Limit Theorem for the Spread of Branching Diffusion with Stabilizing Drift

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ABSTRACT

In the present paper we derive a formula describing the limiting behavior of the position of the rightmost particle in a one-dimensional branching diffusion with a stabilizing drift and generalize the result to a multi-dimensional case.

Key words: branching diffusion, stabilizing drift, action functional, quasipotential

1 INTRODUCTION

The concern of this paper is the asymptotic behavior of the right frontier of a branching diffusion with a stabilizing drift. We start with a one-dimensional case. Suppose a particle positioned at the origin at time 0 begins a diffusion X which satisfies a stochastic differential equation

$$\dot{X}_s = b(X_s) + \dot{W}_s, \quad s \geq 0, \tag{1.1}$$

where $b(x)$ is a smooth stabilizing drift and W is a standard Brownian motion. In a random time having an exponential distribution with parameter 1 the particle splits, and, henceforth, starting at the position of the splitting, the created particles behave independently under the same law as the original particle. And the process continues.

Introduce a potential $U(x) = -\int_0^x b(y)\,dy$. In what follows, the only positive branch of $U(x)$ corresponding to $x > 0$ is involved. We assume that this branch is monotonically increasing, and, therefore, the inverse U^{-1} is uniquely defined. We give general assumptions under which the position R_t of the rightmost particle at time t is governed by the law: for any $\varepsilon > 0$,

$$\mathbf{P}\Big(\Big|\frac{R_t}{U^{-1}(t/2)} - 1\Big| < \varepsilon\Big) \to 1 \text{ as } t \to \infty. \tag{1.2}$$

The problem of determining the spread of a driftless branching Brownian motion had been studied by a number of authors (see, for example, Bramson [2], or Biggins [1]). It had been established that in this case, for any $\varepsilon > 0$,

$$\mathbf{P}\Big(\Big|\frac{R_t}{\sqrt{2}\,t} - 1\Big| < \varepsilon\Big) \to 1 \text{ as } t \to \infty.$$

The present study has been initialized by a conjecture in Lalley and Sellke [7] that claims that the frontier propagation law for the Ornstein-Uhlenbeck (O-U) process ($b(x) = -x$) is $\sqrt{t}$, that is, for any ε, $\mathbf{P}(|R_t - \sqrt{t}| < \varepsilon) \to 1$ as $t \to \infty$. The heuristic analysis in [7] appeals to the observation that the stationary probability density $\exp(-x^2)/\sqrt{\pi}$ of the O-U process determines the approximation of the expected number of particles with positions greater than x

$$e^t \int_x^\infty \exp(-y^2)/\sqrt{\pi}\,dy \sim \exp(t - x^2)/(2\sqrt{\pi}x) \text{ as } x \to \infty.$$

The latter vanishes if $x = x_t = \sqrt{t}$.

An obvious generalization of this conjecture is a result stronger than (1.2): for any $\varepsilon > 0$, $\mathbf{P}\Big(|R_t - U^{-1}(t/2)| < \varepsilon\Big) \to 1$ as $t \to \infty$. One of the methods to tackle this problem would be to use the Poisson Tidal Waves technique of Lalley and Sellke [6], but it is not completely clear how it works in our case. Instead, we use an approach based on the Large Deviations Probabilities (LDPs) theory of Freidlin and Wentzell [4] and prove the weaker result. However, the LDPs are less sensitive to the dimension of diffusion. In this paper a multi-dimensional case is studied. We show that the domain $D(t) = \{X : V(x) < t\}$ is "swept" by the particles at time t, and there are no particles outside of $D(t)$ for large t. Here $V(x)$ is a quasipotential introduced in Freidlin and Wentzell [4] that plays the same role as $2U(x)$ in a one-dimensional case.

A suggested alternative to our method would be the one introduced in Freidlin [3]. We want to emphasize now that the results for our model do not follow from those in Freidlin [3]. Freidlin considers branching diffusions with motions satisfying $\dot{X}_u = b(X_u) + \varepsilon \dot{W}_u$, where ε is small, and the intensity of splitting ε^{-2}. In (1.1), if we make a time-scale change $Y_u = X_{ut}/t$, $\tilde{W}_u = W_{ut}/\sqrt{t}$, $0 \leq u \leq 1$, we arrive at a branching diffusion with motion satisfying $\dot{Y}_u = b(tY_u) + \dot{\tilde{W}}_u/\sqrt{t}$ that splits with the intensity t. It would be the desired model if the drift did not depend on t. Therefore, the case $b(x) = -\mathrm{sgn}(x)$ is the only case covered by this approach. It yields the law: for any $\varepsilon > 0$,

$$\mathbf{P}\left(\left|\frac{R_t}{t/2} - 1\right| < \varepsilon\right) \to 1 \text{ as } t \to \infty.$$

Nonsurprisingly, it coincides with formula (1.2). However, to prove our result we have to assume asymptotical polynomiality of the drift with the power greater than zero. The case of the constant drift does not comply with the mechanism of large deviations. We show that the right frontier reaches the high level $U^{-1}(t/2)$ for large t in the following way: the particles stay in a relatively small neighborhood of the origin, multiplying, and only in a time interval closely preceeding t, one of the particles makes a "large deviation" and attains the high level.

The result (1.2) is proved in Sections 2 and 3. Section 4 is devoted to a multi-dimensional generalization. Proofs of ancillary technical lemmas can be found in Section 5.

2 1-D CASE. UPPER BOUND

In this section the "upper bound" portion of (1.2) is discussed. That is, our objective is to show that under certain assumptions,

$$\mathbf{P}\Big(R_t < (1+\varepsilon)U^{-1}(t/2)\Big) \to 1 \text{ as } t \to \infty.$$

In contrast to Freidlin and Wentzell [4], who study diffusions satisfying $\dot{X}_s = b(X_s) + \varepsilon \dot{W}_s$ for small ε, here there is no "small parameter" in (1.1). However, we make use of the fact that time tends to infinity. In what follows, H_t is a "high" level, $H_t \to \infty$ as $t \to \infty$, and $[0, T_t]$ is a time interval typically increasing as t increases. We study the probabilities

$$\mathbf{P}_x\Big(\max_{0 \le s \le T_t} X_s \ge H_t\Big)$$

for all large t, where x indicates the initial point of diffusion, that is $X_0 = x$. A positive monotonically decreasing function ω_t, $\omega_t \to 0$, is called *slowly vanishing* if for any $c > 0$ the function $\omega_t^c H_t \to \infty$ as $t \to \infty$.
Consider the *action functional* introduced in Freidlin and Wentzell [4]

$$I_{0T_t}(\varphi) = \frac{1}{2}\int_0^{T_t} (\dot{\varphi}_s - b(\varphi_s))^2 \, ds \text{ for any absolutely continuous } \varphi.$$

We want to study the performance of R_t for any polynomially growing potential $U(x) \sim x^\alpha$ as $x \to \infty$, $\alpha > 1$. The technical hurdle is that the behavior of X is essentially different for $\alpha < 2$, $\alpha = 2$, and $\alpha > 2$. For this reason, we impose general assumptions and formulate an upper bound applicable not just to the polynomial case. Then, we specify this upper bound on the case-by-case basis.

ASSUMPTION 1 *The drift $b(x) \searrow -\infty$ as $x \nearrow +\infty$. Let ω_t be a slowly vanishing function, and T_t, $1 \le T_t \le t$, be chosen so that for all large t the inequalities are true*

$$\omega_t |b(\omega_t H_t)| T_t \ge H_t \quad \text{and} \quad \omega_t^2 b^2(\omega_t H_t) T_t \ge U(H_t) \tag{2.1}$$

Lemma 1 *Under Assumption 1, for any φ such that $\omega_t H_t \le \varphi_s \le H_t$, $0 \le s \le T_t$, the action functional*

$$I_{0T_t}(\varphi) \ge U(H_t)/\omega_t.$$

For any X, $X_0 = x$, consider an operator $B : (BX)_s = X_s - x - \int_0^s b(X_u)\,du$.

Assumption 2 (stabilizing drift) *There is a constant C such that for any T_t the inequality holds*

$$\rho_{0T_t}(X^{(1)}, X^{(2)}) \le C\rho_{0T_t}(BX^{(1)}, BX^{(2)})$$

where $\rho_{0T_t}(X^{(1)}, X^{(2)}) = \max_{0 \le s \le T_t} |X_s^{(1)} - X_s^{(2)}|$.

Introduce a set of absolutely continuous functions on the interval $[0, T_t]$ with bounded values of the action functional:

$$\Phi^x = \Phi^x(\lambda_t) = \{\varphi_s,\ 0 \le s \le T_t :\ \varphi_0 = x,\ I_{0T_t}(\varphi) \le \lambda_t\ \}$$

where $\lambda_t \to \infty$ as $t \to \infty$.

