MONOTONE EMPIRICAL BAYES TESTS WITH OPTIMAL RATE OF CONVERGENCE FOR A TRUNCATION PARAMETER

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Technical Report #00-05C

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May 2000

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MONOTONE EMPIRICAL BAYES TESTS WITH OPTIMAL RATE OF CONVERGENCE FOR A TRUNCATION PARAMETER [1]

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Abstract: Monotone empirical Bayes tests for a truncation parameter are considered under a linear loss. We use a new approach to construct a monotone empirical Bayes test δ_n based on the kernel estimate and establish an upper bound for its regret Bayes risk. Also an asymptotic minimax lower bound of the monotone empirical Bayes rules is obtained. Then we find the optimal rate of convergence of the monotone empirical Bayes tests. From the results established here, we conclude that the rule δ_n is optimal in the sense that (1) it has good performance for a small sample size since it possesses weak admissibility and (2) it has good performance for a large sample size since it possesses the optimal convergence rate.

1. Introduction.

Let X denote a random variable having density function

$$f(x|\theta) = a(x)/A(\theta), \qquad 0 < x < \theta, \tag{1.1}$$

where $a(x)$ is a positive, continuous function on $(0, \infty)$, $A(\theta) = \int_0^\theta a(x)dx < \infty$ for every $\theta > 0$, θ is the parameter, which is distributed according to an unknown prior distribution G on $(0, \infty)$. The special case of (1.1) is the Uniform $(0, \theta)$, which corresponds to $a(x) \equiv 1$.

We consider the problem of testing the hypotheses $H_0 : \theta \leq \theta_0$ verses $H_1 : \theta > \theta_0$, where

[1]This research was supported in part by US Army Research Office, Grant DAAH04-95-1-0165 at Purdue University.

AMS Classification: 62C12.

Keywords: Empirical Bayes; regret Bayes risk; optimal rate of convergence, minimax.

θ_0 is a known positive constant. The loss function is $l(\theta, 0) = \max\{\theta - \theta_0, 0\}$ for accepting H_0 and $l(\theta, 1) = \max\{\theta_0 - \theta, 0\}$ for accepting H_1. A test $\delta(x)$ is defined to be a measurable mapping from $(0, \infty)$ into $[0, 1]$ so that $\delta(x) = P\{$ accepting $H_1 | X = x\}$, i.e., $\delta(x)$ is the probability of accepting H_1 when $X = x$ is observed. Let $R(G, \delta)$ denote the Bayes risk of the test δ when G is the prior distribution. Given that $E[\theta] < \infty$, a Bayes test δ_G is found as

$$\delta_G(x) = \begin{cases} 1 & \text{if } E[\theta | X = x] \geq \theta_0, \\ 0 & \text{if } E[\theta | X = x] < \theta_0. \end{cases} \tag{1.2}$$

Because $E[\theta | X = x]$ involves G, the above solution works only if the prior G is given. If G is unknown, this testing problem is formed as a compound decision problem and the empirical Bayes approach is used. Let $X_1, X_2, \cdots, X_n$ be the observations from n independent past experiences and let X be the present observation. Based on $\widetilde{X}_n = (X_1, X_2, \cdots, X_n)$ and X, an empirical Bayes rule $\delta_n(X, \widetilde{X}_n)$ can be constructed. The performance of δ_n is measured by $R(G, \delta_n) - R(G, \delta_G)$, where $R(G, \delta_n) = E[R(G, \delta_n | \widetilde{X}_n)]$. The quantity $R(G, \delta_n) - R(G, \delta_G)$ is referred as the regret Bayes risk (or regret) in the literature.

This approach was introduced by Robbins (1956, 1964). Since then, it has been widely used in statistics. For the distribution family having density (1.1), Gupta and Hsiao (1983) considered the empirical Bayes rule in the selection problem formulation. Later Datta (1991) studied the empirical Bayes rule in the estimation problem formulation.

The case of X $\sim$Uniform $(0, \theta)$ was studied by many authors; see Fox (1978), Van Houwelingen (1987), Nogami (1988), Liang (1990) and Karunamuni(1999). Van Houwelingen (1987) constructed the monotone empirical Bayes test using Grenander's estimator and investigated the rate of convergence of its regret and the limiting distribution of its conditional regret. Liang (1990) also considered another monotone empirical Bayes test and got a better rate. Karunamuni (1999) studied the empirical Bayes test under more general assumptions and obtained more general results. He also tried to find a minimax lower bound of monotone empirical Bayes tests. Unfortunately, a few errors cast doubt on the validity of his result.

In this paper, we study the empirical Bayes test for the problem described above. We use a new approach to construct a monotone empirical Bayes test δ_n based on kernel estimate and establish an upper bound for its regret Bayes risk. We demonstrate that the convergence rate is determined by the order of smoothness of $a(x)$ and $G(\theta)$. The choice of the kernel function does not help much in improving the rate of convergence. This is totally different from the case of the empirical Bayes rules for the exponential family (See Gupta and Li (1999)). Also, an asymptotic minimax lower bound of the monotone empirical Bayes rules is obtained by borrowing an idea from Donoho and Liu (1991a, b). It is proved that δ_n achieves the optimal rate of convergence among all monotone empirical Bayes tests. So the rule δ_n is optimal in the sense that (1) it has good performance for a small sample size since it has weak admissibility and (2) it has good performance for a large sample size since it achieves the optimal rate of convergence.

The rest of the paper is organized as follows: In Section 2 we introduce a few preliminary results. In Section 3 we show the construction of the monotone empirical Bayes test δ_n. The asymptotic upper bound of the regret Bayes risk of δ_n is presented in Section 4. In Section 5, we obtain a minimax lower bound for the regret and then show that δ_n achieves the optimal rate of convergence. The proofs of main results in Section 4 and 5 are given in Section 6. In the appendix, we provide the proofs of a few lemmas used in the previous sections.

