0$ such that the following all hold:

(a) $\Lambda'_{P,Q^*}(\lambda^*) = D$.

(b) If we define, for $a_i, a_j \in A$,

$$W(a_i, a_j) = P_i Q^*(a_j) \frac{e^{\lambda^* \rho_{ij}}}{\sum_{j'} Q^*(a_{j'}) e^{\lambda^* \rho_{ij'}}}$$

then the second marginal of W is Q^*.

(c) If $Q^*(a_j) = 0$ for some j, then

$$\sum_i P_i \frac{e^{\lambda^* \rho_{ij}}}{\sum_{j'} Q^*(a_{j'}) e^{\lambda^* \rho_{ij'}}} \leq 1.$$

Example. Here we present a simple example illustrating the fact that it may happen that $f(X_1) = 0$ for a few isolated values D even when P is not uniform. Take $A = \hat{A} = \{0, 1, 2\}$, let $\alpha = \log_e[3e/(4-e)]$, and consider the distortion measure

$$(\rho_{ij}) = \begin{pmatrix} 0 & 1 & \alpha \\ 1 & 0 & \alpha \\ \alpha & \alpha & 0 \end{pmatrix}.$$

Then, with $P = Q^* = (4/13, 4/13, 5/13)$ and $\lambda^* = -1$, it is straightforward to check that condition (b) of Lemma 3 holds (condition (c) is irrelevant here), and also (13) is satisfied. Therefore, at $D = \Lambda'_{P,Q^*}(\lambda^*) \approx 0.43$, we must have $f(X_1) = 0$ with probability one.

Proof of Theorem 1, (a): Suppose ρ is a permutation distortion measure and P is the uniform distribution on A, $P_i = 1/k$ for all $i = 1, \ldots, k$. First we claim that for any $D \in (0, D_{\max})$ we can take Q^* to also be uniform. With $Q^*(a_j) = 1/k$ for all j, it suffices to find $\lambda^* < 0$ satisfying (a) and (b) of Lemma 3 (part (c) is irrelevant here). We have $D_{\min}^{P,Q^*} = 0$ and

$$D_{\max}^{P,Q^*} = \sum_{i,j} \frac{1}{k}\frac{1}{k}\rho_{ij} = \frac{1}{k}\Sigma$$

where we have written Σ for the quantity $\sum_i \rho_{ij}$, which is independent of j since ρ is a permutation. Also by the permutation property, $D_{\max} = \min_j E_P[\rho(X, a_j)] = \min_j \sum_i (1/k)\rho_{ij} = (1/k)\Sigma$. Choose and fix a $D \in (0, D_{\max})$. Then $D \in (D_{\min}^{P,Q^*}, D_{\max}^{P,Q^*})$ and we can pick $\lambda^* < 0$ as in (10) so that Lemma 3 (a) holds. With this λ^* and Q^* being uniform let W^* be as in Lemma 3 (b); then

$$\sum_i W^*(a_i, a_j) = \sum_i \frac{1}{k}\frac{1}{k}\frac{e^{\lambda^* \rho_{ij}}}{\sum_{j'} \frac{1}{k} e^{\lambda^* \rho_{ij'}}} = \frac{1}{k}\sum_i \frac{e^{\lambda^* \rho_{ij}}}{\sum_{j'} e^{\lambda^* \rho_{ij'}}}.$$

But the sum in the denominator above

$$\sum_{j'} e^{\lambda^* \rho_{ij'}} \quad \text{is independent of } i \tag{14}$$

because ρ is a permutation, so $\sum_i W^*(a_i, a_j) = 1/k = Q^*(a_j)$, and (b) is satisfied. This proves that we can take Q^* to be uniform. Now simply multiplying (14) by $1/k$ we obtain (13), and this implies that $f(x) = 0$ for all $x \in A$. Since $D \in (0, D_{\max})$ was arbitrary, we are done. □