Lemma 2 *If Assumption 2 is fulfilled and $\delta_t = \sqrt{T_t}/\omega_t$ with a slowly vanishing function ω_t, then for any $\gamma > 0$ and any x, the inequality is true*

$$\mathbf{P}_x\Big(\rho_{0T_t}(X, \Phi^x(\lambda_t))\ >\ \delta_t\ \Big) \le \exp\{-\lambda_t(1-\gamma)\}.$$

Lemma 3 *If the assumptions of Lemma 2 hold, then for any $x \le H_t - \delta_t$ and any $\gamma > 0$, the following inequality holds for all t large enough:*

$$\mathbf{P}_x\Big(\max_{0\le s\le T_t} X_s \ge H_t\Big) \le \exp\Big\{-2(U(H_t - \delta_t) - U(x))\,(1-\gamma)\Big\}.$$

Lemma 4 (general upper bound) *Let Assumptions 1 and 2 hold, and let $\delta_t = \sqrt{2T_t}/\omega_t$. Assume $\delta_t \le \omega_t H_t$. Then for any $x \le 2\omega_t H_t$, any $\gamma > 0$, and all large t, we have*

$$\mathbf{P}_x\Big(\max_{0\le s\le t} X_s \ge H_t\Big) \le 1-\Big(1-\exp\Big\{-2(U(H_t-\delta_t)-U(2\omega_t H_t))\,(1-\gamma)\Big\}$$

$$\exp\Big\{-U(H_t)(1-\gamma)/\omega_t\Big\}\Big)^t. \qquad (2.2)$$

PROOF: Let t be so large that $2\omega_t H_t \le H_t - \delta_t$. Introduce Markov stopping times

$$\tau = \inf\{s:\ X_s \ge H_t\} \ \text{ and } \ \eta = \inf\{s \ge T_t:\ X_s \le \omega_t H_t + \delta_t\}.$$

Put

$$A = \{\tau \le 2T_t\} \ \text{ and } \ B = \{\eta > 2T_t\}.$$

From Lemma 3, for all sufficiently large t, we have the inequality

$$\min_{x\le 2\omega_t H_t} \mathbf{P}_x(A) \le \exp\Big\{-2(U(H_t-\delta_t)-U(2\omega_t H_t))(1-\gamma)\Big\}. \qquad (2.3)$$

If the random event $\bar{A}$ occurs, then $X_{T_t} < H_t$, and by the strong Markov property,

$$\mathbf{P}_x(\bar{A}B) \le \max_{y<H_t} \mathbf{P}_y\Big(\omega_t H_t + \delta_t < X_s < H_t,\ 0\le s\le T_t\Big). \qquad (2.4)$$

By Lemma 1, any function φ such that $\varphi_s \le H_t$ for $0 \le s \le T_t$ and $I_{0T_t}(\varphi) \le \lambda_t = U(H_t)/\omega_t$ attains $\omega_t H_t$. Thus, if $X_0 = y$,

$$\{\omega_t H_t + \delta_t < X_s < H_t,\ 0\le s\le T_t\} \subseteq \{\rho_{0T_t}(X, \Phi^y(\lambda_t)) \ > \ \delta_t\}.$$

Therefore, from (2.4) and Lemma 2, one gets

$$\mathbf{P}_x(\bar{A}B) \le \max_{y<H_t} \mathbf{P}_y\Big(\rho_{0T_t}(X, \Phi^y(\lambda_t)) \ > \ \delta_t\Big)$$

$$\le \exp\Big\{-U(H_t)(1-\gamma)/\omega_t\Big\}. \qquad (2.5)$$

Now, we write

$$\mathbf{P}_x\Big(\max_{0\le s\le t} X_s < H_t\Big) \ge \mathbf{P}_x\Big(T_t \le \eta \le 2T_t,\ \max_{0\le s\le \eta} X_s < H_t,\ \max_{\eta\le s\le t} X_s < H_t\Big)$$

$$= \mathbf{E}_x\mathbf{E}\Big[\mathbb{I}\{T_t \le \eta \le 2T_t\}\mathbb{I}\{\max_{0\le s\le \eta} X_s < H_t\}\mathbb{I}\{\max_{\eta\le s\le t} X_s < H_t\}|\mathcal{F}_\eta\Big]$$

where $\mathcal{F}_\eta = \sigma\{X_s,\ 0 \le s \le \eta\}$. Noticing that the first two indicator functions of random events are $\mathcal{F}_\eta$ - measurable and using the strong Markov property, we continue

$$= \mathbf{E}_x\Big[\mathbb{I}\{T_t \le \eta \le 2T_t\}\mathbb{I}\{\max_{0\le s\le \eta} X_s < H_t\}\mathbf{P}_{X_\eta}\Big(\max_{0\le s\le t-\eta} X_s < H_t\Big)\Big].$$

If the random event $\bar{A}\bar{B}$ occurs, then, in particular, X is less than H_t in the interval $[0,\eta]$, and $T_t \le \eta \le 2T_t$. Keeping in mind that $t-\eta < t - T_t$ and $\omega_t H_t + \delta_t \le 2\omega_t H_t$, we have that the above

$$\ge \mathbf{P}_x\Big(T_t \le \eta \le 2T_t,\ \max_{0\le s\le \eta} X_s < H_t\Big) \min_{y\le \omega_t H_t + \delta_t} \mathbf{P}_y\Big(\max_{0\le s\le t-T_t} X_s < H_t\Big)$$

$$\ge \mathbf{P}_x(\bar{A}\bar{B}) \min_{y\le 2\omega_t H_t} \mathbf{P}_y\Big(\max_{0\le s\le t-T_t} X_s < H_t\Big).$$

Now, $\mathbf{P}_x(\bar{A}\bar{B}) = 1 - \mathbf{P}_x(A) - \mathbf{P}_x(\bar{A}B)$, therefore, apply (2.3) and (2.5) to get

$$\min_{x\le 2\omega_t H_t} \mathbf{P}_x(\max_{0\le s\le t} X_s < H_t) \ge \Big(1 - \exp\Big\{-2(U(H_t-\delta_t) - U(2\omega_t H_t))(1-\gamma)\Big\}$$
$$- \exp\Big\{-U(H_t)(1-\gamma)/\omega_t\Big\}\Big) \min_{y\le 2\omega_t H_t} \mathbf{P}_y\Big(\max_{0\le s\le t-T_t} X_s < H_t\Big).$$

Repeat this estimate t times and note that $T_t \ge 1$. The lemma follows. □

Now we show that if the drift $b(x)$ has a polynomial rate of growth and is stabilizing, then the Assumptions 1 and 2 hold and the upper bound (2.2) simplifies to $2t\exp\{-2U(H_t)(1-2\gamma)\}$.

ASSUMPTION 3 (POLYNOMIAL DRIFT) *The drift* $b(x) \sim -x^{\alpha-1}$ *as* $x \to +\infty$, $\alpha > 1$, *and* $b(x) \sim (-x)^{\beta-1}$ *as* $x \to -\infty$, $\beta > 1$.

ASSUMPTION 4 (STABILIZING DRIFT) *For any* $c > 0$ *there exists a positive constant* $\mu = \mu(c)$ *such that*

$$b(X^{(1)}) - b(X^{(2)}) \ge \mu(X^{(2)} - X^{(1)}), \quad -c \le X^{(1)} \le X^{(2)} \le c.$$

Lemma 5 (upper bound for a polynomial drift) *If Assumptions 3 and 4 are satisfied, then (2.2) holds. Moreover, if H_t is such that as $t \to \infty$, $\exp\{-2U(H_t)(1-2\gamma)\} = o(1/t)$, then, for all large t, we have*

$$\mathbf{P}\Big(\max_{0\le s\le t} X_s \ge H_t\Big) \le 2t\exp\Big\{-2U(H_t)(1-2\gamma)\Big\}. \tag{2.6}$$

Proof: First, we check that Assumption 1 holds for polynomial drifts, and that $\delta_t = \sqrt{2T_t}/\omega_t \le \omega_t H_t$ as required in Lemma 4. We claim that under Assumption 3 the inequalities (2.1) are satisfied with $T_t = 1$ if $\alpha > 2$, $T_t = 1/\omega_t^4$ if $\alpha = 2$, and $T_t = H_t^{2-\alpha}/\omega_t^4$ if $1 < \alpha < 2$.

