YET ANOTHER PROOF OF THE RADON-NIKODÝM THEOREM

by

Thomas Sellke
Purdue University

Technical Report #00-09

Department of Statistics
Purdue University
West Lafayette, IN USA

December 2000
Revised March 2001

YET ANOTHER PROOF OF THE RADON-NIKODÝM THEOREM

by

Thomas Sellke

Theorem (Lebesgue-Radon-Nikodým). Let μ and ν be probability measures on $(\Omega, \mathcal{F})$. Then there exists a set $D \in \mathcal{F}$ with $\mu(D) = 0$ and an $\mathcal{F}$-measurable function r: $\Omega \to [0, +\infty)$ for which

$$\nu(E) = \nu(E \cap D) + \int_E r d\mu, \qquad \text{all } E \in \mathcal{F}. \tag{1}$$

We will obtain a Radon-Nikodým derivative r in terms of the solutions of a family of maximization problems. The solutions will in fact be upper truncations of (a version of) r, and D will be the intersection of the truncation sets.

Definition For bounded, measurable f: $\Omega \to [0, \infty]$, define

$$V(f) = \int 2 f d\nu - \int f^2 d\mu. \tag{2}$$

For positive integers m, let $V_m^* = \sup\{V(f), 0 \le f \le m\}$.

Lemma 1 For any $m > 0$, there exists f_m with $0 \le f_m \le m$ and $V(f_m) = V_m^*$.

Lemma 2 For f_m as in Lemma 1, let $C_m = \{\omega : \; f_m(\omega) < m\}$, $D_m = \{\omega : \; f_m(\omega) = m\}$. Then

$$\text{(a) } \nu(E) = \int_E f_m d\mu, \qquad E \subset C_m$$

$$\text{(b) } \nu(E) \ge \int_E f_m d\mu = m\mu(E), \qquad E \subset D_m.$$

Lemma 3 For each positive integer m, let f_m, C_m, and D_m be as in Lemma 2. Let I_A be the indicator function of set A, and let

$$A_m = C_m \cap (\bigcup_{i=1}^{m-1} C_i)^c, \qquad r = \sum_{m=1}^{\infty} I_{A_m} f_m, \qquad D = \bigcap_{m=1}^{\infty} D_m.$$

Then D and r are as in the LRN theorem.

Before proving the lemmas "from scratch", it will be enlightening to see why they *follow* from the LRN theorem. Let D and r be as in the theorem, and let $\tilde{r} = r + \infty - I_D$. We can write

$$V(f) = \int_D 2f d\nu + \int_{D^c} (2rf - f^2) d\mu \tag{3}$$

Note that $(2rf - f^2) = r^2 - (r-f)^2$. So for each $\omega \in D^c$, the second integrand in (3) is strictly maximized, subject to $0 \leq f \leq m$, by $f_m(\omega) = r(\omega) \wedge m$. The first integrand in (3) is strictly maximized for each $\omega \in D$, subject to $0 \leq f \leq m$, by $f_m(\omega) = m$. Thus, $f_m = \tilde{r} \wedge m$ has $V(f_m) = V_m^*$, and it is easy to see that this maximizer is $(\mu+\nu)$-essentially unique. Lemmas 2 and 3 are easy consequences of the fact that each f_m is $(\mu+\nu)$-almost everywhere equal to $\tilde{r} \wedge m$.

Proof of Lemma 1: Suppose $0 \leq g_i \leq m$ and $V(g_i) \geq V_m^* - \varepsilon$ for $i = 1, 2$. Let $\bar{g} = (g_1 + g_2)/2$. Then

$$V_m^* \geq V(\bar{g}) = \frac{1}{2}\{V(g_1) + V(g_2)\} + \frac{1}{4}\int (g_1 - g_2)^2 d\mu$$
$$\geq V_m^* - \varepsilon + \frac{1}{4}\int (g_1 - g_2)^2 d\mu,$$

so $\int (g_1 - g_2)^2 d\mu \leq 4\varepsilon$.

Now suppose that $0 \leq g_k \leq m$ and $V(g_k) \geq V_m^* - 4^{-(k+1)}$ for $k = 1, 2, \ldots$. The above calculation shows $\int (g_k - g_{k+1})^2 d\mu \leq 4^{-k}$, and by the Lyapounov inequality (or by Jensen or Schwarz) we have $S|g_k - g_{k-1}| d\mu \leq 2^{-k}$. Then $\int \sum_{k=1}^{\infty} |g_k - g_{k+1}| d\mu \leq 1$, which implies that g_k converges μ-almost surely to $f_m \triangleq \limsup g_k$

Hence, $g_k(\omega)$ is μ-almost surely a Cauchy sequence and therefore convergent to f_m. The bounded convergence theorem implies $\int g_k^2 d\mu \longrightarrow \int f_m^2 d\mu$. By Fatou (applied to $m - g_k$)

$$\int f_m d\nu = \int \limsup g_k d\nu \geq \limsup \int g_k d\nu.$$

Hence,

$$V(f_m) = \int 2f_m d\nu - \int f_m^2 d\mu \geq \limsup V(g_k) = V_m^*.$$

□

Proof of Lemma 2 (a): If not, there exists $E^* \subset C_m$ with $\nu(E^*) - \int_{E^*} f_m d\mu \neq 0$ and with f_m bounded away from 0 and m on E^*. Then

$$V(f_m + \alpha I_{E^*}) - V(f_m) = 2\alpha\{\nu(E^*) - \int_{E^*} f_m d\mu\} - \alpha^2 \mu(E^*),$$

which contradicts the maximality of $V(f_m)$. The proof of Lemma 2(b) is similar. □

Proof of Lemma 3: By Lemma 2(b), $1 \geq \nu(D_m) \geq m\mu(D_m)$, and so $\mu(D_m) \leq m^{-1}$. Consequently, $\mu(D) = \mu(\cap D_m) = 0$. Lemma 2(a) implies $\nu(E) = \int_E f_m d\mu = \int_E r d\mu$ for $E \subset A_m \subset C_m$. The sets A_m partition $\cup C_m = D^c$, so for *any* $E \in \mathcal{F}$,

$$\begin{aligned}
\nu(E) &= \nu(E \cap D) + \nu(E \cap D^c) \\
&= \nu(E \cap D) + \sum_{m=1}^{\infty} \nu(E \cap A_m) \\
&= \nu(E \cap D) + \sum_{m=1}^{\infty} \int_{E \cap A_m} r d\mu \\
&= \nu(E \cap D) + \int_{E \cap D^c} r d\mu \\
&= \nu(E \cap D) + \int_E r d\mu.
\end{aligned}$$

□

ACKNOWLEDGEMENT. This work was partially supported by NSF grant DMS-9971767.

Statistics Department
Purdue University
West Lafayette, IN 47907-1399
tsellke@stat.purdue.edu