Proof of Theorem 1, (b): Let D_n, $n \geq 1$, be a sequence of of distortion values in $(0, D_{\max})$ for which $f(X_1) = 0$ with probability one, and such that $D_n \downarrow 0$. By Lemma 1 and Proposition 1, for each D_n there is a Q_n and a $\lambda_n < 0$ such that $R(D_n) = (\log e)\Lambda^*_{P,Q_n}(\lambda_n)$ and $D_n = \Lambda'_{P,Q_n}(\lambda_n)$. By (11), $\lambda_n \to -\infty$ as $n \to \infty$. Let

$$\widetilde{D} = (\min_i P_i)(\min_{i \neq j} \rho_{ij}) > 0.$$

Then for all n large enough so that $D_n < \widetilde{D}$, we must have $Q_n(a_i) > 0$ for all i (otherwise it is trivial to check that $\Lambda'_{P,Q_n}(\lambda) \geq \widetilde{D}$ for any $\lambda < 0$, contradicting the choice of λ_n). From now on we restrict attention to these large enough n's. As discussed above, $f(X_1) = 0$ with probability one if and only if condition (13) holds, which, in this case, becomes

$$\sum_{j=1}^{k} Q_n(a_j) e^{\lambda_n \rho_{ij}} \quad \text{is independent of } i. \tag{15}$$

By Lemma 3 (b) we have that for all j

$$\sum_i P_i \frac{e^{\lambda_n \rho_{ij}}}{\sum_{j'} Q_n(a_{j'}) e^{\lambda^* \rho_{ij'}}} = 1,$$

but by (15) the denominator is independent of i so

$$\sum_i P_i e^{\lambda_n \rho_{ij}} = c_n, \quad \text{independent of } j. \tag{16}$$

Since $\lambda_n \to -\infty$, letting $n \to \infty$ yields $P_j = \lim_n c_n$ for all j, so P is the uniform distribution (recall our assumption that $\rho_{ij} = 0$ if and only if $i = j$). Moreover, from (16) it follows that

$$\sum_i e^{\lambda_n \rho_{ij}} = kc_n, \quad \text{independent of } j. \tag{17}$$

To show that ρ is a permutation, fix two arbitrary indices $j \neq j'$ and reorder the vectors $(\rho_{1j}, \ldots, \rho_{kj})$ and $(\rho_{1j'}, \ldots, \rho_{kj'})$ so that their elements are nondecreasing. Let $(\sigma_1, \ldots, \sigma_k)$ and $(\sigma'_1, \ldots, \sigma'_k)$ be the corresponding ordered vectors. Then $\sigma_1 = \sigma'_1 = 0$ and (17) implies that

$$\sum_{i=2}^{k} e^{\lambda_n (\sigma_i - \sigma'_2)} = \sum_{i=2}^{k} e^{\lambda_n (\sigma'_i - \sigma'_2)}.$$

Next we show that if $\sigma_2 \neq \sigma'_2$, say $\sigma_2 > \sigma'_2$, we get a contradiction. Since $\sigma_i - \sigma'_2 > 0$ for all $i \geq 2$, as $n \to \infty$ the left-hand-side above tends to 0 but the right-hand-side is ≥ 1. Therefore $\sigma_2 = \sigma'_2$. Continuing inductively, $\sigma_i = \sigma'_i$ for all i, so $(\rho_{1j}, \ldots, \rho_{kj})$ and $(\rho_{1j'}, \ldots, \rho_{kj'})$ are permutations of one another. Since j and j' were arbitrary, this completes the proof. □

Proof of Corollary 1: As before, let D_n, $n \geq 1$, be a sequence of of distortion values in $(0, D_{\max})$ for which $f(X_1) = 0$ with probability one, and let Q_n and $\lambda_n < 0$ be chosen such that $R(D_n) = (\log e)\Lambda^*_{P,Q_n}(\lambda_n)$. Since $R(D)$ is differentiable on $(0, D_{\max})$ (see [4, Theorem 2.5.1]), from Lemma 2 we get that $\lambda_n = (\log_e 2) R'(D_n)$. Moreover, since we assume that $R(D)$ is strictly convex on $(0, D_{\max})$, the λ_n are all distinct.

If the sequence $\{\lambda_n\}$ is unbounded, i.e., it has a subsequence that tends to $-\infty$, then we can proceed exactly as in the proof of Theorem 1. So assume that the sequence $\{\lambda_n\}$ is bounded. Since for each n, $R(P, Q_n, D_n) = R(D_n) > 0$, there must be a subset S of $\{1, 2, \ldots, k\}$ of size $N = |S| \geq 2$, such that infinitely many of the Q_n are supported on $\{a_j : j \in S\}$. Without loss of generality we can relabel the elements of A so that $S = \{1, 2, \ldots, N\}$. If $N = k$ then we can again repeat the argument in the proof of Theorem 1.