Case $\alpha > 2$. Using the definition of the slowly vanishing function ω_t, we find that

$$\omega_t(\omega_t H_t)^{\alpha-1}T_t = \omega_t^\alpha H_t^{\alpha-1} = (\omega_t^{\alpha/(\alpha-2)}H_t)^{\alpha-2}H_t \ge H_t.$$

The second inequality in (2.1) is also true:

$$\omega_t^2(\omega_t H_t)^{2\alpha-2}T_t = (\omega_t^{2\alpha/(\alpha-2)}H_t)^{\alpha-2}H_t^\alpha \ge H_t^\alpha = U(H_t).$$

Now, $\delta_t = \sqrt{2T_t}/\omega_t = \sqrt{2}/\omega_t \le \omega_t H_t$.

Case $\alpha = 2$. We have that $\omega_t(\omega_t H_t)T_t = \omega_t^{-2}H_t \ge H_t$, and $\omega_t^2(\omega_t H_t)^2 T_t = H_t^2 \ge H_t^2 = U(H_t)$. In this case, $\delta_t = \sqrt{2T_t}/\omega_t = \sqrt{2}/\omega_t^3 \le \omega_t H_t$.

Case $1 < \alpha < 2$. We find that

$$\omega_t(\omega_t H_t)^{\alpha-1}T_t = H_t/\omega_t^{4-\alpha} \ge H_t,$$

and

$$\omega_t^2(\omega_t H_t)^{2\alpha-2}T_t = H_t^\alpha/\omega_t^{4-2\alpha} \ge H_t^\alpha = U(H_t).$$

Finally,

$$\delta_t = \sqrt{2T_t}/\omega_t = \sqrt{2}H_t/(\omega_t^3 H_t^{\alpha/2}) \le \omega_t H_t.$$

If Assumption 3 holds, then there are positive constants C_* and C^* such that

$$C_* \le |xb(x)|/U(x) \le C^*. \tag{2.7}$$

For this reason,

$$U(H_t) - U(H_t - \delta_t) \leq \delta_t C^* U(H_t)/H_t \leq C^* \omega_t U(H_t).$$

Since $U(H_t) \sim H_t^\alpha$ as $t \to \infty$, there exists a constant $C_1 > 0$ such that

$$U(2\omega_t H_t) \leq C_1 (2\omega_t)^\alpha U(H_t).$$

Therefore, for all t large enough, we bound the first exponent on the right-hand side of (2.2) by

$$\exp\Big\{-2(U(H_t - \delta_t) - U(2\omega_t H_t))(1-\gamma)\Big\}$$

$$\leq \exp\Big\{-2U(H_t)(1-\gamma)(1 - C^*\omega_t - C_1(2\omega_t)^\alpha)\Big\} \leq \exp\Big\{-2U(H_t)(1-2\gamma)\Big\}.$$

Since $\gamma > 0$ is arbitrarily small, the upper bound in (2.2) does not exceed

$$1 - \Big(1 - 2\exp\{-2U(H_t)(1-2\gamma)\}\Big)^t \sim 2t\exp\{-2U(H_t)(1-2\gamma)\} \ \text{ as } \ t \to \infty.$$

In the above we used the assumption that $\exp\{-2U(H_t)(1-2\gamma)\} = o(1/t)$ for large t. Finally, we verify that Assumption 4 implies the stability condition in Assumption 2. Denote $\rho = \rho_{0T_t}(BX^{(1)}, BX^{(2)})$. For simplicity, let $v = X^{(2)} - X^{(1)}$ and $w = BX^{(2)} - BX^{(1)}$. Fix an arbitrary $s \in [0, T_t]$. Denote s_0 the last time $v = 0$ and suppose, first, v is positive in the interval (s_0, s). Using Assumption 4 with specified $c = \max_{s_0 \leq s \leq T_t}(|X_s^{(1)}|, |X_s^{(2)}|)$, we have

$$v_s = w_s - w_{s_0} - \int_{s_0}^{s} (b(X_u^{(1)}) - b(X_u^{(2)}))\, du \leq 2\rho - \mu \int_{s_0}^{s} v_u\, du \leq 2\rho.$$

Suppose now v is negative in the interval (s_0, s). Hence, $-v$ is positive. We have

$$-v_s = -(w_s - w_{s_0}) - \int_{s_0}^{s} (b(X_u^{(2)}) - b(X_u^{(1)}))\, du \leq 2\rho - \mu \int_{s_0}^{s} (-v_u)\, du \leq 2\rho.$$

Thus, we have shown that for any $s \in [0, T_t]$, $-2\rho \leq v_s \leq 2\rho$, that is, the inequality in Assumption 2 holds with $C = 2$. □

Theorem 1 *If Assumptions 3 and 4 are fulfilled, then for an arbitrarily small* $\varepsilon > 0$, *we have*

$$\mathbf{P}\Big(R_t < (1+\varepsilon)U^{-1}(t/2)\Big) \to 1 \text{ as } t \to \infty. \tag{2.8}$$

Proof: In formula (2.6) put $H_t = (1+\varepsilon)U^{-1}(t/2)$ and choose $\gamma = \varepsilon C_*/4$ where C_* satisfies (2.7). Let N_t denote the number of particles in $[H_t, \infty)$ at time t. Since the number of particles in existence at time t has a geometric distribution with parameter $p = e^{-t}$ (see, for example, Sevast'yanov [8]), we obtain

$$\mathbf{E}[N_t] = e^t \mathbf{P}(X_t \geq H_t)$$

$$\leq 2te^t \exp\Big\{-2(1-\varepsilon C_*/2)U\Big(U^{-1}(t/2) + \varepsilon U^{-1}(t/2)\Big)\Big\}. \tag{2.9}$$

As follows from (2.7),

$$2U\Big(U^{-1}(t/2) + \varepsilon U^{-1}(t/2)\Big) \geq t + 2|b(U^{-1}(t/2))|\varepsilon U^{-1}(t/2)$$

$$\geq t + 2\varepsilon C_* U\Big(U^{-1}(t/2)\Big) = t(1+\varepsilon C_*).$$

Thus, from (2.9), for all large t, the upper bound holds

$$\mathbf{E}[N_t] \leq 2t \exp\Big\{t - t(1+\varepsilon C_*)(1-\varepsilon C_*/2)\Big\}$$

$$\leq 2t\exp\{-t\varepsilon C_*/4\} \leq \exp\{-t\varepsilon C_*/8\}. \tag{2.10}$$

The Chebyshev inequality and (2.10) imply (2.8). □

Remark 1 Lemma 4 shows that the random event $\{\max_{0\leq s\leq t} X_s \geq H_t\}$ is "composed" of t/T_t attempts to reach the level H_t over time intervals of the length T_t . In Lemma 5, the length T_t is finite if $\alpha > 2$, near finite if $\alpha = 2$, and about $H_t^{2-\alpha}$ if $\alpha < 2$, since ω_t is a slowly vanishing function. Note that in the case of a polynomial drift, $H_t \sim t^{1/\alpha}$, so that $T_t = o(t)$ as $t \to \infty$ for any $\alpha > 1$.

3 1-D CASE. LOWER BOUND

The main interest of this section is showing that

$$\mathbf{P}\Big(R_t > (1-\varepsilon)U^{-1}(t/2)\Big) \to 1 \text{ as } t \to \infty.$$

Consider an interval $[-L_t, M_t]$ where the endpoints L_t and M_t are positive functions such that $U(-L_t) = U(M_t) = \log t$. Consider a time $T_t = o(t)$ for large t. The rate of growth of T_t will be specified in the proof of Lemma 9 (see Remark 2).

In order to prove the lower bound statement, we will show that a particle located at time $t - T_t$ inside the interval $[-L_t, M_t]$ has an exponentially small probability to reach the "high" level by time t, but there are exponentially many candidates. Therefore, with probability close to 1 at least one of the particles succeeds.

Let N_{t-T_t} and $\bar{N}_{t-T_t}$ denote the number of particles inside and outside of the interval $[-L_t, M_t]$ at time $t - T_t$, respectively.

