# Spread of Branching Process with Independent Jump Increments under Stabilizing Drift

by

Olga A. Korosteleva
Purdue University

Technical Report # 02-02

Department of Statistics
Purdue University
West Lafayette, IN USA

February 2002

# Spread of Branching Process with Independent Jump Increments under Stabilizing Drift

Olga A. Korosteleva
Purdue University

ABSTRACT

The asymptotic behavior of the right endpoint of a one-dimensional branching diffusion is studied. The motion of a single particle in the model is assumed having a stabilizing drift and independent jump increments.
*Key words:* branching process, stabilizing drift, cumulant, the Legendre transform, action functional, quasipotential

## 1 INTRODUCTION

In a one-dimensional branching process, the initial particle positioned at the origin commences a motion described by the stochastic differential equation

$$\dot{X}_s = b(X_s) + \dot{\xi}_s, \quad s \geq 0, \tag{1.1}$$

where $b(x)$ is a stabilizing drift, and $\xi$ is a homogeneous Markov process with independent jump increments. The distribution of the length of a jump is given by a measure $\mu$. We suppose that $\mu$ has a compact support $K$ and $\mu(h) > 0$, where $h$ is the size of a maximal jump. We also assume that the measure $\mu$ has zero mean and a finite second moment, that is, $\int_K y\ \mu(dy) = 0$, and $\int_K y^2 \mu(dy) = d < \infty$. The times between consecutive jumps are exponential with the unit rate.

The original particle lives a random time which has an exponential distribution with parameter one and is independent of the travel of the particle, and then splits into two daughter particles. The created particles then behave independently with the same life history as the original particle.

The objective of the present paper is to describe the asymptotic behavior of the right frontier of the branching process. Let $R_t$ denote the position of the rightmost particle at time $t$. Consider an asymptotically polynomial drift $b(x)$, that is, $b(x) \sim -|x|^\alpha \mathrm{sgn}(x)$, $x \to \pm\infty$, $\alpha > 0$. We show that under an additional assumption on the drift $b(x)$, for any $\varkappa > 2$, as $t \to \infty$,

$$\mathbf{P}\left( \max(1, \frac{h}{\alpha}) \frac{t}{\log t} \le R_t \le (\varkappa + \frac{h}{\alpha}) \frac{t}{\log t} \right) \to 1. \tag{1.2}$$

To prove formula (1.2), we use the large deviations technique introduced in Freidlin and Wentzell [3]. In Korostelev and Korosteleva [5], we considered a similar branching process except that the random term of the movement was described by a standard Brownian motion. There, an individual trajectory attained a certain high level along a continuous extremal. In the present model, a similar mechanism of large deviations works for movements with not very strong drifts (that is, for $\alpha < 1$). In this case, a trajectory performs a large number of relatively small jumps in order to reach a high level. On the other hand, the mechanism is principally different if $\alpha \ge 1$. In that instance, it suffices for a particle to make one huge jump at time close to the terminal time $t$.

An interesting question is whether a sharper upper bound can be proved in the formula (1.2). We show the result only for $\varkappa$ arbitrarily close to 2, and can not do better due to intrinsic difficulties arising in the theory of large deviations.

It is appropriate here to say a few words about the existing results on the asymptotic behavior of the right frontier of a branching process without a drift. The methods of Freidlin [2] or Biggins [1] can be used to prove that for the driftless branching process, for any $\varepsilon > 0$, as $t$ tends to infinity,

$\mathbf{P}(|R_t/(at) - 1| < \varepsilon) \to 1$, where $a$ is a solution of the functional equation $L(a) = 1$, $L$ is the Legendre transform of the cumulant of the motion process. If, for example, the jump process is defined to be a symmetric Poisson process with jumps of unit size, then $a \approx 1.5088$ solves the equation $a \log(a + \sqrt{a^2+1}) = \sqrt{a^2+1}$.