2. Preliminary. We assume $P(\theta \leq \theta_0) < 1$ and $P(\theta > \theta_0) < 1$ so that the testing problem is meaningful. This will not be mentioned further in this paper. Also to ensure that the Bayes analysis can be carried out, we assume $\int_0^\infty \theta dG(\theta) < \infty$ without further mention. Let $f_G(x) = a(x)\int_0^\infty dG(\theta)/A(\theta)$ be the marginal density of X and $\phi_G(x) = E[\theta|X = x]$ be the posterior mean of θ given $X = x$. Denote $\alpha_G(x) = \int_0^\infty dG(\theta)/A(\theta)$ and $w(x) = \alpha_G(x)[\theta_0 - \phi_G(x)]$. The existence of $\phi_G(x)$ and $\alpha_G(x)$ is justified by $\int_0^\infty \theta dG(\theta) < \infty$. Note that $\alpha_G(x) > 0$ and $\phi_G(x)$ is increasing for $x > 0$, the Bayes rule, stated in Section 1, can be represented as

$$\delta_G(x) = \begin{cases} 1 & \text{if } \phi_G(x) \geq \theta_0 \Longleftrightarrow w(x) \leq 0 \Longleftrightarrow x \geq c_G, \\ 0 & \text{if } \phi_G(x) < \theta_0 \Longleftrightarrow w(x) > 0 \Longleftrightarrow x < c_G, \end{cases} \tag{2.1}$$

where $c_G = \inf\{x > 0 : \phi_G(x) = \theta_0\}$. c_G is called the critical point corresponding to G. Noting that the Bayes rule δ_G is characterized by a single number c_G, a monotone empirical Bayes test (MEBT) can be constructed through estimating c_G by $c_n(X_1, X_2, \cdots, X_n)$, say, and defining

$$\delta_n = \begin{cases} 1 & \text{if} \quad x \geq c_n, \\ 0 & \text{if} \quad x < c_n. \end{cases} \tag{2.2}$$

Then the regret of δ_n is

$$R(G, \delta_n) - R(G, \delta_G) = E\int_{c_n}^{c_G} w(x)a(x)dx. \tag{2.3}$$

To consider the rate of convergence of $R(G,\delta_n)-R(G,\delta_G)$, we assume that for some $r\geq 1$, $\alpha_G(x)$ is r-times continuously differentiable and for $i=0,1,\cdots,r$,

$$\sup_{0<x\leq\theta_0+1}|\alpha_G^{(i)}(x)|\leq B_r<\infty. \tag{2.4}$$

Furthermore, we assume that

$$g(c_G)=G'(c_G)\neq 0. \tag{2.5}$$

3. Construction of Monotone Empirical Bayes Test. The kernel method is used to construct the empirical Bayes test. Let $K_0(y)$ be a Borel-measurable, bounded function vanishing outside the interval $[0,1]$ such that

$$\int_0^1 y^j K_0(y)dy=\begin{cases}1 & \text{if} \quad j=0,\\ 0 & \text{if} \quad j=1,2,\cdots,r-1,\end{cases} \tag{3.1}$$

with $B_{1k}=\frac{1}{r!}\int_0^1 y^r|K_0(y)|dy<\infty$ and $B_{2k}=\int_0^1[K_0(y)]^2dy<\infty$. Let $B_k>0$ be a number such that $|K_0(y)|\leq B_k$ for $y\in[0,1]$. This type of kernel function has been used by many authors. Denote $K_1(y)=\int_0^y K_0(s)ds$ and $u_n=n^{-1/(2r+1)}$. For any $x\in(0,\infty)$, define

$$W_n(x)=\frac{\theta_0-x}{nu_n}\sum_{j=1}^n\frac{K_0(\frac{X_j-x}{u_n})}{a(X_j)}-\frac{1}{n}\sum_{j=1}^n\frac{K_1(\frac{X_j-x}{u_n})}{a(X_j)}. \tag{3.2}$$

It is shown later that $W_n(x)$ is an asymptotically unbiased and consistent estimator of $w(x)$. Note that $c_G=\int_0^{\theta_0}I_{[w(x)>0]}dx$. Let

$$c_n=\int_{d_n}^{\theta_0}I_{[W_n(x)>0]}dx+d_n, \tag{3.3}$$

where $d_n=0$ if $a(0+)>0$ and $d_n=\max\{x\leq\theta_0: a(x)<\sqrt{u_n}\}$ if $a(0+)=0$. Note that $d_n\to 0$. Naturally, we propose a monotone empirical Bayes test $\delta_n(x)$ by

$$\delta_n=\begin{cases}1 & \text{if} \quad x\geq c_n,\\ 0 & \text{if} \quad x<c_n.\end{cases} \tag{3.4}$$

Then as $d_n<c_G$,

$$c_n-c_G=-\int_{d_n}^{c_G}I_{[W_n(x)\leq 0]}dx+\int_{c_G}^{\theta_0}I_{[W_n(x)>0]}dx. \tag{3.5}$$

Remark 3.1. The construction of c_n is the key to improve the rate of convergence of the empirical Bayes rule. Van Houwelingen (1987) constructed $\phi_n(x)$, an estimator of $\phi_G(x)$, and then let $c_n=\sup\{x: x>0, \phi_n(x)<\theta_0\}$. It is interesting that he proved that either

$c_n = \theta_0$ or $c_n =$ *the smallest minimizer of* $[1 - F_n(x)]/[\theta_0 - x]$, where $F_n(x)$ is the empirical c.d.f. of $X_1, \cdots, X_n$. Liang (1990) used a similar method to construct c_n in an elegant way. Karunamuni (1999) defined $c_n = \inf\{x \in (0, \theta_0), 1 - \hat{F}_n(x) + \hat{f}(x)(x - \theta_0)\}$, where $\hat{f}(x)$ and $\hat{F}(x)$ are the kernel estimators of $f_G(x)$ and its c.d.f. $F_G(x)$. In this paper, we use c_n as defined by (3.3). The existence of c_n is obvious. Moreover, the quantity $c_n - c_G$ can be expressed clearly by $W_n(x)$ through (3.5) as $d_n < c_G$. So the rate of convergence of $c_n - c_G$ is easy to be investigated through the property of $W_n(x)$. This idea was used by Brown, Cohen and Strawderman (1976) to prove the completeness of the monotone rules. Van Houwelingen(1976) and Stijnen (1985) used a similar idea to study the monotone empirical Bayes rules for the continuous exponential family.

Remark 3.2. From (3.5), we see that, for fixed n, we only need to consider the property of $W_n(x)$ and $w(x)$ on $[d_n, \theta_0]$ by introducing d_n. If $a(0+) = 0$, the use of d_n allows us to efficiently control the quantity $a(x)$. Here "efficiently" means that we control $a(x)$ through u_n so that the values of $a(x)$ are in a tolerable range. This localization technique helps us to get a better bound of the regret Bayes risk. It will be much clearer after Remark 6.3.