LEMMA 6 *As* $t \to \infty$, $\mathbf{P}\Big(\bar{N}_{t-T_t} \geq \frac{e^{t-T_t}}{2\log t}\Big) \to 0.$

LEMMA 7 *As* $t \to \infty$, $\mathbf{P}\Big(N_{t-T_t} > \frac{e^{t-T_t}}{2\log t}\Big) \to 1.$

LEMMA 8 *Let W be a standard Brownian motion on the interval $[0, T_t]$. Take any $\delta > 0$ and denote $\nu = \frac{1}{2}(\frac{\pi}{2\delta})^2$. Then, for any stochastic integral $\int_0^{T_t} f(s)\, dW_s$, for sufficiently large t,*

$$\mathbf{P}\Big(\int_0^{T_t} f(s)\, dW_s > -\sqrt{4\nu T_t \int_0^{T_t} f(s)^2\, ds},\ \ \rho_{0T_t}(W, 0) < \delta\Big) > \frac{1}{4}e^{-\nu T_t}.$$

The drift $b(x)$ is assumed smooth, therefore, it satisfies the Lipschitz condition: for any $c > 0$ there exists $K = K(c)$ such that

$$|b(X^{(1)}) - b(X^{(2)})| \leq K|X^{(1)} - X^{(2)}|, \quad c = \max_{a \leq s \leq b}(|X_s^{(1)}|, |X_s^{(2)}|).$$

LEMMA 9 *Fix* $\delta > 0$ *and let* $\nu = \frac{1}{2}(\frac{\pi}{2\delta})^2$. *Then, for any* $x \in [-L_t, M_t]$, *and any "high" level* H_t, *if* t *is large enough,*

$$\mathbf{P}_x\Big(X_{T_t} > H_t\Big) > \frac{1}{4}\exp\Big\{-F_t - \nu T_t - \sqrt{8\nu T_t F_t}\Big\}$$

where $F_t = 2U(H_t+\delta)+\sqrt{2}\delta K_t\sqrt{T_t}\sqrt{2U(H_t+\delta)}+\delta^2 K_t^2 T_t/2+\zeta_t+\sqrt{2}\delta K_1\sqrt{\zeta_t}+\delta^2 K_1^2/2$, $\zeta_t = (M_t+L_t)^2 + b^2(M_t) \vee b^2(-L_t)$, K_t *is some function, and* K_1 *is a constant.*

PROOF: Consider a function φ such that $\varphi_s = x + (M_t - x)s$ for $0 \le s \le 1$, $\varphi_{T_t} = H_t + \delta$, and $\dot\varphi_s = -b(\varphi_s)$ for $1 \le s \le T_t$. Consider an auxiliary process $Y = X - \varphi$. Recalling the expression for the Radon-Nikodym derivative of the measure of Y with respect to the measure of X, we write

$$\mathbf{P}_x\Big(X_{T_t} > H_t\Big) \ge \mathbf{P}_x\Big(\rho_{0T_t}(X,\varphi) < \delta\Big) = \mathbf{P}\Big(\rho_{0T_t}(Y,0) < \delta\Big)$$

$$= \mathbf{E}\Big[\exp\Big\{-\int_0^{T_t}(\dot\varphi_s - b(X_s))\,dW_s - \frac{1}{2}\int_0^{T_t}(\dot\varphi_s - b(X_s))^2\,ds\Big\}\mathbb{I}\{\rho_{0T_t}(W,0) < \delta\}\Big].$$

Applying now the Chebyshev inequality and Lemma 8, we obtain

$$\mathbf{P}_x\Big(X_{T_t} > H_t\Big)$$

$$> \frac{1}{4}\exp\Big\{-\frac{1}{2}\int_0^{T_t}(\dot\varphi_s - b(X_s))^2\,ds - \nu T_t - \sqrt{4\nu T_t\int_0^{T_t}(\dot\varphi_s - b(X_s))^2\,ds}\Big\}. \quad (3.1)$$

Further, we estimate

$$\frac{1}{2}\int_1^{T_t}(\dot\varphi_s - b(X_s))^2\,ds \le \frac{1}{2}\int_1^{T_t}(\dot\varphi_s - b(\varphi_s))^2\,ds$$

$$+\int_1^{T_t}|\dot\varphi_s - b(\varphi_s)|\,|b(\varphi_s) - b(X_s)|\,ds + \frac{1}{2}\int_1^{T_t}(b(\varphi_s) - b(X_s))^2\,ds. \quad (3.2)$$

Note that the function φ is chosen to satisfy the Euler equation, and, therefore, it minimizes the action functional I_{1T_t}. By Theorem 3.1 in Chapter 4 of Freidlin and Wentzell [4], $I_{1T_t}(\varphi) = 2(U(\varphi_{T_t}) - U(\varphi_1))$. Hence,

$$\frac{1}{2}\int_1^{T_t}(\dot\varphi_s - b(\varphi_s))^2\,ds = 2U(H_t+\delta) - 2U(M_t) \le 2U(H_t+\delta). \tag{3.3}$$

The Lipschitz condition with $c_t = \max_{1\le s\le T_t}(|\varphi_s|, |X_s|)$ and $K_t = K(c_t)$, the Cauchy-Schwarz inequality, and (3.3) imply

$$\int_1^{T_t} |\dot\varphi_s - b(\varphi_s)|\,|b(\varphi_s) - b(X_s)|\,ds$$

$$\le K_t\rho_{1T_t}(X,\varphi)\sqrt{T_t}\sqrt{\int_1^{T_t}(\dot\varphi_s - b(\varphi_s))^2\,ds}$$

$$\le \sqrt{2}\delta K_t\sqrt{T_t}\sqrt{2U(H_t+\delta)}. \tag{3.4}$$

By the Lipschitz condition,

$$\frac{1}{2}\int_1^{T_t}(b(\varphi_s) - b(X_s))^2\,ds \le \frac{1}{2}K_t^2\rho_{1T_t}^2(X,\varphi)T_t \le \frac{1}{2}\delta^2K_t^2T_t. \tag{3.5}$$

From (3.2)-(3.5) we have

$$\frac{1}{2}\int_1^{T_t}(\dot\varphi_s - b(X_s))^2\,ds \le 2U(H_t+\delta)$$

$$+\sqrt{2}\delta K_t\sqrt{T_t}\sqrt{2U(H_t+\delta)} + \frac{1}{2}\delta^2K_t^2T_t. \tag{3.6}$$

Similarly shown,

$$\frac{1}{2}\int_0^1(\dot\varphi_s - b(X_s))^2\,ds \le \frac{1}{2}\int_0^1(\dot\varphi_s - b(\varphi_s))^2\,ds$$

$$+\sqrt{2}\delta K_1\sqrt{\frac{1}{2}\int_0^1(\dot\varphi_s - b(\varphi_s))^2\,ds} + \frac{1}{2}\delta^2K_1^2 \tag{3.7}$$

where K_1 is the Lipschitz constant corresponding to $c = \max_{0\le s\le 1}(|\varphi_s|, |X_s|)$. Finally,

$$\frac{1}{2}\int_0^1(\dot\varphi_s - b(\varphi_s))^2\,ds = \frac{1}{2}\int_0^1\Big(M_t - x - b(x+(M_t-x)s)\Big)^2\,ds$$

$$\le (M_t-x)^2 + \int_0^1 b^2(x+(M_t-x)s)\,ds \le (M_t+L_t)^2 + b^2(M_t)\vee b^2(-L_t). \tag{3.8}$$

Combining (3.1), (3.6)-(3.8) gives the result. □

THEOREM 2 *Suppose Assumption 3 and* (2.7) *hold. Then, for any arbitrarily small* $\varepsilon > 0$,

$$\mathbf{P}\Big(R_t > (1-\varepsilon)U^{-1}(t/2)\Big) \to 1 \text{ as } t \to \infty.$$

PROOF: Put $H_t = (1-\varepsilon)U^{-1}(t/2)$ and let $\delta_t = \sqrt{T_t/(\omega_t t)}$ where ω_t is a slowly vanishing function. Then, for sufficiently large t,

$$2U\Big((1-\varepsilon)U^{-1}(t/2)+\delta_t\Big) \le 2U\Big((1-\varepsilon)U^{-1}(t/2)\Big) + 3\delta_t\Big|b\Big((1-\varepsilon)U^{-1}(t/2)\Big)\Big|$$

$$\le t - \varepsilon U^{-1}(t/2)\Big|b(U^{-1}(t/2))\Big| + 3\delta_t\Big|b\Big((1-\varepsilon)U^{-1}(t/2)\Big)\Big|.$$

From (2.7), the second term is less than or equal to $-\varepsilon C_* t/2$. By Assumption 3, there exists a constant $c = c(\delta, \varepsilon, \alpha)$ such that the third term does not exceed $ct^{\frac{\alpha-1}{\alpha}}$. Therefore,

$$2U\Big((1-\varepsilon)U^{-1}(t/2)+\delta_t\Big) \le t - \frac{\varepsilon C_*}{2}t + ct^{\frac{\alpha-1}{\alpha}}.$$

Next, by Assumption 3, for large t, $M_t \sim (\log t)^{\frac{1}{\alpha}}$, $L_t \sim (\log t)^{\frac{1}{\beta}}$,

$$b^2(M_t) \sim (\log t)^{\frac{2\alpha-2}{\alpha}}, \text{ and } b^2(-L_t) \sim (\log t)^{\frac{2\beta-2}{\beta}}.$$

