

**ON TIME CHANGING CONTINUOUS MARTINGALES
TO BROWNIAN MOTION**

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Summary

The embedding of coin-tossing random walk in continuous unbounded martingales is used to give a new proof that they may be time changed to Brownian motion.

We give a proof of the result of Dambis and Dubins-Schwartz (see [3]) that continuous martingales with unbounded paths can be time changed to standard Brownian motion. We first consider the case that the paths of M are not constant on any open interval, and then discuss the general case.

Theorem. *Let $M_t, t \geq 0$, be a continuous martingale satisfying $M_0 = 0$, $\sup_t |M_t| = \infty$, and $P(M_s = M_a, a < s < b) = 0$ for all $0 < a < b$. Then there are stopping times $\eta_t, t \geq 0$, which strictly and continuously increase from 0 to infinity, such that $M_{\eta_t}, t \geq 0$, is Brownian motion.*

Proof: Let $u_0^M = 0$, and $u_{k+1}^M = \inf\{t > u_k : |M_t - M_{u_k}| = 1\}$, $k > 0$, and let $v_{n,j}^M = u_j^{2^n M}$, if $n, j \geq 0$. We drop the superscript M for the rest of this paragraph. Then $M_{u_j}, j \geq 0$, is a fair random walk, and $M_{v_{n,j}}, j \geq 0$, has the distribution of a fair random walk divided by 2^n . Of course the distribution of the $v_{n,j}$ is different for different martingales, but the distribution of the ordering of these times is not. To be precise, the probability of any event in the algebra of events generated by the events $\{v_{i,j} < v_{k,l}\}$ has the same probability for all martingales M . To see this, it helps to first check that $P(v_{1,3} < v_{0,1}) = 1/2$, since the random walk $M_{v_{0,j}}, j \geq 0$, is embedded in the random walk $M_{v_{1,j}}, j \geq 0$ by the discrete analog of the times $v_{0,k}$, and the probability of the analogous event for these walks is $1/2$. Now since the walks $M_{v_{k,j}}, j \geq 0$, can for $0 \leq k < n$ all be embedded in the walk $M_{v_{n,j}},$

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$j \geq 0$, which is of course the same walk for any M , the probability of an event in the algebra is the probability of an event for discrete random walk.

Lemma. For $0 \leq n < \infty$, let $t_{n,j}, 0 \leq j < \infty$, be a sequence, and suppose

i) $0 = t_{0,n}, n \geq 0$,

ii) $t_{n,j} < t_{n,j+1}, j \geq 0$,

iii) for all j and n , $t_{n,j}$ is one of the numbers $t_{n+1,k}, k \geq 0$,

iv) the set of all the $t_{n,j}$ is dense in $[0, \infty)$.

Then a sequence $a_n, n \geq 0$, of nonnegative numbers converges if and only if given m there is a j such that $t_{m,j} < a_k < t_{m,j+2}$ for all large enough k . Furthermore if K is a positive integer, an increasing nonnegative function f on $[0, K]$ is continuous if and only if given $n > 0$ there is m such that for each $i, 0 \leq i < Km$, there is $j = j(i)$ such that $t_{n,j} < f(i/m)$ and $f((i+1)/m) < t_{n,j+2}$.

This lemma is obvious. Now let $v_{n,2^{2n}}^M$ play the role of a_n and $v_{n,j}^M$ have the role of $t_{n,j}$ in this lemma. The conditions i)-iv) are easy to check, using the absence of flat spots for iv). The lemma implies that whether or not $v_{n,2^{2n}}^M$ converges (a.s.) depends only on the distribution of the order of the $v_{i,j}^M$. Since this latter distribution does not depend on M , we have either convergence for all M or no M . But if M is a Brownian motion B , we do have convergence, to 1. For following Skorohod (see[1]), $v_{n,2^{2n}}^B$ has the distribution of the average of 2^{2n} iid random variables each having the distribution of $u_1^B := u$. Since $Eu = 1$ and since the variance of u is finite, easily shown upon noting that $P(u > k+1 | u > k) < P(|Z| < 2), k > 0$, where Z is standard normal, Chebyshev's inequality gives this convergence to 1. Similarly $\lim_{n \rightarrow \infty} v_{n, [t2^{2n}]}^M := \eta_t^M$ exists, where $[\]$ denotes the greatest integer function. Now the distribution of $M_{\eta_1^M}$ is the limit of the distributions of $M_{v_{n,2^{2n}}^M}$, since M has continuous paths, and thus is the same for all martingales M , and this limit can be identified, by taking $M = B$, as standard normal. All the joint distributions can be similarly treated, and so $M_{\eta_t^M}$ is Brownian motion. This implies that η_t^M is strictly increasing. To see that it is continuous, use the last sentence of the lemma. An argument like that just given shows that continuity on $[0, K]$ for any K , and thus continuity on $[0, \infty)$, either holds for all or no M . And $\eta_t^B = t$. Finally, since

η_t^M is continuous and strictly increasing, $\eta_t^M = \sup_{k>0} \lim_{n \rightarrow \infty} v_{n, [(1-\frac{1}{k})2^{2n}t]}^M$, and so is a stopping time. \square

In case the paths of M have flat spots, remove them. Let A stand for the union of the open intervals on which M is constant. Let $h(t) = \inf\{y : |(0, y) \cap A^c| = t\}$, $0 \leq t \leq \infty$, where $||$ is Lebesgue measure and the c denotes complement, so that if we define $N_t = M_{h(t)}$, N is continuous with no flat spots. Whether or not N is a martingale, random walks can be embedded in it, since they can be embedded M . Thus just as above, $N_{\eta_t^N}$ is Brownian motion. Put $\mu_t = h(\eta_t^N)$. Then μ is left continuous and strictly increasing, and M_{μ_t} is Brownian motion. And $\mu_t = \sup_{k>0} \limsup_{n \rightarrow \infty} v_{n, [(1-\frac{1}{k})2^{2n}t]}^M$, so μ_t is a stopping time.

The times $v_{n,j}^M$ were used in [2], differently than here, to show the existence of quadratic variation for continuous martingales. We remark that the Dambis and Dubins-Schwartz theorem could be used as the starting point of an exposition of continuous martingales.

References

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