4. Rate of convergence of δ_n.

In this section, we will investigate the performance of δ_n by finding the convergence rate of the regret Bayes risk of δ_n and obtaining a rate of uniform convergence over some class of prior distributions.

From $P(\theta > \theta_0) > 1$, we know $c_G < \theta_0$(See Van Honwelingen (1987)). Also, from (2.4) and (2.5), we know $c_G > 0$. Therefore $0 < c_G < \theta_0$. Suppose that $\epsilon_G > 0$ satisfies $0 < c_G - \epsilon_G < c_G + \epsilon_G < \theta_0$. From (2.3),

$$R(G, \delta_n) - R(G, \delta_G) \leq E[I_{[|c_n - c_G| > \epsilon_G]} \int_{c_n}^{c_G} w(x)a(x)dx] + E[I_{[|c_n - c_G| \leq \epsilon_G]} \int_{c_n}^{c_G} w(x)a(x)dx].$$

It is easy to see that $E[I_{[|c_n - c_G| > \epsilon_G]} \int_{c_n}^{c_G} w(x)a(x)dx] \leq (\theta_0 + E[\theta])\epsilon_G^{-4} E(c_n - c_G)^4$ since $\int_{c_n}^{c_G} w(x)a(x)dx \leq (\theta_0 + E[\theta])$ and $I_{[|c_n - c_G| \leq \epsilon_G]} \int_{c_n}^{c_G} w(x)a(x)dx \leq \bar{a} I_{[|c_n - c_G| \leq \epsilon_G]} \int_{c_n}^{c_G} w(x)dx$, where $\bar{a} = \max\{a(x) : x \in [c_G - \epsilon_G, c_G + \epsilon_G]\}$. (2.4) indicates that $w(x)$ is differentiable and therefore $w'(x) = g(x)(x - \theta_0)/A(x)$. Using Tayler expansion of $\int_{c_n}^{c_G} w(x)dx$ at c_G,

$$I_{[|c_n - c_G| \leq \epsilon_G]} \int_{c_n}^{c_G} w(x)dx = -\frac{1}{2} \times w'(\hat{c}_n)(c_n - c_G)^2 I_{[|c_n - c_G| \leq \epsilon_G]}, \tag{4.1}$$

where $\hat{c}_n$ is an intermediate value between c_n and c_G. Let $\bar{w} = \max\{-\frac{1}{2}w'(x) : x \in [c_G -$

$\epsilon_G, c_G + \epsilon_G]\}$. Then

$$R(G,\delta_n) - R(G,\delta_G) \le (\theta_0 + E[\theta])\epsilon_G^{-4} E(c_n - c_G)^4 + \bar{a}\bar{w}E(c_n - c_G)^2. \tag{4.2}$$

The upper bound of the regret Bayes risk is controlled by $E(c_n - c_G)^2$ and $E(c_n - c_G)^4$. We investigate them and have the following results.

Proposition 4.1. Let $M_{r,k} = [4B_r B_{1k} a(c_G)]^2 + 3(B_{2k}/B_{1k})^2$. Then

$$\lim_{n\to\infty} n^{\frac{2r}{2r+1}} E(c_n - c_G)^2 \le M_{r,k}\theta_0^2[a(c_G)w'(c_G)]^{-2} \quad \text{and} \quad \lim_{n\to\infty} n^{\frac{2r}{2r+1}} E(c_n - c_G)^4 = 0. \tag{4.3}$$

The proof of Proposition 4.1 is given in Section 6. From (4.2) and (4.3),

$$\lim_{n\to\infty} n^{\frac{2r}{2r+1}}[R(G,\delta_n) - R(G,\delta_G)] \le \bar{a}\bar{w}M_{r,k}\theta_0^2[a(c_G)w'(c_G)]^{-2}.$$

Note that the left-hand-side of the previous inequality is independent of ϵ_G and $\bar{a}\bar{w} \to \frac{1}{2}a(c_G)|w'(c_G)|$ as $\epsilon_G \to 0$. Therefore we have the following theorem.

Theorem 4.2. *Let $M_{r,k}$ be the same as defined in Proposition 4.1. Then*

$$\lim_{n\to\infty} n^{\frac{2r}{2r+1}}[R(G,\delta_n) - R(G,\delta_G)] \le M_{r,k}\theta_0^2/[2a(c_G)|w'(c_G)|]. \tag{4.4}$$

Remark 4.3. The rate of δ_n depends on the regularity of $\alpha_G(x)$ or the regularity of $a(x)$ and $G(\theta)$ through (2.4). The order r of the regularity determines the order of the convergence rate. The value of r may be obtained from the experiment. For the uniform distribution (which is the case of $a(x) = 1$ here), Van Houwelingen (1987) and Liang (1990) constructed the empirical Bayes rule for $r = 1$. Karunamuni(1999) studied it for $r \ge 1$. So in terms of the convergence rate, (4.4) is a generalization of their results with general $a(x)$ and $r \ge 1$.

Next we consider the uniform convergence rate of δ_n over a class of prior distributions. Denote

$$\mathcal{G} = \{G : G \text{ satisfies } \int_0^\infty \theta dG(\theta) < \infty, (2.4), c_0 \le c_G \le \theta_0, \min_{x\in[c_0,\theta_0]} |w'(x)| \ge L\}, \tag{4.5}$$

where $c_0 = 0$ if $a(0+) > 0$ and $c_0 > 0$ if $a(0+) = 0$, $L > 0$. We assume in the following that L is properly chosen so that $\mathcal{G}$ is not empty.

Theorem 4.4. *Let* $M'_{r,k} = [4B_r B_{1k} A(\theta_0)]^2 + 3(B_{2k}/B_{1k})^2$ *and* $\underline{a} = \min\{a(x) : x \in (c_0, \theta_0]\}$. *Then*

$$\lim_{n\to\infty} n^{\frac{2r}{2r+1}} \sup_{G\in\mathcal{G}} [R(G,\delta_n) - R(G,\delta_G)] \le M'_{r,k}\theta_0^2/[2\underline{a}L]. \tag{4.6}$$

Remark 4.5. Theorem 4.4 considers the rate of uniform convergence. We require that the uniform lower bound of c_G be a positive number in case of $a(0+) = 0$. For single G, we use d_n to keep the estimator $W_n(x)$ away from blowing up rapidly at $x = 0$. For $\mathcal{G}$, a set of G, we have to specify a uniform lower bound for c_G. For $a(0+) > 0$, $W_n(x)$ has a relatively small variance at $x = 0$, and it converges to $w(x)$ in a tolerable manner. It is allowed that $c_0 = 0$. This will be clearer after Remark 6.3.