Thus, in Lemma 9, $\zeta_t \le (\log t)^\gamma$ where $\gamma = \max(\frac{2}{\alpha}, \frac{2}{\beta}, \frac{2\alpha-2}{\alpha}, \frac{2\beta-2}{\beta}) + 1$. Consequently,

$$F_t \le t - \frac{\varepsilon C_*}{2}t + ct^{\frac{\alpha-1}{\alpha}} + \sqrt{2}\delta_t K_t\sqrt{T_t}\sqrt{t - \frac{\varepsilon C_*}{2}t + ct^{\frac{\alpha-1}{\alpha}}} + \delta_t^2 K_t^2 T_t/2$$

$$+(\log t)^\gamma + \sqrt{2}\delta_t K_1(\log t)^{\gamma/2} + \delta_t^2 K_1^2/2.$$

Further, the function δ_t has been chosen specifically that $\delta_t^2 K_t^2 T_t = o(t)$ as $t \to \infty$ (see Remark 2 for the growth rate of T_t). Hence, for large enough t, $F_t \le t - \varepsilon C_* t/4$. Thus, for all $x \in [-L_t, M_t]$,

$$\mathbf{P}_x\Big(X_{T_t} > (1-\varepsilon)U^{-1}(t/2)\Big) > \frac{1}{4}\exp\Big\{-t + \frac{\varepsilon C_*}{4}t - \nu_t T_t - \sqrt{8\nu_t T_t}\sqrt{t - \frac{\varepsilon C_*}{4}t}\Big\}$$

where $\nu_t = \frac{1}{2}(\frac{\pi}{2\delta_t})^2$. Again, δ_t is chosen so that $\nu_t T_t = o(t)$ as t increases.

Whence it follows that for all sufficiently large t,

$$\mathbf{P}_x\Big(X_{T_t} > (1-\varepsilon)U^{-1}(t/2)\Big) > \exp\{-t + \frac{\varepsilon C_*}{8}t\}.$$

Taking into account this result and the result of Lemma 7, we compute

$$\mathbf{P}\Big(R_t > (1-\varepsilon)U^{-1}(t/2)\Big)$$

$$\geq \mathbf{P}\Big(R_t > (1-\varepsilon)U^{-1}(t/2)\Big|\, N_{t-T_t} > \frac{e^{t-T_t}}{2\log t}\Big)\times$$

$$\times\mathbf{P}\Big(N_{t-T_t} > \frac{e^{t-T_t}}{2\log t}\Big)$$

$$= \mathbf{P}_x\Big(\text{at least one particle reaches the level in time } T_t\Big)\times$$

$$\times\mathbf{P}\Big(N_{t-T_t} > \frac{e^{t-T_t}}{2\log t}\Big)$$

$$> \Big[1-\Big(1-e^{-t+\frac{\varepsilon C_*}{8}t}\Big)^{\frac{e^{t-T_t}}{2\log t}}\Big]\,\mathbf{P}\Big(N_{t-T_t} > \frac{e^{t-T_t}}{2\log t}\Big) \to 1$$

as $t \to \infty$. □

Remark 2 The time interval T_t is determined entirely by the differential equation satisfied by the extremal φ in Lemma 9. Under the assumption of a polynomial growth of the drift, for large t, $T_t \sim t^{\frac{2-\alpha}{\alpha}}$ if $\alpha < 2$, $T_t \sim \log t$ if $\alpha = 2$, and T_t is finite if $\alpha > 2$. Note that in either of the three cases, $T_t = o(t)$ as $t \to \infty$.

Remark 3 If the potential $U(x)$ is a polynomial, $U(x) = |x|^\alpha$, $\alpha > 1$, Theorem 2 can be proved also by means of a non-random time change as the following heuristic illustrates. Put $Y_s = h_s^{-1}X_s$ where $h_s = U^{-1}(s/2) = (s/2)^{1/\alpha}$. Then the process Y satisfies

$$\dot{Y}_s = h_s^{\alpha-2}b(Y_s) - (\alpha s)^{-1}Y_s + h_s^{-1}\dot{W}_s.$$

For large s, the term $(\alpha s)^{-1}Y_s$ is negligible, hence, $\dot{Y}_s \approx h_s^{\alpha-2}b(Y_s) + h_s^{-1}\dot{W}_s$.

Now, the time substitution $ds_1 = h_s^{\alpha-2}\,ds$ defines a new stochastic process $Z_{s_1} = Y_{s(s_1)}$ such that

$$\dot{Z}_{s_1} \approx b(Z_{s_1}) + \sigma_{s_1}\dot{\tilde{W}}_{s_1}$$

where $\tilde{W}$ is a standard Brownian motion and $\sigma_{s_1} = ((\alpha-1)s_1/\alpha)^{-\alpha/(4\alpha-4)}$. Indeed, the transition from $h_s^{-1}\dot{W}_s$ to $\sigma_{s_1}\dot{\tilde{W}}_{s_1}$ is justified by the identity $h_s^{-2}\,ds = \sigma_{s_1}^2\,ds_1$.

The terminal instant $s = t$ corresponds to $t_1 = s_1(t)$ such that $t/2 = ((\alpha-1)t_1/\alpha)^{\alpha/(2\alpha-2)}$. Thus, $\sigma_{t_1} = ((\alpha-1)t_1/\alpha)^{-\alpha/(4\alpha-4)} = \sqrt{2/t}$. If s_1 is close to the terminal time t_1, then $\sigma_{s_1} \approx \sigma_{t_1}$, therefore, the process Z_{s_1} is a small random perturbation of a dynamical system satisfying

$$\dot{Z}_{s_1} \approx b(Z_{s_1}) + \sigma_{t_1}\dot{\tilde{W}}_{s_1}.$$

Hence, by Freidlin and Wentzell [4], for sufficiently large t,

$$\mathbf{P}(Y_t > 1-\varepsilon) \approx \mathbf{P}(Z_{t_1} > 1-\varepsilon) \asymp \exp\{-2\sigma_{t_1}^{-2}U(1-\varepsilon)\}$$
$$= \exp\{-(1-\varepsilon)^{\alpha}t\} \approx \exp\{-t+\alpha\varepsilon t\}.$$

Further, at time close to t there are at least $\exp\{t-\log t\}$ particles with probability close to 1. Therefore, as in the proof of the above theorem, the probability that at least one of the particles reaches the level $(1-\varepsilon)U^{-1}(t/2)$ tends to 1 as time increases.

4 MULTI-DIMENSIONAL CASE

In this section we study a d-dimensional branching diffusion, $d > 1$, with individual trajectories governed by a stochastic differential equation

$$\dot{X}_s = b(X_s) + \dot{W}_s, \quad s \geq 0,$$

where $b(x)$ is a smooth vector field in $\mathbf{R}^d$, and W denotes a standard d-dimensional Brownian motion. An essential difference from a one-dimensional case is that $b(x)$ is not necessarily potential.

A multi-dimensional analogue of the action functional is

$$I_{0T}(\varphi) = \frac{1}{2}\int_0^T |\dot{\varphi}_s - b(\varphi_s)|^2\, ds$$

where φ is absolutely continuous on $[0,T]$, and $|\cdot|$ denotes the Euclidean norm of a vector. Following Freidlin and Wentzell [4], introduce a *quasipotential* $V(x)$ by

$$V(x) = \inf\{I_{0T}(\varphi) \mid T > 0,\ \varphi_0 = 0,\ \varphi_T = x\},\quad x \in \mathbf{R}^d.$$

For any $\lambda > 0$, define a set $D(\lambda) = \{x,\ x \in \mathbf{R}^d :\ V(x) < \lambda\}$ bounded by a level surface of the quasipotential.

It is convenient to state a multi-dimensional analogue of (1.2) separately for upper and lower bounds: for any $\varepsilon > 0$, as $t \to \infty$,

$$\mathbf{P}\Big(\text{at time } t \text{ there exists a particle outside of } D((1+\varepsilon)t)\Big) \to 0, \tag{4.1}$$

and

$$\mathbf{P}\Big(\text{at time } t \text{ there exists a particle outside of } D((1-\varepsilon)t)\Big) \to 1. \tag{4.2}$$

REMARK 4 By Freidlin and Wentzell [4], in a one-dimensional case, $V(x) = 2U(x)$. Therefore, $D(t) = \{x :\ V(x) = 2U(x) < t\}$ and the right endpoint of $D(t)$ is $U^{-1}(t/2)$. In (1.2) we claimed that with probability close to 1, for large t, R_t is bounded by $(1 \pm \varepsilon)U^{-1}(t/2)$. This statement is still valid if the bounds are replaced by $U^{-1}((1 \pm \varepsilon)t/2)$ since, in view of (2.7), for large t,

$$(1-\frac{\varepsilon C^*}{2})t \le V((1-\varepsilon)U^{-1}(t/2)) \le V(R_t) \le V((1+\varepsilon)U^{-1}(t/2)) \le (1+\frac{3\varepsilon C^*}{2})t.$$

This shows that the behavior described by (4.1) and (4.2) takes place in a one-dimensional case as well.