Formula (1.2) does not cover the case $\alpha = 0$. However, Freidlin's theory [2] can be used for getting a result for the drift $b(x) = -\mathrm{sgn}(x)$. It states that for any $\varepsilon > 0$, $\mathbf{P}(|R_t/(bt) - 1| < \varepsilon) \to 1$ as $t \to \infty$, where $b^{-1}$ is a solution of the Hamilton-Jacobi equation $H(b^{-1}) = 0$, $H$ is the cumulant of the process of the motion. For the situation of the symmetric Poisson process with unit jumps (and the drift $-\mathrm{sgn}(x)$), $b \approx 0.6188$ is obtained by solving the equation $\cosh(b^{-1}) = 1 + b^{-1}$.

Freidlin's approach can not be applied directly to the models with a more general polynomial drift we are considering in this paper. His small parameter wave propagation theory accommodates processes with small intensity of jumps and high rate of splitting. Making a time-scale change $Y_u = X_{ut}/t,\ 0 \le u \le 1$, in (1.1), we obtain a new model with motion $Y$ satisfying the equation $\dot{Y}_u = b(tY_u) + \dot{\xi}_{ut}/t$, and splitting occurring with intensity $t$. For large values of $t$, this model meets the requirements of Freidlin's model except that the drift depends on large parameter $t$. Therefore, Freidlin's theory is not applicable here unless $b(x) = -\mathrm{sgn}(x)$.

In Korostelev and Korosteleva [5], a multi-dimensional generalization of a one-dimensional branching diffusion was studied. The proofs in both cases are similar. Here, a multi-dimensional branching process could also have been considered. Explicit form of the result would depend on possibility to solve the Hamilton-Jacobi equation for the gradient of the quasipotential.

The proof of the statement (1.2) is given in the other two sections of the paper. The upper and the lower bounds of the result are proved separately.

## 2 UPPER BOUND

In this section, we formulate an assumption on $b(x)$ under which the upper bound of (1.2) holds, that is, for any $\varkappa > 2$,

$$\mathbf{P}\Big(R_t \leq (\varkappa + \frac{h}{\alpha})\frac{t}{\log t}\Big) \to 1 \text{ as } t \to \infty.$$

For simplicity, we assume $h = 1$ throughout this section.

ASSUMPTION 1 (STABILIZING DRIFT) *For any $c > 0$, there exists a positive constant $\lambda = \lambda(c)$ such that*

$$b(X^{(1)}) - b(X^{(2)}) \geq \lambda(X^{(2)} - X^{(1)}), \ -c \leq X^{(1)} \leq X^{(2)} \leq c.$$

Denote $\rho_{0t}(X^{(1)},\ X^{(2)}) = \max\limits_{0\leq s\leq t}|X_s^{(1)} - X_s^{(2)}|$. For any $X$ such that $X_0 = x$, define an operator $B$ by $(BX)_s = X_s - x - \int_0^s b(X_u)\,du,\ s \geq 0$.

LEMMA 1 *Under Assumption 1,*

$$\rho_{0t}(X^{(1)},\ X^{(2)}) \leq 2\ \rho_{0t}(BX^{(1)},\ BX^{(2)}).$$

PROOF: For simplicity, introduce notation $\rho = \rho_{0t}(BX^{(1)},\ BX^{(2)})$, $v = X^{(2)} - X^{(1)}$, and $\zeta = BX^{(2)} - BX^{(1)}$. Fix some time $s$ in the interval $[0, t]$. Let $s_0$ be the last time prior to $s$ at which $v = 0$. Suppose first that $v$ is positive in $(s_0, s)$. By Assumption 1 with $c = \max\limits_{s_0\leq s\leq t}(|X_s^{(1)}|,\ |X_s^{(2)}|)$,

$$v_s = \zeta_s - \zeta_{s_0} - \int_{s_0}^{s}(b(X_u^{(1)}) - b(X_u^{(2)}))\,du \leq 2\,\rho - \lambda\int_{s_0}^{s} v_u\,du \leq 2\,\rho.$$