Remark 4.6. From (4.6), we see that an upper bound of the monotone empirical Bayes tests over $\mathcal{G}$ is $ln^{-2r/(2r+1)}$ for some $l > 0$. $O(n^{-2r/(2r+1)})$ is the slowest achievable order of uniform convergence rate. In the following, we will see that this is also the fastest we can hope.

5. Asymptotic Minimax Property of δ_n. We shall obtain a minimax lower bound for the regrets of all monotone empirical Bayes tests first. Then we show that δ_n achieves this minimax lower bound within a constant.

As presented in Section 2, the problem of constructing a monotone empirical Bayes rule is essentially to find an estimator c_n^* of c_G, a functional of the marginal distribution $f_G(x)$ of X, based on the i.i.d. sample $X_1, \cdots, X_n$. Donoho and Liu (1991a, b) found that a minimax lower bound of c_n^* going to c_G can be obtained by finding the best possible lower bound of the hardest two-point subproblem. And they found the lower bounds for many interesting problems. Fan (1991) found the lower bounds (as well as the optimal rates) for various deconvolution problems. This approach was tried by Karunamuni (1999) for the empirical Bayes test problem in the uniform distribution family.

In the following parts of this paper, $l_1, l_2, \cdots$ stand for the positive constants, which may have different values on different occasions.

Let $\mathcal{C}$ be the set of all estimators c_n^* with $c_n^* \ge 0$ and let $\mathcal{D}$ be the set of all empirical Bayes rules of type (2.2) with $c_n = c_n^* \in \mathcal{C}$. Denote $\bar{\mathcal{C}} = \{\bar{c}_n = c_n^* \vee c_0 \wedge \theta_0 : c_n^* \in \mathcal{C}\}$. For $c_n^* \in \mathcal{C}$, $\bar{c}_n = c_n^* \vee c_0 \wedge \theta_0 \in \bar{\mathcal{C}}$. Then

$$\int_{c_n^*}^{c_G} w(x)a(x)dx \ge \int_{\bar{c}_n}^{c_G} w(x)a(x)dx \ge \underline{a}\int_{\bar{c}_n}^{c_G} w(x)dx = -\frac{\underline{a}}{2} w'(\hat{\bar{c}}_n)(\bar{c}_n - c_G)^2, \tag{5.1}$$

where $\underline{a}$ is the same as defined in Theorem 4.4, and $\hat{\bar{c}}_n$ is an intermediate value between $\bar{c}_n$ and c_G. Clearly, $\hat{\bar{c}}_n \in [c_0, \theta_0]$. Therefore

$$|w'(\hat{\bar{c}}_n)| \geq L. \tag{5.2}$$

From (5.1) and (5.2), we obtain $\int_{c_n^*}^{c_G} w(x)a(x)dx \geq l_1(\bar{c}_n - c_G)^2$. The previous inequality leads to

$$\inf_{c_n^* \in \mathcal{C}} \sup_{G \in \mathcal{G}} E[\int_{c_n^*}^{c_G} w(x)a(x)dx] \geq l_2 \inf_{\bar{c}_n \in \bar{\mathcal{C}}} \sup_{G \in \mathcal{G}} E(\bar{c}_n - c_G)^2. \tag{5.3}$$

Note that $\bar{\mathcal{C}} \subset \mathcal{C}$. It holds that

$$\inf_{\bar{c}_n \in \bar{\mathcal{C}}} \sup_{G \in \mathcal{G}} E(\bar{c}_n - c_G)^2 \geq \inf_{c_n^* \in \mathcal{C}} \sup_{G \in \mathcal{G}} E(c_n^* - c_G)^2. \tag{5.4}$$

From the results in Donoho and Liu (1991a) (Theorem 3.1 and the remark after Lemma 3.3),

$$\inf_{c_n^* \in \mathcal{C}} \sup_{G \in \mathcal{G}} E(c_n^* - c_G)^2 \geq l_1 \sup\{(c_{f_1} - c_{f_2})^2 : \int_0^\infty [\sqrt{f_1(x)} - \sqrt{f_2(x)}]^2 dx \leq \frac{l_2}{n}, \forall f_1, f_2 \in \mathcal{F}\}, \tag{5.5}$$

where $\mathcal{F} = \{f_G(x) = a(x)\int_x^\infty dG(\theta)/A(\theta) : G \in \mathcal{G}\}$ and c_{f_1}, c_{f_2} are the critical points corresponding to f_1 (or G_1) and f_2 (or G_2) respectively. Next we shall prove for large n

$$\sup\{(c_{f_1} - c_{f_2})^2 : \int_0^\infty (\sqrt{f_1(x)} - \sqrt{f_2(x)})^2 dx \leq \frac{l_2}{n}, \forall f_1, f_2 \in \mathcal{F}\} \geq l_3 n^{-\frac{2r}{2r+1}}. \tag{5.6}$$

This can be done by identifying the hardest two-point subproblem. That is, we prove (5.6) by finding f_1, $f_2 \in \mathcal{F}$ such that

$$\int_0^\infty [\sqrt{f_1(x)} - \sqrt{f_2(x)}]^2 dx \leq \frac{l_2}{n} \quad \text{and} \quad (c_{f_1} - c_{f_2})^2 \geq l_3 n^{-\frac{2r}{2r+1}}. \tag{5.7}$$

Suppose that $G_1 \in \mathcal{G}$ with density $g_1(\theta)$ and $c_{f_1} \in (c_0, \theta_0)$, where c_{f_1} is the critical point corresponding to G_1. Let $g_2(\theta) = (1 + u_n^r \mu_n)^{-1}[g_1(\theta) + u_n^{r-1} A(\theta) H(\frac{\theta - c_{f_1}}{u_n})] I_{[\theta > 0]}$, where $\mu_n = \int_{-1}^1 A(c_{f_1} + tu_n) H(t) dt$ and $H(t)$ is a function such that (1) it has support $[-1, 1]$, (2) $\int_{-1}^1 H(t)dt = 0$ and $\int_0^1 H(t)dt \neq 0$, and (3) it has bounded derivatives upto order r. Let $f_i(x) = a(x)\int_x^\infty \frac{g_i(\theta)}{A(\theta)} d\theta$ for $i = 1, 2$. It is easy to see that $c_0 < c_{f_1} - u_n < c_{f_1} + u_n < \theta_0$ and $g_2(\theta) \geq 0$ as n is large. We assume $c_0 < c_{f_1} - u_n < c_{f_1} + u_n < \theta_0$ and $g_2(\theta) \geq 0$ for any $n \geq 1$ without loss of generality. Let c_{f_2} be the critical point corresponding to $g_2(\theta)$. As $n \to \infty$, $c_{f_2} \to c_{f_1}$ since $g_2(\theta) \to g_1(\theta)$. We assume $c_{f_2} \in (c_0, \theta_0)$ without loss of generality. Then it is easy to see that $f_2 \in \mathcal{F}$. In Section 6 we prove that (5.7) holds for such f_1 and f_2.