In some multi-dimensional instances, the quasipotential $V(x)$ can be written out in explicit form.

EXAMPLE (ORNSTEIN-UHLENBECK PROCESS) Let the diffusion process satisfy $\dot{X}_s = AX_s + \dot{W}_s$, $s \geq 0$, where A is a $d \times d$ stability matrix (i.e., the real part of each of its eigenvalues is negative). Then, the quasipotential $V(x) = \frac{1}{2}x'[\int_0^\infty e^{As}e^{A's}\,ds]^{-1}x$. If A is assumed normal ($AA' = A'A$), then $V(x) = -\frac{1}{2}x'(A+A')x$, and if A is symmetric, $V(x) = -x'Ax$.

To prove (4.1) and (4.2), we have to impose assumptions similar to Assumptions 1 and 2. But first, we start with a technical assumption on the geometry of domains $D(\lambda)$.

ASSUMPTION 5 *For all large λ, the domains $D(\lambda)$ have smooth boundaries $\partial D(\lambda)$ diffeomorphic to a sphere in $\mathbf{R}^d$.*

Denote by $H(\lambda)$ the diameter of $D(\lambda)$. Let $\nabla V(x)$ be the gradient of $V(x)$, and put

$$\beta_*(\lambda) = \min_{x \in \partial D(\lambda)} |\nabla V(x)| \quad \text{and} \quad \beta^*(\lambda) = \max_{x \in \partial D(\lambda)} |\nabla V(x)|.$$

ASSUMPTION 6 *Let $\beta_*(\lambda)$ have a monotonic polynomial rate of growth in λ, so that*

$$\beta_*(\lambda) = \min_{x \notin D(\lambda)} |\nabla V(x)|,$$

and let there exist positive constants C_* and C^* such that for all λ large enough, the following inequalities hold:

$$C_* \leq \frac{\beta_*(\lambda)}{\beta^*(\lambda)} \quad \text{and} \quad C_* \leq \frac{\beta_*(\lambda)H(\lambda)}{\lambda} \leq C^*. \tag{4.3}$$

REMARK 5 Note that the latter inequalities in (4.3) are analogous to (2.7), while the former inequality is a regularity condition on $V(x)$ that assures the uniform rate of growth of the quasipotential in all directions.

Let $\lambda_t \to \infty$ as $t \to \infty$. Call a positive monotonically decreasing function ω_t, $\omega_t \to 0$, *slowly vanishing* if for every $a, b > 0$, $\omega_t^a H(\omega_t^b \lambda_t) \to \infty$ as $t \to \infty$.

ASSUMPTION 7 *For any $\lambda_t \to \infty$, there exist T_t, $0 < T_t \leq t$, and a slowly vanishing function ω_t such that, for all large t, the inequalities*

$$\omega_t^2 \beta_*(\omega_t \lambda_t) T_t \geq H(\omega_t \lambda_t) \text{ and } \omega_t^2 \beta_*^2(\omega_t \lambda_t) T_t \geq \lambda_t$$

are true.

LEMMA 10 *Suppose Assumptions 5 - 7 hold. Then for any function φ such that $\varphi_s \in D(\lambda_t) \setminus D(\omega_t \lambda_t)$, $0 \leq s \leq T_t$, the action functional*

$$I_{0T_t}(\varphi) \geq \lambda_t / \omega_t.$$

LEMMA 11 *Let Assumptions 5 - 7 and a multi-dimensional version of Assumption 2 be valid. Then for any $x \in D(3\omega_t \lambda_t)$, any arbitrarily small $\gamma > 0$, and all large t, we have*

$$\mathbf{P}_x\Big(X_t \notin D(\lambda_t)\Big) \leq 1 - \Big(1 - \exp\Big\{-\lambda_t(1-4\omega_t)(1-\gamma)\Big\} - \exp\Big\{-\lambda_t(1-\gamma)/\omega_t\Big\}\Big)^t$$

$$\leq 1 - \Big(1 - 2\exp\{-\lambda_t(1-2\gamma)\}\Big)^t \sim 2t\exp\{-\lambda_t(1-2\gamma)\}.$$

THEOREM 3 *If the conditions of Lemma 11 are satisfied, then the upper bound* (4.1) *holds.*

PROOF: The proof is identical to the proof of Theorem 1. Let N_t denote the number of particles outside of $D((1+\varepsilon)t)$. Applying Lemma 11 with $\gamma = \varepsilon/4$, one gets

$$\mathbf{E}_x[N_t] \leq 2t\exp\Big\{t - t(1+\varepsilon)(1-2\gamma)\Big\} \leq \exp\{-t\varepsilon/8\},$$

and the Chebyshev inequality completes the proof. □

Remained to show that the lower bound (4.2) holds. For some instant $t-T_t$, denote N_{t-T_t} the number of particles inside $D(2\log t)$ and $\bar{N}_{t-T_t}$ the number of particles outside of the set. In this notation, the results of Lemmas 6 and 7 are valid and the proofs translate directly. The statement of Lemma 8 is also true in a multi-dimensional situation, if the stochastic integral is understood to be $\int_0^{T_t}\langle f(s), dW_s\rangle$ where $\langle\cdot,\cdot\rangle$ is the Euclidean scalar product of vectors.

LEMMA 12 *Suppose $\lambda_t \to \infty$ as $t \to \infty$. Let ω_t be a slowly vanishing function. Take $\delta_t \le \omega_t^2 H(\lambda_t)$ and denote $\nu_t = \frac{1}{2}(\frac{\pi}{2\delta_t})^2$. Then, for any $x \in D(2\log t)$, for all large t,*

$$\mathbf{P}_x\Big(X_{T_t} \notin D(\lambda_t)\Big) > \frac{1}{4}\exp\Big\{-F_t - \nu_t T_t - \sqrt{8\nu_t T_t F_t}\Big\}$$

where $F_t = (1+\omega_t)\lambda_t + \sqrt{2}\delta_t K_t\sqrt{T_t}\sqrt{(1+\omega_t)\lambda_t} + \delta_t^2 K_t^2 T_t/2 + 2\log t + \delta_t^2 k_t^2 \tau_t/2 + 2\delta_t k_t\sqrt{\tau_t}\sqrt{\log t}$, K_t, k_t, and τ_t are some functions.

PROOF: Specify $x \in D(2\log t)$ and $y \in \partial D((1+\omega_t)\lambda_t)$. Let φ be a combination of two extremals φ_1 and φ_2 such that $\varphi_1(0) = 0$, $\varphi_1(\tau_t) = x$ for some τ_t, and $\varphi_2(0) = 0$, $\varphi_2(T_t) = y$. That is, formally, $\varphi_s = \varphi_1(\tau_t - s)$, if $0 \le s \le \tau_t$, and $\varphi_s = \varphi_2((s-\tau_t)T_t/(T_t-\tau_t))$, if $\tau_t \le s \le T_t$. Denote k_t and K_t the Lipschitz constants corresponding to $c_t = \max_{0\le s\le\tau_t}(|\varphi_s|, |X_s|)$ and $c_t = \max_{\tau_t\le s\le T_t}(|\varphi_s|, |X_s|)$, respectively. Note that

$$\frac{1}{2}\int_0^{\tau_t} |\dot{\varphi}_s - b(\varphi_s)|^2\,ds = V(\varphi_1(\tau_t)) - V(\varphi_1(0)) = V(x) < 2\log t,$$

$$\frac{1}{2}\int_{\tau_t}^{T_t} |\dot{\varphi}_s - b(\varphi_s)|^2\,ds = V(\varphi_2(T_t)) - V(\varphi_2(0)) = V(y) = (1+\omega_t)\lambda_t.$$

The rest of the proof is the proof of Lemma 9 with the above modifications. □

REMARK 6 By Lemma 3.1 in Chapter 4 of Freidlin and Wentzell [4], the optimum speed along an extremal is the modulus of the drift. If the drift is polynomial, the speed outside of a sufficiently large domain exceeds any given constant, and the time of motion is below the length of the extremal over an arbitrarily large constant. The length of the extremal is not larger than $O(|y|)$ where y is a terminal point. This reasoning shows that $\tau_t = o(\log t)$ and $T_t = o(t)$ as $t \to \infty$.