If we assume now that $v$ is negative in the same interval, then, since $-v$ is positive, we get

$$-v_s = -(\zeta_s - \zeta_{s_0}) - \int_{s_0}^{s}(b(X_u^{(2)}) - b(X_u^{(1)}))\,du \leq 2\,\rho - \lambda\int_{s_0}^{s}(-v_u)\,du \leq 2\,\rho.$$

The assertion follows. □

LEMMA 2 *Suppose Assumption 1 is true. Let $H_t = (\varkappa + \frac{1}{\alpha})t/\log t$. Then,*

$$\mathbf{P}\left(\max_{0\le s\le t} X_s \ge H_t\right) \le \exp\{-t(1+\varepsilon/8)\}.$$

PROOF: The cumulant of the process $X$, $X_0 = x$, satisfying (1.1), is

$$H(x,\theta) \stackrel{\text{def}}{=} \lim_{t\searrow 0} \frac{1}{t} \log \mathbf{E}\, e^{\theta(X_t - x)} = \lim_{t\searrow 0} \frac{1}{t} \log \mathbf{E} \exp\{\theta \int_0^t b(X_s)\, ds + \theta\, \xi_t\}$$

$$= \theta\, b(x) + \int_{[-1,1]} (e^{\theta y} - 1)\mu(\, dy) = \theta\, b(x) + \mu\{1\}e^{\theta} + o(e^{\theta}) \text{ for large } \theta.$$

The Legendre transformation of $H$ is

$$L(x,\beta) \stackrel{\text{def}}{=} \sup_{\theta} \Big[\theta\beta - H(x,\theta)\Big] = \sup_{\theta} \Big[\theta(\beta - b(x)) - \mu\{1\}e^{\theta} + o(e^{\theta})\Big]$$

$$= \begin{cases} (\beta - b(x))\left[\log \frac{\beta - b(x)}{\mu\{1\}} - 1\right] + o(\beta - b(x)),\ \beta - b(x) \to \infty & \text{if } \beta - b(x) > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

The action functional, as introduced in Freidlin and Wentzell [3], is defined for any time $T$ to be equal to $I_{0T}(\varphi) = \int_0^T L(\varphi_t, \dot{\varphi}_t)\, dt$ for absolutely continuous $\varphi$ and $+\infty$ otherwise.

By definition, a quasipotential $V(x) = \inf\{I_{0T}(\varphi) :\ T > 0,\ \varphi_0 = 0,\ \varphi_T = x\}$. As proved in the foregoing book, the quasipotential $V(x)$ satisfies the Hamilton-Jacobi equation $H(x, V'(x)) = 0$. From here,

$V'(x)b(x) + \mu\{1\}e^{V'(x)} + o(e^{V'(x)}) = 0$, so $V'(x) = \log|b(x)| + o(\log|b(x)|) = \alpha \log x + o(\log x)$ for large $x$. Therefore, $V(x) = \alpha x \log x + o(x \log x)$.

Since $\varkappa$ in the definition of $H_t$ can be taken arbitrarily close to 2, there exists $\varepsilon > 0$ such that $H_t = (1+\varepsilon)(2+\varepsilon+1/\alpha)t/\log t$.

Let $\delta_t = (1+\varepsilon)(2+\varepsilon)t/\log t$ and consider a function $\varphi$ such that $\varphi_0 = 0$, and $\varphi_t = H_t - \delta_t = (1+\varepsilon)t/(\alpha \log t)$. As shown in Freidlin and Wentzell,

$I_{0t}(\varphi) \geq V(\varphi_t) - V(\varphi_0)$. In this case,

$$I_{0t}(\varphi) \geq V(\frac{(1+\varepsilon)t}{\alpha \log t}) = \alpha \frac{(1+\varepsilon)t}{\alpha \log t} \log\left(\frac{(1+\varepsilon)t}{\alpha \log t}\right) + o(t) > (1+\frac{\varepsilon}{2})t, \quad t \to \infty.$$

Consider a set of absolutely continuous functions on the interval $[0,t]$ with bounded values of the action functional:

$$\Phi(\lambda) = \{\varphi_s,\ 0 \leq s \leq t: \ \int_0^t L(\dot{\varphi}_s - b(\varphi_s))\, ds \leq \lambda\}.$$

We have

$$\mathbf{P}\left(\max_{0 \leq s \leq t} X_s \geq H_t\right) \leq \mathbf{P}\left(\rho_{0t}(X,\ \Phi(\lambda_t)) > \delta_t\right) \tag{2.1}$$

where $\lambda_t = (1+\varepsilon/2)t$ for large $t$.