Assuming $N \leq k - 1$, we proceed to get a contradiction. Since $f(x) = 0$ with probability one, condition (13) implies that

$$\sum_{j=1}^{k} Q_n(a_j) e^{\lambda_n \rho_{ij}} = \sum_{j=1}^{N} Q_n(a_j) e^{\lambda_n \rho_{ij}} = c_n, \quad \text{independent of } i.$$

Defining $\rho_{i0} = 0$ for all i and letting $T(\lambda)$ denote the $(N+1) \times (N+1)$ matrix with entries $\exp(\lambda \rho_{ij})$ for $i = 1, \ldots, N+1$ and $j = 0, 1, \ldots, N$, the above conditions imply that

$$\underbrace{\begin{pmatrix} 1 & 1 & e^{\lambda_n \rho_{12}} & \ldots & \ldots & e^{\lambda_n \rho_{1N}} \\ 1 & e^{\lambda_n \rho_{21}} & 1 & e^{\lambda_n \rho_{23}} & \ldots & e^{\lambda_n \rho_{2N}} \\ \vdots & \vdots & & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & \vdots \\ 1 & e^{\lambda_n \rho_{N1}} & \ldots & \ldots & e^{\lambda_n \rho_{N(N-1)}} & 1 \\ 1 & e^{\lambda_n \rho_{(N+1)1}} & \ldots & \ldots & \ldots & e^{\lambda_n \rho_{(N+1)N}} \end{pmatrix}}_{T(\lambda_n)} \begin{pmatrix} -c_n \\ Q_n(a_1) \\ \vdots \\ \vdots \\ Q_n(a_N) \end{pmatrix} = \mathbf{0} \in \mathbb{R}^{N+1}.$$

Therefore $\det(T(\lambda_n)) = 0$ for all λ_n. The sequence $\{\lambda_n\}$ is bounded so it must have an accumulation point, and since $\det(T(\lambda))$ is an analytic function of λ it can only have isolated zeroes unless it is identically zero (see, e.g., the discussion in [1, Section 4.3.2]). So here we must have that $\det(T(\lambda)) \equiv 0$ for all $\lambda \leq 0$. But as $\lambda \to -\infty$, $T(\lambda)$ converges to the matrix

$$T_\infty = \begin{pmatrix} 1 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & \vdots \\ 1 & 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}$$

which has determinant equal to 1 or -1, and this provides the desired contradiction. □

3.3 Proofs in the Continuous Case

Proof of Theorem 2: We argue by contradiction. Suppose $f(X_1) = 0$ with probability one for some $D \in (0, D_{\max})$. Choose a Q^* and a $\lambda^* < 0$ as in (10). Then (13) implies that

$$\sum_{j=1}^{k} Q^*(a_j) e^{\lambda^* \rho(x, a_j)} = \text{Constant} \quad \text{for } P-\text{almost all } x,$$

but since P has an absolutely continuous component with positive density on I, and since the functions $\rho_j(\cdot)$ are assumed to be continuous, this holds for all $x \in I$, and therefore contradicts the linear independence assumption of Theorem 2. □

Proof of Theorem 3: First we observe that condition (a) immediately implies condition (b). Therefore it suffices to show that if condition (b) holds, $f(X_1)$ cannot be equal to zero with probability one for distortion levels $D > 0$ arbitrarily close to zero. We proceed as in the proof of Corollary 1. Assuming that there is a sequence D_n, $n \geq 1$, of distortion values in $(0, D_{\max})$ for which $f(X_1) = 0$ with probability one, and such that $D_n \downarrow 0$, we will derive a contradiction.