THEOREM 4 *Under conditions of Lemma 12, the lower bound* (4.2) *holds.*

PROOF: In Lemma 12, take $\lambda_t = (1-\varepsilon)t$. For large enough t, $(1+\omega_t)(1-\varepsilon) \le 1 - \omega_t\varepsilon$. Now, in view of the above remark, δ_t can be always chosen so that $2\log t + 2\delta_t k_t \sqrt{\tau_t}\sqrt{\log t} + \delta_t^2 k_t^2 \tau_t/2 \le (\log t)^\gamma$ for some $\gamma > 1$, $\delta_t^2 K_t^2 T_t = o(t)$, and $T_t/\delta_t^2 = o(t)$. Hence, for sufficiently large t,

$$F_t \le (1-\omega_t\varepsilon)t + \sqrt{2}\delta_t K_t \sqrt{T_t}\sqrt{(1-\omega_t\varepsilon)t} + \frac{1}{2}\delta_t^2 K_t^2 T_t + (\log t)^\gamma \le t - \frac{\omega_t\varepsilon}{2}t,$$

and

$$\mathbf{P}_x\Big(X_{T_t} \notin D((1-\varepsilon)t)\Big) \ge \frac{1}{4}\exp\Big\{-t + \frac{\omega_t\varepsilon}{2}t - \nu_t T_t - \sqrt{8\nu_t T_t}\sqrt{(1-\frac{\omega_t\varepsilon}{2})t}\Big\}$$

$$> \exp\{-t + \frac{\omega_t\varepsilon}{4}t\}.$$

The calculation carried out at the end of Theorem 2 finishes the proof. □

5 APPENDIX

PROOF OF LEMMA 1 By the Cauchy-Schwarz inequality,

$$I_{0T_t}(\varphi) = \frac{1}{2}\int_0^{T_t} (\dot\varphi_s - b(\varphi_s))^2\, ds \ge \frac{1}{2T_t}\Big(\int_0^{T_t} (\dot\varphi_s - b(\varphi_s))\, ds\Big)^2$$

$$= \frac{1}{2T_t}\Big((\varphi_{T_t} - \varphi_0) + \int_0^{T_t} |b(\varphi_s)|\, ds\Big)^2 \ge \frac{1}{2T_t}\Big(|b(\omega_t H_t)|T_t - H_t\Big)^2.$$

Here we used the fact that $\varphi_{T_t} - \varphi_0 \ge -(1-\omega_t)H_t > -H_t$ and $\int_0^{T_t} |b(\varphi_s)|\, ds \ge |b(\omega_t H_t)|T_t$. The first inequality in (2.1) implies that for all large t,

$|b(\omega_t H_t)|T_t/2 \geq H_t$. Thus, from the second inequality in (2.1), we obtain that

$$I_{0T_t}(\varphi) \geq \frac{1}{8} b^2(\omega_t H_t) T_t \geq U(H_t)/(8\omega_t^2) \geq U(H_t)/\omega_t. \quad \square$$

PROOF OF LEMMA 2 It can be shown (cf. Theorem 2.2 in Chapter 3 of Freidlin and Wentzell [4]) that for any $\gamma > 0$ and any $\varepsilon > 0$ sufficiently small, a standard Brownian motion W_u, $0 \leq u \leq 1$, satisfies

$$\mathbf{P}\Big(\rho_{01}(\varepsilon W, \tilde{\Psi}(1)) > d\Big) \leq \exp\{-(1-\gamma)\varepsilon^{-2}\} \tag{5.1}$$

where

$$\tilde{\Psi}(\lambda) = \{\tilde{\psi}_u,\ 0 \leq u \leq 1 :\ \frac{1}{2}\int_0^1 (\frac{d\tilde{\psi}_u}{du})^2\, du \leq \lambda\}, \quad \lambda > 0,$$

and $d = d(\varepsilon)$ is such that $d(\varepsilon)/\varepsilon \to \infty$ as $\varepsilon \to 0$.

Put $\varepsilon = \varepsilon_t = 1/\sqrt{\lambda_t} \to 0$ as $t \to \infty$, and define a set of functions

$$\Psi(\lambda_t) = \{\psi_s,\ 0 \leq s \leq T_t :\ \frac{1}{2}\int_0^{T_t} \dot{\psi}_s^2\, ds \leq \lambda_t\}.$$

If we introduce $W_s = (BX)_s$ and $\psi_s = (B\varphi)_s$, $0 \leq s \leq T_t$, then Assumption 2 implies

$$\mathbf{P}_x\Big(\rho_{0T_t}(X, \Phi^x(\lambda_t)) > \delta_t\Big) \leq \mathbf{P}\Big(\rho_{0T_t}(W, \Psi(\lambda_t)) > \delta_t/C\Big).$$

With the time-space rescaling, we define another standard Brownian motion $\tilde{W}_u$ in the interval $[0,1]$ where

$$\tilde{W}_u = W_{uT_t}/\sqrt{T_t}, \ \ 0 \leq u \leq 1.$$

Put $\tilde{\psi}_u = \psi_{uT_t}/\sqrt{T_t}$ and note that

$$\mathbf{P}\Big(\rho_{0T_t}(W, \Psi(\lambda_t)) > \delta_t/C\Big) = \mathbf{P}\Big(\rho_{01}(\tilde{W}, \tilde{\Psi}(\lambda_t)) > \delta_t/(C\sqrt{T_t})\Big)$$

$$= \mathbf{P}\Big(\rho_{01}(\varepsilon_t \tilde{W}, \tilde{\Psi}(1)) > \delta_t/(C\sqrt{T_t\lambda_t})\Big).$$

Finally, using that $\delta_t = \sqrt{T_t}/\omega_t$, we note that $d_t = \delta_t/(C\sqrt{T_t\lambda_t})$ satisfies

$$d_t/\varepsilon_t = \delta_t/(C\sqrt{T_t}) = 1/(C\omega_t) \to \infty \text{ as } t \to \infty.$$

Thus, (5.1) applies, and the lemma follows. □

PROOF OF LEMMA 3 As shown in Freidlin and Wentzell [4], for any function φ such that $\varphi_0 = x \leq H_t - \delta_t$ and $\varphi_{T_t} \geq H_t - \delta_t$, the action functional $I_{0T_t}(\varphi)$ satisfies the inequality

$$I_{0T_t}(\varphi) \geq 2(U(H_t - \delta_t) - U(x)) = \lambda_t.$$

Consequently,

$$\{\max_{0 \leq s \leq T_t} X_s \geq H_t\} \subseteq \{\rho_{0T_t}(X, \Phi^x(\lambda_t)) > \delta_t\},$$

and Lemma 2 applies. □

PROOF OF LEMMA 6 By the Chebyshev inequality,

$$\mathbf{P}\Big(\bar{N}_{t-T_t} \geq \frac{e^{t-T_t}}{2\log t}\Big) \leq 2\ \log t\ e^{-t+T_t}\mathbf{E}\bar{N}_{t-T_t}$$

$$= 2\ \log t\ e^{-t+T_t}e^{t-T_t}\mathbf{P}\Big(X_{t-T_t} \notin [-L_t, M_t]\Big) = 2\ \log t\ \mathbf{P}\Big(X_{t-T_t} \notin [-L_t, M_t]\Big).$$

Now, recalling the assumption $U(-L_t) = U(M_t) = \log t$ and applying Lemma 5, one gets

$$\mathbf{P}(X_{t-T_t} \geq M_t) \leq \mathbf{P}\Big(\max_{0 \leq s \leq t-T_t} X_s \geq M_t\Big)$$

$$\leq 2(t - T_t)\exp\{-2U(M_t)(1 - 2\gamma)\} \leq 2/t^{1-4\gamma}.$$

Similarly shown, $\mathbf{P}\Big(X_{t-T_t} \leq -L_t\Big) \leq 2/t^{1-4\gamma}$. Thus,

$$\mathbf{P}\Big(X_{t-T_t} \notin [-L_t, M_t]\Big) \leq 4/t^{1-4\gamma}.$$

Finally,

$$\mathbf{P}\Big(\bar{N}_{t-T_t} \geq \frac{e^{t-T_t}}{2\ \log t}\Big) \leq \frac{8\ \log t}{t^{1-4\gamma}} \to 0 \text{ as } t \to \infty. \quad \square$$

PROOF OF LEMMA 7 Denote $N^*_{t-T_t}$ the total number of particles alive at time $t - T_t$. Simple computation gives

$$\mathbf{P}\Big(N^*_{t-T_t} > \frac{e^{t-T_t}}{\log t}\Big) = e^{-(t-T_t)} \sum_{n=1+e^{t-T_t}/\log t}^{\infty} \Big(1 - e^{-(t-T_t)}\Big)^{n-1}$$

$$= \Big(1 - e^{-(t-T_t)}\Big)^{e^{t-T_t}/\log t} \sim e^{-\frac{1}{\log t}} \to 1 \text{ as } t \to \infty.$$