Let $\tilde{X} = BX$ and $\tilde{\varphi} = B\varphi$. By Lemma 1,

$$\mathbf{P}\left(\rho_{0t}(X,\ \Phi(\lambda_t)) > \delta_t\right) \leq \mathbf{P}\left(\rho_{0t}(\tilde{X},\ \tilde{\Phi}(\lambda_t)) > \frac{\delta_t}{2}\right) \tag{2.2}$$

where $\tilde{\Phi}(\lambda) = \{\tilde{\varphi}_s,\ 0 \leq s \leq t: \ \int_0^t L(\dot{\tilde{\varphi}}_s)\, ds \leq \lambda\}$.

Introduce a time-scale transformation $Y_u = \tilde{X}_{ut}/t,\ \psi_u = \tilde{\varphi}_{ut}/t,\ 0 \leq u \leq 1$.

We write

$$\int_0^1 L(\dot{\psi}_u)\, du = \int_0^1 \left[ (\frac{d\tilde{\varphi}_{ut}}{du}) \left( \log(\frac{d\tilde{\varphi}_{ut}}{du}) - 1 \right) + o(\frac{d\tilde{\varphi}_{ut}}{du}) \right] du$$

$$= \int_0^t \left[ (\frac{d\tilde{\varphi}_s}{ds}) \left( \log(\frac{d\tilde{\varphi}_s}{ds}) - 1 \right) + o(\frac{d\tilde{\varphi}_s}{ds}) \right] \frac{ds}{t} = \frac{1}{t} \int_0^t L(\dot{\tilde{\varphi}}_s)\, ds.$$

Therefore,

$$\mathbf{P}\left(\rho_{0t}(\tilde{X}, \tilde{\Phi}(\lambda_t)) > \frac{\delta_t}{2}\right) \leq \mathbf{P}\left(\rho_{01}(Y, \Psi(\lambda_t/t)) > \frac{\delta_t}{2t}\right) \tag{2.3}$$

where $\Psi(\lambda) = \{\psi_u,\ 0 \leq u \leq 1: \ \int_0^t L(\dot{\psi}_u)\, du \leq \lambda\}$.

Now, we would like to show

$$\mathbf{P}\left(\rho_{01}(Y, \Psi(1+\varepsilon/2)) > \frac{\delta_t}{2t}\right) \leq \exp\{-t(1+\varepsilon/8)\}. \tag{2.4}$$

To do so, we reproduce the prove of formula (2.6) of Chapter 5 in Freidlin and Wentzell [3], but with proper modifications.

Let $\gamma$ be arbitrarily small and take $\Delta = t^{\gamma-1}$. How small $\gamma$ should be chosen will be specified later. Consider a random polygon $l_s,\ 0 \leq s \leq 1$, joining vertices $(k\Delta, Y_{k\Delta}),\ k = 0, \cdots, n-1,\ n = \frac{1}{\Delta} = t^{1-\gamma}$. Introduce events

$$A_k = \{\max_{k\Delta \leq s \leq (k+1)\Delta} |Y_s - Y_{k\Delta}| < \frac{\delta_t}{2t},\ \max_{k\Delta \leq s \leq (k+1)\Delta} |Y_s - Y_{(k+1)\Delta}| < \frac{\delta_t}{2t}\}$$