Theorem 5.1. *For some $l > 0$,*

$$\inf_{\delta_n^* \in \mathcal{D}} \sup_{G \in \mathcal{G}} [R(G, \delta_n^*) - R(G, \delta_G)] \geq l n^{-\frac{2r}{2r+1}}. \tag{5.8}$$

Remark 5.2. Theorem 5.1 says that $n^{-\frac{2r}{2r+1}}$ is the best possible order of the rate of convergence. None of the monotone empirical Bayes tests can beat this order. From Theorem 4.4, we see that δ_n attains this order of the rate of convergence.

Remark 5.3. From Theorem 4.4 and Theorem 5.1, we conclude that δ_n has good performance for a large sample size. Since it is a monotone rule, it has weak admissibility. Thus it has a good performance for a small sample size.

6. Proofs of Proposition 4.1 and (5.7). Note that $w'(x) = g(x)(x-\theta_0)/A(x)$. Then $w'(c_G) < 0$ since $g(c_G) > 0$ and $0 < c_G < \theta_0$. Let $A_\epsilon = \min\{-w'(x) : c_G - \epsilon \leq x \leq c_G + \epsilon\}$ for $\epsilon > 0$. Then, as $\epsilon \to 0$,

$$A_\epsilon \to -w'(c_G) = \frac{g(c_G)}{A(c_G)}(\theta_0 - c_G) > 0. \tag{6.1}$$

Assume that ϵ_0 is a number such that $0 < c_G - \epsilon < c_G + \epsilon < \theta_0$ and $A_\epsilon > 0$ for all $\epsilon < \epsilon_0$. Let $\eta_1 = c_G - \epsilon$ and $\eta_2 = c_G + \epsilon$ for $\epsilon < \epsilon_0$.

Lemma 6.1. *The following statements hold.*

(i) *For* $x \in (0, \theta_0]$, $|w(x)| \leq 2\theta_0 B_r$.

(ii) *For* $x \in (0, \eta_1) \cup (\eta_2, \theta_0)$, $|w(x)| \geq \epsilon A_\epsilon$ *as* $\epsilon < \epsilon_0$.

Note that $W_n(x) = \frac{1}{n}\sum_{j=1}^n V_n(X_j, x)$, where

$$V_n(X_j, x) = \frac{\theta_0 - x}{u_n} \times \frac{K_0(\frac{X_j - x}{u_n})}{a(X_j)} - \frac{K_1(\frac{X_j - x}{u_n})}{a(X_j)}. \tag{6.2}$$

Let $w_n(x) = E[V_n(X_j, x)]$, $Z_{jn} = V_n(X_j, x) - w_n(x)$ and $\sigma_n^2 = EZ_{jn}^2$. Then Z_{jn} are i.i.d. with mean 0 for fixed x and n.

Lemma 6.2. *Let* $p_{n,\epsilon} = \max\{1/a(x) : x \in (\eta_1, \eta_2 + u_n)\}$. *Denote* $C_n(x) = (\theta_0 - x + u_n)B_r B_{1k} u_n^r$ *and* $D_n(x) = (\theta_0 - x)^2 B_{2k}\alpha_G(x) + u_n[2(\theta_0 - x) + u_n]B_k^2\alpha_G(x)$ *for* $x \in (0, \theta_0)$. *Let* γ *be some positive constant. Then we have the following:*

(i) *For all* $x \in (0, \theta_0)$, $|w_n(x) - w(x)| \leq C_n(x)$.

(ii) *For* $x \in [\eta_1, \eta_2]$, $E[|Z_{jn}|^3] \leq \gamma p_{n\epsilon}^2 u_n^{-2}$ *and* $\sigma_n^2 \leq D_n(\eta_1)p_{n,\epsilon}u_n^{-1}$.

(iii) *For* $x \in [d_n, \theta_0]$, $E[|Z_{jn}|^3] \leq \gamma u_n^{-3}$ *and* $\sigma_n^2 \leq D_n(x)u_n^{-3/2}$.

(iv) *For* $x \in (0, \theta_0)$, $w(x) > 2C_n(x) \Longrightarrow w_n(x) \geq \frac{1}{2}w(x)$.

(v) *For* $x \in (0, \theta_0)$, $w(x) < -2C_n(x) \Longrightarrow w_n(x) \le -\frac{1}{2}w(x)$.

Remark 6.3. For the case of $a(0+) > 0$, $a(x)$ has a lower bound at $(0, \theta_0]$. Therefore $E[|Z_{jn}|^3] \sim u_n^{-2}$, and $\sigma_n^2 \sim u_n^{-1}$ for any $x \in (0, \theta_0]$. For the case of $a(0+) = 0$, if x is near 0, $|Z_{jn}|$ and σ_n^2 become infinite even for fixed n in most situations. Therefore, we have to isolate 0 progressively through a positive sequence d_n, so that $|Z_{jn}|$ and σ_n^2 go to infinity in a controlled way. When the uniform convergence rate is considered, we have to specify a lower uniform bound for c_G over $\mathcal{G}$.