Using this and the result of Lemma 6, we obtain

$$\mathbf{P}\Big(N_{t-T_t} > \frac{e^{t-T_t}}{2\,\log t}\Big)$$

$$\geq \mathbf{P}\Big(N^*_{t-T_t} > \frac{e^{t-T_t}}{\log t},\ \bar{N}_{t-T_t} < \frac{e^{t-T_t}}{2\,\log t}\Big)$$

$$\geq \mathbf{P}\Big(N^*_{t-T_t} > \frac{e^{t-T_t}}{\log t}\Big) + \mathbf{P}\Big(\bar{N}_{t-T_t} < \frac{e^{t-T_t}}{2\,\log t}\Big) - 1 \to 1 \text{ as } t \to \infty. \quad \Box$$

PROOF OF LEMMA 8 For simplicity, denote $I = (\int_0^{T_t} f(s)^2\,ds)^{1/2}$ and $c = \sqrt{4\nu T_t}$. Let $z > 0$ be a variable. The Chebyshev inequality and a simple fact that if $w \sim N(0,\sigma^2)$, then $\mathbf{E}e^{z|w|} < 2e^{z^2\sigma^2/2}$ imply

$$\mathbf{P}\Big(\int_0^{T_t} f(s)\,dW_s \leq -cI\Big) \leq \mathbf{P}\Big(z|\int_0^{T_t} f(s)\,dW_s| \geq zcI\Big)$$

$$\leq e^{-zcI}\mathbf{E}\exp\{z|\int_0^{T_t} f(s)\,dW_s|\} < 2\exp\{z^2I^2/2 - zcI\}.$$

The exponent is minimized for $z = c/I$ and the minimal value is $2e^{-c^2/2} = 2e^{-2\nu T_t}$.

Further, it is known that $\mathbf{P}\Big(\rho_{0T_t}(W,0) < \delta\Big) > \frac{1}{2}e^{-\nu T_t}$ (see, for example, Gikhman and Skorokhod [5]). Therefore, for large enough t,

$$\mathbf{P}\Big(\int_0^{T_t} f(s)\,dW_s > -cI, \rho_{0T_t}(W,0) < \delta\Big)$$

$$\geq \mathbf{P}\Big(\int_0^{T_t} f(s)\,dW_s > -cI\Big) + \mathbf{P}\Big(\rho_{0T_t}(W,0) < \delta\Big) - 1$$

$$> 1 - 2e^{-2\nu T_t} + \frac{1}{2}e^{-\nu T_t} - 1 > \frac{1}{4}e^{-\nu T_t}. \quad \Box$$

Proof of Lemma 10 At each x, $x \neq 0$, define $e(x)$ to be the unit vector in the direction of the gradient $\nabla V(x)$, i.e., $e(x) = \nabla V(x)/|\nabla V(x)|$. Note that

$$|\dot{\varphi}_s - b(\varphi_s)| \geq |\langle \dot{\varphi}_s - b(\varphi_s), e(\varphi_s)\rangle|$$

Hence, as in Lemma 1, we have the inequality

$$I_{0T_t}(\varphi) = \frac{1}{2}\int_0^{T_t} \Big(\langle \dot{\varphi}_s, e(\varphi_s)\rangle - \langle b(\varphi_s), e(\varphi_s)\rangle\Big)^2 ds$$

$$\geq \frac{1}{2T_t}\Big(\int_0^{T_t} \langle \dot{\varphi}_s, e(\varphi_s)\rangle\, ds - \int_0^{T_t} \langle b(\varphi_s), e(\varphi_s)\rangle\, ds\Big)^2. \qquad (5.2)$$

The quasipotential $V(x)$ satisfies the Hamilton-Jacobi equation (see Freidlin and Wentzell [4]):

$$\frac{1}{2}|\nabla V(x)|^2 + \langle b(x), \nabla V(x)\rangle = 0.$$

For this reason,

$$-\langle b(\varphi_s), e(\varphi_s)\rangle = \frac{1}{2}|\nabla V(x)| \geq \frac{1}{2}\min_{x \notin D(\omega_t\lambda_t)} |\nabla V(x)| = \frac{1}{2}\beta_*(\omega_t\lambda_t),$$

and

$$-\int_0^{T_t} \langle b(\varphi_s), e(\varphi_s)\rangle\, ds \geq \frac{1}{2}\beta_*(\omega_t\lambda_t)T_t.$$

Next, recalling that $\varphi_s \in D(\lambda_t) \setminus D(\omega_t\lambda_t)$ when $0 \leq s \leq T_t$, we estimate the first integral in (5.2):

$$|\int_0^{T_t} \langle \dot{\varphi}_s, e(\varphi_s)\rangle\, ds| = |\nabla V(x)|^{-1}|\int_0^{T_t} \langle \dot{\varphi}_s, \nabla V(\varphi_s)\rangle\, ds|$$

$$\leq |\int_0^{T_t} \langle \dot{\varphi}_s, \nabla V(\varphi_s)\rangle\, ds|/\beta_*(\omega_t\lambda_t) = |V(\varphi_{T_t}) - V(\varphi_0)|/\beta_*(\omega_t\lambda_t)$$

$$\leq \frac{(1-\omega_t)\lambda_t}{\beta_*(\omega_t\lambda_t)} \leq \frac{\lambda_t}{\beta_*(\omega_t\lambda_t)} \leq \frac{H(\omega_t\lambda_t)}{C_*\omega_t}$$

where (4.3) is used in the final stage. By Assumption 7, for all large t,

$$\frac{1}{4}\beta_*(\omega_t\lambda_t)T_t \geq \frac{H(\omega_t\lambda_t)}{C_*\omega_t}.$$

Hence,

$$I_{0T_t}(\varphi) \geq \frac{1}{8}\beta_*^2(\omega_t\lambda_t)T_t \geq \lambda_t/(8\omega_t^2) \geq \lambda_t/\omega_t. \quad \square$$

PROOF OF LEMMA 11 The proof of this lemma essentially follows the lines of the proof of Lemma 4. We have to verify an upper bound analogous to that in Lemma 3. We prove the following version of Lemma 3: for any $\gamma > 0$ and any $x \in D((1-\omega_t)\lambda_t)$,

$\mathbf{P}_x\Big(X_s \notin D(\lambda_t) \text{ for some } s,\ \ 0 \leq s \leq T_t\Big) \leq \exp\Big\{-((1-\omega_t)\lambda_t - V(x))(1-\gamma)\Big\}.$

Consider a set $\{\varphi \ : \ \varphi_0 = x,\ \varphi_s \notin D((1-\omega_t)\lambda_t) \text{ for some } s,\ \ 0 \leq s \leq T_t\}$. Note that for any φ in this set, the inequality holds

$$I_{0T_t}(\varphi) \geq (1-\omega_t)\lambda_t - V(x) = \tilde{\lambda}_t.$$

Put $\delta_t = \sqrt{T_t}/\omega_t$ and assume that $\delta_t \leq \omega_t^2 H(\omega_t\lambda_t)$. We show that the δ_t-neighborhood of $D((1-\omega_t)\lambda_t)$ is in $D(\lambda_t)$. Indeed, the distance between $\partial D(\lambda_t)$ and $\partial D((1-\omega_t)\lambda_t)$ is no less than

$$\frac{\omega_t\lambda_t}{\min_{x\notin D((1-\omega_t)\lambda_t)} |\nabla V(x)|} = \frac{\omega_t\lambda_t}{\beta_*((1-\omega_t)\lambda_t)}$$

$$\geq \frac{\omega_t H((1-\omega_t)\lambda_t)}{(1-\omega_t)C^*} \geq \omega_t^2 H(\omega_t\lambda_t) \geq \delta_t.$$

Thus, if a trajectory X leaves $D(\lambda_t)$, then $\rho_{0T_t}(X, \Phi^x(\tilde{\lambda}_t)) > \delta_t$ where $\Phi^x(\tilde{\lambda}_t) = \{\varphi_s,\ 0 \leq s \leq T_t \ : \ \varphi_0 = x,\ I_{0T_t} \leq \tilde{\lambda}_t\}$, and the upper bound from Lemma 2 applies.

The proof is finished by making proper adjustments in the proof of Lemma 4 with $\tau = \inf\{s : \ X_s \notin D(\lambda_t)\}$ and $\eta = \inf\{s \geq T_t : \ X_s \in D(2\omega_t\lambda_t)\}$. □

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