where $k = 0, \cdots, n-1$. If all of the events $A_k$ occur, then $\rho_{01}(Y, l) < \delta_t/(2t)$ and, so, the event $\{\rho_{01}(Y, \Psi(1+\varepsilon/2)) > \delta_t/(2t)\}$ implies that $l \notin \Psi(1+\varepsilon/2)$. Thus, we have

$$\mathbf{P}\left(\rho_{01}(Y, \Psi(1+\varepsilon/2)) > \frac{\delta_t}{2t}\right) \leq \mathbf{P}\left(\bigcup_{k=0}^{n-1} A_k^c\right) + \mathbf{P}\left(\bigcap_{k=0}^{n-1} A_k \bigcap \{l \notin \Psi(1+\varepsilon/2)\}\right). \tag{2.5}$$

We estimate the first term

$$\mathbf{P}\left(\bigcup_{k=0}^{n-1} A_k^c\right) \leq \sum_{k=0}^{n-1} \mathbf{P}\left(A_k^c\right)$$

$$\leq \sum_{k=0}^{n-1} \left[\mathbf{P}\left(\max_{k\Delta \leq s \leq (k+1)\Delta} |Y_s - Y_{k\Delta}| > \frac{\delta_t}{2t}\right) + \mathbf{P}\left(\max_{k\Delta \leq s \leq (k+1)\Delta} |Y_s - Y_{(k+1)\Delta}| > \frac{\delta_t}{2t}\right)\right]$$

$$\leq 2\,n \max_y \mathbf{P}_y\left(\max_{0 \leq s \leq \Delta} |Y_s - y| > \frac{\delta_t}{2t}\right). \tag{2.6}$$

The last inequality is justified by the Markov property of the process $Y$ with respect to times $k\,\Delta,\ \ k = 0, \cdots, n$.

Next, $\mathbf{E}\, Y_s = s \int \beta\, \mu(d\beta) = 0$ for $0 \leq s \leq \Delta$, and $var\, Y_\Delta = \Delta \int \beta^2\, \mu(d\beta) = \Delta\, d$. By the Lévy inequality (see Gikhman and Skorokhod [4]),

$$\mathbf{P}_y\left(\max_{0 \leq s \leq \Delta} |Y_s - y| > \frac{\delta_t}{2t}\right) \leq 2\, \mathbf{P}_y\left(|Y_\Delta - y| > \frac{\delta_t}{2t} - 2\sqrt{\Delta\, d}\right). \tag{2.7}$$

From the exponential Chebyshev inequality, the probability in (2.7)

$$\leq \left[\mathbf{E} \exp\{C\, t\, (Y_\Delta - y)\} + \mathbf{E} \exp\{-C\, t\, (Y_\Delta - y)\}\right] \exp\{-C\, t\, \frac{\delta_t}{2t} + C\, t\, 2\sqrt{\Delta\, d}\}. \tag{2.8}$$

From formula (2.8) in Chapter 5 of Freidlin and Wentzell [3] it follows that

$$\mathbf{E} \exp\{C\, t\, (Y_\Delta - y)\} = \exp\{t\Delta\, H(C)\} = \exp\{t\Delta(e^C - 1)\} \leq \exp\{t\Delta\, e^C\},$$

and, similarly,

$$\mathbf{E} \exp\{-C\, t\, (Y_\Delta - y)\} = \exp\{t\Delta\, H(-C)\} = \exp\{t\Delta(e^{-C} - 1)\} \leq \exp\{t\Delta\, e^{-C}\}.$$

Thus, (2.8)

$$\leq \left(\exp\{t\Delta\, e^C\} + \exp\{t\Delta\, e^{-C}\}\right) \exp\{-C\, t\, \frac{\delta_t}{2t} + C\, t\, 2\sqrt{\Delta\, d}\}. \tag{2.9}$$

Choose $C = 2(1+\varepsilon)t/\delta_t = \log t/(1+\varepsilon/2)$. Then, (2.9) becomes

$$= \left(\exp\{t^{\gamma + \frac{1}{1+\varepsilon/2}}\} + \exp\{t^{\gamma - \frac{1}{1+\varepsilon/2}}\}\right) \exp\{-(1+\varepsilon)t + \frac{2\sqrt{d}}{1+\varepsilon/2}\, t^{\frac{1}{2}+\frac{\gamma}{2}} \log t\}$$

$$\leq \frac{1}{8n} \exp\{-(1+\varepsilon/8)t\} \text{ for large enough } t, \tag{2.10}$$

if $\gamma$ is such that $\gamma + 1/(1+\varepsilon/2) < 1$.