Proof of Proposition 4.1. Noting that $c_G > 0$, we assume that $d_n < c_G$ for all n without loss of generality. From (3.5), $c_n - c_G = -\int_{d_n}^{c_G} I_{[W_n(x)\le 0]}dx + \int_{c_G}^{\theta_0} I_{[W_n(x)>0]}dx$ and

$$E(c_n - c_G)^2 \le E[\int_{d_n}^{c_G} I_{[W_n(x)\le 0]}dx]^2 + E[\int_{c_G}^{\theta_0} I_{[W_n(x)>0]}dx]^2 \equiv r_{1n} + r_{2n}. \tag{6.3}$$

It turns out by Holder inequality and a little algebra that

$$r_{1n} \le E[\int_{d_n}^{\eta_1 \vee d_n} I_{[W_n(x)\le 0]}dx + \int_{\eta_1 \vee d_n}^{c_G} I_{[W_n(x)\le 0]}dx]^2 \le 2c_G I_1 + 2I_2 + 2I_3, \tag{6.4}$$

where $\eta_1 = c_G - \epsilon$, $\epsilon < \epsilon_0$, $I_1 = [\int_{d_n}^{\eta_1 \vee d_n} P(W_n(x) \le 0)dx$, $I_2 = [\int_{\eta_1}^{c_G} I_{[w(x)\le 2C_n(\eta_1)]}dx]^2$, $I_3 = E[\int_{\eta_1}^{c_G} I_{[W_n(x)\le 0, w(x)>2C_n(\eta_1)]}dx]^2$. For $w(x) > 2C_n(x)$, $w_n(x) > 1/2w(x)$ from (iv) of Lemma 6.2. Then we have

$$P(W_n(x) \le 0) = P(\frac{1}{\sqrt{n\sigma_n^2}}\sum_{j=1}^n Z_{jn} \le \frac{-\sqrt{n}w_n(x)}{\sigma_n}) \le P(\frac{1}{\sqrt{n\sigma_n^2}}\sum_{j=1}^n Z_{jn} \le \frac{-\sqrt{n}w(x)}{2\sigma_n}).$$

Applying Theorem 14 on page 125 in Petrov (1975) to the left-hand-side of the above inequality,

$$P(W_n(x) \le 0) \le \Phi(-\frac{\sqrt{n}w(x)}{2\sigma_n}) + \frac{8AE|Z_{jn}|^3}{\sqrt{n}[2\sigma_n + \sqrt{n}w(x)]^3} = S_n(x) + T_n(x), \tag{6.5}$$

where A is a constant. For $x \in [d_n, \eta_1 \vee d_n]$, $w(x) \ge \epsilon A_\epsilon$ and certainly $w(x) > 2C_n(x)$ as n is large. Also note that $\sigma_n^2 \le D_n(x)u_n^{-3/2}$ and $E[|Z_{jn}|^3] \le \gamma u_n^{-3}$. It follows that $S_n(x) \le \Phi(-n^{1/3})$ and $T_n(x) = 1/(n\sqrt{u_n})$ as n is large. Thus

$$I_1 = \int_{d_n}^{c_1} P(W_n(x) \le 0)dx = o(n^{-2r/(2r+1)}). \tag{6.6}$$

For $x \in [\eta_1, c_G]$, $|w'(x)| \ge A_\epsilon$. Thus $I_2 \le A_\epsilon^{-2}[\int_{\eta_1}^{c_G} I_{[w(x)\le 2C_n(\eta_1)]}w'(x)dx]^2$. Letting $y = w(x)/[2C_n(\eta_1)]$, $I_2 \le 4A_\epsilon^{-2}[C_n(\eta_1)]^2 \int_0^\infty I_{[y\le 1]}dy = 4A_\epsilon^{-2}[C_n(\eta_1)]^2$. Therefore

$$\lim_{\epsilon\to 0}\lim_{n\to\infty} n^{\frac{2r}{2r+1}} I_2 \le [\frac{2B_r B_{1k}(\theta_0 - c_G)}{w'(c_G)}]^2, \tag{6.7}$$

where it is used that $n^{r/(2r+1)}C_n(\eta_1) \to (\theta_0 - \eta_1)B_r B_{1k}$ as $n \to \infty$ and $\eta_1 \to c_G$, $A_\epsilon \to -w'(c_G)$ as $\epsilon \to 0$. By Holder inequality again,

$$I_3 \le \int_{\eta_1}^{c_G} P(W_n(x) \le 0) w^3(x) I_{[w(x)>2C_n(\eta_1)]} dx \times \int_{\eta_1}^{c_G} w^{-3}(x) I_{[w(x)>2C_n(\eta_1)]} dx.$$

Letting $y = w(x)/[2C_n(\eta_1)]$, $\int_{\eta_1}^{c_G} w^{-3}(x) I_{[w(x)>2C_n(\eta_1)]} dx \le 1/[8A_\epsilon C_n^2(\eta_1)]$. Using the previous two inequalities and (6.5), we have

$$I_3 \le \frac{1}{8A_\epsilon C_n^2(\eta_1)}[\int_{\eta_1}^{c_G} S_n(x) w^3(x) dx + \int_{\eta_1}^{c_G} T_n(x) w^3(x) dx]. \tag{6.8}$$

For $x \in [\eta_1, c_G]$, $\sigma_n \le \sqrt{D_n(\eta_1) p_{n,\epsilon} u_n^{-1}}$, $E[|Z_{jn}|^3] \le \gamma p_{n,\epsilon}^2 u_n^{-2}$ and $w(x) > 2C_n(\eta_1)$. Therefore

$$\int_{\eta_1}^{c_G} S_n(x) w^3(x) dx \le \frac{1}{A_\epsilon}\int_{\eta_1}^{c_G} \Phi(-\frac{\sqrt{nu_n} w(x)}{2\sqrt{p_{n,\epsilon} D_n(\eta_1)}}) w^3(x) dw(x) \le \frac{6[p_{n,\epsilon} D_n(\eta_1)]^2}{A_\epsilon (nu_n)^2}, \tag{6.9}$$

and

$$\int_{\eta_1}^{c_G} T_n(x) w^3(x) dx \le \frac{8A\gamma p_{n,\epsilon}^2}{(nu_n)^2}\epsilon. \tag{6.10}$$

Combining (6.8), (6.9) and (6.10), we have

$$\lim_{\epsilon\to 0}\lim_{n\to\infty} n^{\frac{2r}{2r+1}} I_3 \le \frac{3}{4}\cdot[\frac{B_{2k}(\theta_0 - c_G)}{B_{1k} a(c_G) w'(c_G)}]^2, \tag{6.11}$$

where it is used that $p_{n,\epsilon} \to p_\epsilon = \max\{1/a(x) : \eta_1 \le x \le \eta_2\}$ and $D_n(\eta_1) \to (\theta_0 - \eta_1)^2 B_{2k} \alpha_G(\eta_1)$ as $n \to \infty$, and $p_\epsilon \to 1/a(c_G)$, $\eta_1 \to c_G$ as $\epsilon \to 0$. From (6.4), (6.6), (6.7) and (6.11), we obtain

$$\lim_{n\to\infty} n^{\frac{2r}{2r+1}} r_{1n} \le 0.5 M_{r,k}\theta_0^2[a(c_G)w'(c_G)]^{-2}. \tag{6.12}$$