Pick Q_n and $\lambda_n < 0$ such that $R(D_n) = (\log e)\Lambda^*_{P,Q_n}(\lambda_n)$. By (11), $\lambda_n \to -\infty$ as $n \to \infty$, and by (13),

$$\sum_{j=1}^{k} Q_n(a_j) e^{\lambda_n \rho_j(x)} = c_n \quad \text{for } P\text{-almost all } x \in I. \tag{18}$$

Since P has an absolutely continuous component with positive density on I, and since the functions $\rho_j(\cdot)$ are assumed to be continuous, (18) holds for all $x \in I$. In particular, for the points $x_0, \ldots, x_k$ in condition (b), (18) becomes

$$\widetilde{T}(\lambda_n)\,(-c_n, Q_n(a_1), \ldots, Q_n(a_k))' = \mathbf{0} \in \mathbb{R}^{k+1},$$

where $\widetilde{T}(\lambda)$ is the $(k+1)\times(k+1)$ matrix with entries $\exp(\lambda\rho_j(x_i))$, $0 \le i, j \le k$, and v' denotes the transpose of a vector v. Therefore, since the entries of the vector $(Q_n(a_1), \ldots, Q_n(a_k))$ sum to 1, it follows that $\det(\widetilde{T}(\lambda_n)) = 0$ for all n, or, equivalently,

$$\det(\widetilde{T}(\lambda_n)) = \sum_{\pi} (-1)^{\text{sign}(\pi)} e^{\lambda_n \sum_{j=0}^{k} \rho_j(x_{\pi(j)})} = \sum_{\pi} (-1)^{\text{sign}(\pi)} e^{\lambda_n s_\pi} = 0, \tag{19}$$

where the sums are over all permutations π of the set $\{0, 1, \ldots, k\}$, and the constants s_π are given by $\sum_{j=0}^{k} \rho_j(x_{\pi(j)})$. Therefore, for any real number $s \ge 0$, we must have that

$$\sum_{\pi\,:\,s_\pi = s} (-1)^{\text{sign}(\pi)} = 0. \tag{20}$$

To see this, let $\{s(1), s(2), \ldots\}$ be the (finite) increasing sequence of all possible values for the constants s_π. Then (19) implies that

$$\sum_{\pi\,:\,s_\pi = s(1)} (-1)^{\text{sign}(\pi)} e^{\lambda_n s(1)} + \sum_{\pi\,:\,s_\pi > s(1)} (-1)^{\text{sign}(\pi)} e^{\lambda_n s_\pi} = 0.$$

Multiplying both sides by $e^{-\lambda_n s(1)}$ and letting $n \to \infty$ yields (20) with $s = s(1)$. Continuing this way with $s(2)$, then $s(3)$ and so on, proves (20) for all s.

But now notice that condition (b) implies that, if π^* denotes the identity permutation, then $s_\pi \ne s_{\pi^*}$ for all other permutations π. Therefore, taking $s = s_{\pi^*}$ in (20) we get the desired contradiction. □

Proof of Corollary 2: Let D_n, $n \ge 1$, be a sequence of distortion values in $(0, D_{\max})$ for which $f(X_1) = 0$ with probability one, and pick Q_n and $\lambda_n < 0$ as in the proof of Theorem 3.

If the sequence $\{\lambda_n\}$ is unbounded, we can repeat the exact same proof as for Theorem 3. So assume that $\{\lambda_n\}$ is bounded. Since we also assume that $R(D)$ is differentiable and strictly convex, it follows from Lemma 2 that the $\lambda_n = (\log_e 2)R'(D_n)$ are all distinct. Proceeding as in the proof of Theorem 3, we get that $\det(\widetilde{T}(\lambda)) = 0$ for all $\lambda = \lambda_n$. The sequence $\{\lambda_n\}$ is bounded so it must have an accumulation point, and $\det(\widetilde{T}(\lambda))$ is an analytic function of λ. Therefore, arguing as in the proof of Corollary 1, $\det(T(\lambda)) \equiv 0$ for all $\lambda \le 0$. So we can find a sequence $\lambda'_m \to -\infty$ for which $\det(\widetilde{T}(\lambda'_m)) = 0$. With λ'_m in place of λ_n, the argument proceeds exactly as in the proof of Theorem 3. □