Combining (2.6)-(2.10), one gets

$$\mathbf{P}\Big(\bigcup_{k=0}^{n-1} A_k^c\Big) \leq \frac{1}{2}\exp\{-(1+\varepsilon/8)t\} \text{ for large } t. \tag{2.11}$$

Next, we would like to prove exactly the same bound for the second probability in (2.5). By the exponential Chebyshev inequality,

$$\mathbf{P}\Big(\bigcap_{k=0}^{n-1} A_k \bigcap \{l \notin \Psi(1+\varepsilon/2)\}\Big) = \mathbf{P}\Big(\bigcap_{k=0}^{n-1} A_k \bigcap \{I_{01}(l) > 1+\varepsilon/2\}\Big)$$

$$\leq \mathbf{E}\Big[\exp\{t\, I_{01}(l) - t\,(1+\varepsilon/2)\}\mathbb{I}_{\cap A_k}\Big]. \tag{2.12}$$

The action functional of the polygon is equal to

$$I_{01}(l) = \sum_{k=0}^{n-1}\int_{k\Delta}^{(k+1)\Delta} L(\dot{l}_s)\, ds$$

$$= \sum_{k=0}^{n-1}\int_{k\Delta}^{(k+1)\Delta} L\Big(\frac{Y_{(k+1)\Delta} - Y_{k\Delta}}{\Delta}\Big)\, ds = \sum_{k=0}^{n-1}\Delta\, L\Big(\frac{Y_{(k+1)\Delta} - Y_{k\Delta}}{\Delta}\Big).$$

Therefore, by the Markov property of the process $Y$ at times $k\,\Delta$, $k = 0, \cdots, n-1$, the expectation in (2.12) does not exceed

$$\exp\{-(1+\varepsilon/2)\,t\}\,\mathbf{E}\Big[\mathbb{I}_{\cap A_k}\prod_{k-0}^{n-1}\exp\{t\,\Delta\, L\Big(\frac{Y_{(k+1)\Delta} - Y_{k\Delta}}{\Delta}\Big)\}\Big]$$

$$\leq \exp\{-(1+\varepsilon/2)\,t\}\,\Big[\max_y \mathbf{E}\Big(\mathbb{I}_{A_0}\exp\{t\,\Delta\, L\Big(\frac{Y_\Delta - y}{\Delta}\Big)\}\Big)\Big]^n. \tag{2.13}$$

If the event $A_0$ occurs, then $|Y_\Delta - y| < \delta_t/(2t)$. For any $\beta$, $|\beta| < \delta_t/(2t\Delta)$, consider a polygon $\tilde{L}(\beta)$ such that $L(\beta) - \tilde{L}(\beta) \leq \varepsilon/4$.

Denote by $\beta_i$ the points at which $L(\beta_i) = i\varepsilon/4$, and define

$$\tilde{L}(\beta) = \max_{-i_0 \le i \le i_0} \Big[ L(\beta_i) + L'(\beta_i)(\beta - \beta_i) \Big] \quad \text{where}$$

$$i_0 = \left\lceil \frac{4}{\varepsilon} L\Big(\frac{\delta_t}{2t\Delta}\Big) \right\rceil \le c \frac{t^{1-\gamma}}{\log t} \log\left(\frac{t^{1-\gamma}}{\log t}\right) \le c\, t^{1-\gamma}, \quad t \to \infty, \quad \text{for some } c > 0.$$