By symmetry, we have a similar result for r_{2n}. Therefore

$$\lim_{n\to\infty} n^{\frac{2r}{2r+1}} E(c_n - c_G)^2 \le M_{r,k}\theta_0^2[a(c_G)w'(c_G)]^{-2}. \tag{6.13}$$

To prove (4.3), we need to show next that $E(c_n - c_G)^4 = o(n^{-2r/(2r+1)})$. Note that

$$\begin{aligned} E(c_n - c_G)^4 &\le l \times \{[\int_{d_n}^{\eta_1 \vee d_n} P(W_n(x) \le 0) dx] + [\int_{\eta_2}^{\theta_0} P(W_n(x) > 0) dx] \\ &+ [\int_{\eta_1}^{c_G} I_{[|w(x)| \le 2C_n(\eta_1)]} dx]^4 + [\int_{c_G}^{\eta_2} I_{[|w(x)| \le 2C_n(\eta_1)]} dx]^4 \\ &+ [\int_{\eta_1}^{c_G} I_{[W_n(x) \le 0, w(x) > 2C_n(\eta_1)]} dx]^4 + [\int_{c_G}^{\eta_2} I_{[W_n(x) > 0, w(x) < -2C_n(\eta_1)]} dx]^4\}. \end{aligned}$$

Similar to (6.6), $\int_{d_n}^{\eta_1 \vee d_n} P(W_n(x) \le 0) dx = o((nu_n)^{-1})$, $\int_{\eta_2}^{\theta_0} P(W_n(x) > 0) dx = o((nu_n)^{-1})$. Similar to (6.7), $[\int_{\eta_1}^{c_G} I_{[|w(x)| \le 2C_n(\eta_1)]} dx]^4 = o((nu_n)^{-1})$, $[\int_{c_G}^{\eta_2} I_{[|w(x)| \le 2C_n(\eta_1)]} dx]^4 = o((nu_n)^{-1})$. So to prove $E(c_n - c_G)^4 = o(n^{-\frac{2r}{2r+1}})$, it is sufficient to show that

$$E[\int_{\eta_1}^{c_G} I_{[W_n(x) \le 0, w(x) > 2C_n(\eta_1)]} dx]^4 = o(n^{-\frac{2r}{2r+1}}), \tag{6.14}$$

and

$$E[\int_{c_G}^{\eta_2} I_{[W_n(x)>0,w(x)<-2C_n(\eta_1)]}dx]^4 = o(n^{-\frac{2r}{2r+1}}). \tag{6.15}$$

We only prove (6.14) here. (6.15) can be obtained by symmetry. By Holder inequality,

$$\begin{aligned}
& E[\int_{\eta_1}^{c_G} I_{[W_n(x)\le 0,w(x)>2C_n(\eta_1)]}dx]^4 \\
\le\ & E[\int_{\eta_1}^{c_G} I_{[W_n(x)\le 0]}w^3(x)dx \cdot \int_{\eta_1}^{c_G} w^{-3}(x)I_{[w(x)>2C_n(\eta_1)]}dx]^2 \\
\le\ & (c_G-\eta_1)\int_{\eta_1}^{c_G} P(W_n(x)\le 0)w^6(x)dx \cdot [\int_{\eta_1}^{c_G} w^{-3}(x)I_{[w(x)>2C_n(\eta_1)]}dx]^2.
\end{aligned}$$

We already know that

$$\int_{\eta_1}^{c_G} w^{-3}(x)I_{[w(x)>2C_n(\eta_1)]}dx = O(nu_n),$$

and similar to (6.9) and (6.10), we have

$$\int_{\eta_1}^{c_G} P(W_n(x)\le 0)w^6(x)dx = O((nu_n)^{-7/2}).$$

Then (6.14) is proved from the previous three inequalities. Thus the proof of (4.3) is complete.

Proof of (5.7). Note that $\theta > \theta_0 \Longrightarrow H((\theta - c_{f_1})u_n^{-1}) = 0$ and $f_1(x) = 0 \Longrightarrow f_2(x) = 0$. Then $[\sqrt{f_1(x)} - \sqrt{f_2(x)}]^2 \le I_{[f_1(x)>0]}[f_1(x) - f_2(x)]^2/f_1(x)$. After a few calculations, we have

$$\int_0^\infty [\sqrt{f_1(x)} - \sqrt{f_2(x)}]^2 dx \le l_1\{u_n^{2r-2}[\int_{\theta_0}^\infty \frac{g_1(\theta)}{A(\theta)}d\theta]^{-1}\int_0^{\theta_0}[\int_x^\infty H(\frac{\theta - c_{f_1}}{u_n})d\theta]^2 dx + u_n^{2r}\mu_n^2\}. \tag{6.16}$$

Let $\theta = c_{f_1} + tu_n$ and $x = c_{f_1} + yu_n$. And note that $H(t)$ has the support $[-1,1]$ and $\int_{-1}^1 H(t)dt = 0$. Then

$$\int_0^{\theta_0}[\int_x^\infty H(\frac{\theta - c_{f_1}}{u_n})d\theta]^2 dx \le u_n^2\int_0^{\theta_0}[\int_{(x-c_{f_1})/u_n}^\infty H(t)dt]^2 dx \le l_3 u_n^3\int_{-1}^1[\int_y^\infty H(t)dt]^2 dy.$$

Also

$$\frac{\mu_n}{u_n} = \int_{-1}^1 \frac{A(c_{f_1}+tu_n)}{u_n}H(t)dt \to a(c_{f_1})\int_{-1}^1 tH(t)dt.$$

From previous two results, (6.16) yields

$$\int_0^\infty (\sqrt{f_1(x)} - \sqrt{f_2(x)})^2 dx \le l_1[O(u_n^{2r+1}) + O(u_n^{2r+2})] \le \frac{l_2}{n}. \tag{6.17}$$

On the other hand, we have $[w_2(c_{f_1})]^2 = [w_2(c_{f_2}) - w_2(c_{f_1})]^2 = [w_2'(\hat{c}_{f_1})]^2(c_{f_2} - c_{f_1})^2$, where $\hat{c}_{f_1}$ is an intermediate value between c_{f_1} and c_{f_2}. It is easy to see that $[w_2'(\hat{c}_{f_1})]^2 \le 1/l_1$. Then $(c_{f_2} - c_{f_1})^2 \ge l_1[w_2(c_{f_1})]^2$. Note that

$$[w_2(c_{f_1})]^2 \ge l_2 u_n^{2r-2}[\int_{c_{f_1}}^\infty (\theta_0 - \theta)H(\frac{\theta - c_{f_1}}{u_n})d\theta]^2 \ge l_2 u_n^{2r}[\int_0^1 (\theta_0 - c_{f_1} + tu_n)H(t)dt]^2$$

and $\int_0^1 H(t)dt \neq 0$. Therefore

$$(c_{f_2} - c_{f_1})^2 \geq l_3 n^{-\frac{2r}{2r+1}}. \tag{6.18}$$

This result together with (6.17) leads to (5.7). The proof is complete now.