Appendix

Proof of (11): Suppose (11) is false. Then it is possible to pick a constant $K < \infty$ and a sequence of $D_n \in (0, D_{\max})$ with corresponding $\lambda_n^* < 0$, such that $D_n \to 0$ as $n \to \infty$ but $\lambda_n^* \geq -K$ for all n. Let Q_n^* achieve (9) with $D = D_n$, so that

$$\Lambda'_{P,Q_n^*}(\lambda_n^*) = D_n. \tag{21}$$

For each n, recalling that $\rho(x,y) \leq M$ for all x, y,

$$\begin{aligned} \Lambda'_{P,Q_n}(\lambda_n^*) &= E_P\left[\frac{E_{Q_n}\left(\rho(X,Y)e^{\lambda_n^*\rho(X,Y)}\right)}{E_{Q_n}\left(e^{\lambda_n^*\rho(X,Y)}\right)}\right] \\ &\geq E_P\left[E_{Q_n}\left(\rho(X,Y)e^{\lambda_n^*\rho(X,Y)}\right)\right] \\ &\geq E_{Q_n}\left[E_P\left(\rho(X,Y)e^{-KM}\right)\right] \\ &\geq e^{-KM}D_{\max}, \end{aligned}$$

which is bounded away from zero. Since the $D_n \downarrow 0$, this contradicts (21). □

Proof of Lemma 2: Part (i) immediately follows from the minimax representation in [10, Lemma 2]. For (ii) note that, since $\Lambda_{P,Q}(\lambda)$ is continuous and convex in λ (Lemma 1), $\Gamma(\lambda)$ is lower semicontinuous and convex. Then by convex duality (see, e.g., Lemma 4.5.8 in [7]), it follows that $\Gamma(\lambda) = \sup_{x\geq 0}[\lambda x - (\log_e 2)R(x)]$. For $D \in (0, D_{\max})$ and λ^* as in (10), we have

$$\Gamma(\lambda^*) = \lambda^* D - (\log_e 2)R(D) = \sup_{x\geq 0}[\lambda^* x - (\log_e 2)R(x)].$$

But since $R(\cdot)$ is convex and (by assumption) differentiable at D, it must be that the derivative of $[\lambda^* x - (\log_e 2)R(x)]$ vanishes at $x = D$, i.e., $\lambda^* = (\log_e 2)R'(D)$. □

Proof of Lemma 3: First suppose that for some $\lambda^* < 0$, (a), (b) and (c) all hold. For $i = 1, \ldots, k$, let

$$B_i = \frac{P_i}{\sum_j Q^*(a_j)e^{\lambda^*\rho_{ij}}}.$$

Then (b) and (c) imply that equations (3.19) and (3.20) in [6, p. 145] are satisfied with $\delta = -\lambda^*$, so by [6, Theorem 3.7] equation (3.18) is satisfied by W^*. This, together with Lemma 3.1 in [6, Chapter 2] imply that $R(D) = H(W \| P \times W_Y^*)$, where W_Y^* is the second marginal of W^*. But $W_Y^* = Q^*$, so $R(D) = E_P[H(W^*(\cdot|X) \| Q^*(\cdot))]$, and by the definition of W^* and Proposition 1, $E_P[H(W^*(\cdot|X) \| Q^*(\cdot))] = R(P, Q^*, D)$.

Conversely, suppose Q^* achieves the infimum in (9). Then by Lemma 1 there is a (unique) $\lambda^* < 0$ such that (a) holds, and letting W^* be defined as in (b) we also have

$$\begin{aligned} R(D) &\overset{(a)}{=} R(P, Q^*, D) \\ &\overset{(b)}{=} H(W^* \| P \times Q^*) \end{aligned}$$

$$\begin{aligned} &\overset{(c)}{=} \quad H(W^*\|P\times W_Y^*) + H(W_Y^*\|Q^*) \\ &\overset{(d)}{\geq} \quad H(W^*\|P\times W_Y^*) \\ &\overset{(e)}{\geq} \quad R(D) \end{aligned}$$

where (a) follows by assumption; (b) from (10), Proposition 1 and the definition of W^*; (c) by the chain rule for relative entropy (see [5, Theorem 2.5.3]); (d) is because relative entropy is nonnegative; and (e) follows from the definition of $R(D)$ in (8). Therefore $H(W_Y^*\|Q^*) = 0$, implying (b). Finally note that the above argument shows that W^* achieves $R(D)$. Then by Theorem 3.7 in [6, p. 145] W^* satisfies equation (3.18) of [6, p. 145] with $\delta = -\lambda^*$, and by the uniqueness of the constants B_i and equation (3.19) of [6, p. 145] we get (c). □

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