Appendix. We prove Lemma 6.1 and Lemma 6.2 in this appendix.

Proof of Lemma 6.1. From (2.4), $\alpha_G(0+) \leq B_r$. Since $c_G > 0$, $\phi_G(0+) < \phi_G(c_G) = \theta_0$. Note that both $\alpha_G(x)$ and $\alpha_G(x)\phi_G(x)$ are decreasing. Then for $x \in (0, \theta_0]$

$$|w(x)| \leq \theta_0 \alpha_G(x) + \alpha_G(x)\phi_G(x) \leq 2\theta_0 B_r.$$

That is (i). To prove (ii), noting that $w(x)$ is decreasing on $(0, \theta_0)$ and $w(c_G) = 0$, $|w(x)| \geq |w(c_G - \epsilon)| \wedge |w(c_G + \epsilon)|$. Since $|w(c_G - \epsilon)| = |w(c_G - \epsilon) - w(c_G)| = |w'(c_G^*)|\epsilon \geq A_\epsilon \epsilon$, where $c_G^* \in (c_G - \epsilon, c_G)$, and $|w(c_G + \epsilon)| \geq A_\epsilon \epsilon$ similarly , (ii) is obtained.

Proof of Lemma 6.2. It is easy to verify the following equation

$$w(x) = (\theta_0 - x)\alpha_G(x) - \int_x^\infty \alpha_G(s)ds.$$

Using Taylor expansion and (3.1), a straight forward computation shows that

$$E[\frac{K_0(\frac{X_j - x}{u_n})}{u_n a(X_j)}] = \alpha_G(x) + u_n^r \times \frac{1}{r!}\int_0^1 K_0(t)t^r \alpha_G^{(r)}(x + u_n t_1^*)dt,$$

where $0 \leq t_1^* \leq 1$. Also

$$E[\frac{K_1(\frac{X_j - x}{u_n})}{a(X_j)}] = \int_x^\infty \alpha_G(s)ds + u_n^r \times \frac{u_n}{(r+1)!}\int_0^1 K_0(t)t^{r+1}\alpha_G^{(r)}(x + u_n t_2^*)dt,$$

for some $0 \leq t_2^* \leq 1$. Thus, $|E[V_n(X_j, x)] - w(x)| \leq u_n^r(\theta_0 - x + u_n)B_r B_{1k}$. Then (i) is proved. Note that $\alpha_G(x)$ is a decreasing function for $x > 0$. Then

$$\sigma_n^2 \leq E[V(X_j, x, n)]^2 = \int_0^1 \frac{1}{u_n a(x + u_n t)}[(\theta_0 - x)K_0(t) - u_n K_1(t)]^2 \alpha_G(x + u_n t)dt.$$

Therefore $\sigma_n^2 \leq D_n(c_1)p_{n,\epsilon}u_n^{-1}$ for $x \in [c_1, c_2]$ and $\sigma_n^2 \leq D_n(c_1)u_n^{-3/2}$ for $x \in [d_n, \theta_0]$. The results for $E[|Z_{jn}|^3]$ can be proved similarly. This completes the proofs of (ii) and (iii).

If $w(x) > 2C_n(x)$,

$$w_n(x) = w(x) + [w_n(x) - w(x)] > 2C_n(x) - C_n(x) = C_n(x) > 0$$

and

$$\frac{w(x)}{w_n(x)} = \frac{w(x)}{w(x) + [w_n(x) - w(x)]} \leq \frac{w(x) - 2C_n(x) + 2C_n(x)}{w(x) - 2C_n(x) + C_n(x)} \leq 2.$$

Then (iv) is proved. (v) can be proved in a similar way.

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REPORT DOCUMENTATION PAGE

Form Approved OMB NO. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE
May, 2000

3. REPORT TYPE AND DATES COVERED
Technical Report, May, 2000

4. TITLE AND SUBTITLE
MONOTONE EMPIRICAL BAYES TESTS WITH OPTIMAL RATE OF CONVERGENCE FOR A TRUNCATION PARAMETER

5. FUNDING NUMBERS
DAAH04-95-1-0165

6. AUTHOR(S)
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7. PERFORMING ORGANIZATION NAMES(S) AND ADDRESS(ES)
Purdue University
Department of Statistics
West Lafayette, IN 47907-1399

8. PERFORMING ORGANIZATION REPORT NUMBER
Technical Report #00-05C

9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)
U.S. Army Research Office
P.O. Box 12211
Research Triangle Park, NC 27709-2211

10. SPONSORING / MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES
The views, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy or decision, unless so designated by other documentation.

12a. DISTRIBUTION / AVAILABILITY STATEMENT
Approved for public release; distribution unlimited.

12 b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*
Monotone empirical Bayes tests for a truncation parameter are considered under a linear loss. We use a new approach to construct a monotone empirical Bayes test δ_n based on the kernel estimate and established an upper bound for its regret Bayes risk. Also an asymptotic minimax lower bound of the monotone empirical Bayes rules is obtained. Then we find the optimal rate of convergence of the montone empirical Bayes tests. From the results established here, we conclude that the rule δ_n is optimal in the sense that (1) it has good performance for a small sample size since it possesses weak admissibility and (2) it has good performance for a la4ge sample size since it possesses the optimal convergence rate.

14. SUBJECT TERMS
Empirical Bayes; regret Bayes risk; optimal rate of convergence; minimax

15. NUMBER IF PAGES
15

16. PRICE CODE

17. SECURITY CLASSIFICATION OR REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UL