Since the functions $H$ and $L$ are conjugate, we can rewrite

$$\tilde{L}(\beta) = \max_{-i_0 \le i \le i_0} \Big[ L'(\beta_i)\,\beta - H(L'(\beta_i)) \Big].$$

Thus, the expected value in (2.13) does not exceed

$$\mathbf{E} \exp\left\{ \frac{t\,\Delta\,\varepsilon}{4} + t\,\Delta\,\tilde{L}\Big(\frac{Y_\Delta - y}{\Delta}\Big) \right\}$$

$$= \mathbf{E} \exp\left\{ \frac{t\,\Delta\,\varepsilon}{4} + t\,\Delta \max_{-i_0 \le i \le i_0} \Big[ L'(\beta_i)\,\frac{Y_\Delta - y}{\Delta} - H(L'(\beta_i)) \Big] \right\}$$

$$\le \exp\{\frac{t\,\Delta\,\varepsilon}{4}\} \sum_{i=-i_o}^{i=i_0} \mathbf{E} \exp\Big\{ t\,L'(\beta_i)\,(Y_\Delta - y) - t\,\Delta\,H(L'(\beta_i)) \Big\}$$

$$= \exp\{\frac{t\,\Delta\,\varepsilon}{4}\}(2i_0 + 1). \tag{2.14}$$

To justify the above equality, use formula (2.8) in Chapter 5 of Freidlin and Wentzell [3]:

$$\mathbf{E} \exp\Big\{ t\,L'(\beta_i)\,(Y_\Delta - y) - t\,\Delta\,H(L'(\beta_i)) \Big\} = 1.$$

Finally, from (2.12) - (2.14), the second term in (2.5)

$$\mathbf{P}\Big(\bigcap_{k=0}^{n-1} A_k \bigcap \{l \notin \Psi(1+\varepsilon/2)\}\Big)$$

$$\leq \exp\{-(1+\varepsilon/2)\,t + n\,t\,\Delta\,\varepsilon/4 + n\,\log(2\,i_0+1)\}$$

$$\leq \exp\{-(1+\varepsilon/2)\,t + \varepsilon\,t/4 + c_1\,t^{1-\gamma}\log t\} \ \ (\text{for some } \ c_1 > 0)$$

$$\leq \frac{1}{2}\exp\{-(1+\varepsilon/8)\,t\} \ \text{ for all large } t. \tag{2.15}$$

From (2.11) and (2.15) the formula (2.4) holds, and, thus, the result of the lemma follows. □

THEOREM 1 *Suppose Assumption 1 holds. Then, for any* $\varkappa > 2$,

$$\mathbf{P}\left(R_t \leq (\varkappa + \frac{1}{\alpha})\frac{t}{\log t}\right) \to 1 \ \text{ as } \ t \to \infty.$$

PROOF: Denote $H_t = (\varkappa + \frac{1}{\alpha})t/\log t$. Let $N_t$ be the number of particles located at time $t$ above the level $H_t$. It is not difficult to show that the total number of particles alive at time $t$ has a geometric distribution with mean $\exp\{t\}$ (see, for example, Sevast'yanov [6]).

Therefore, $\mathbf{E}\,N_t = e^t\,\mathbf{P}\,(X_t \geq H_t)$. Hence, using the Markov inequality and the result of Lemma 2, we have

$$\mathbf{P}\Big(R_t \geq H_t\Big) = \mathbf{P}\Big(N_t \geq 1\Big) \leq \mathbf{E}\,N_t = e^t\,\mathbf{P}\,(X_t \geq H_t)$$

$$\leq \exp\{t \ - \ (1+\varepsilon/8)\,t\} = \exp\{-\,\varepsilon\,t/8\} \to 0 \ \text{ as } \ t \to \infty. \quad \square$$

## 3 LOWER BOUND

Here we focus on proving that as $t$ tends to infinity,

$$\mathbf{P}\left(R_t \geq \max(1, \frac{h}{\alpha})\frac{t}{\log t}\right) \to 1.$$

As in the previous section, we let $h = 1$ everywhere in the proof.

Consider an interval $[-L_t, L_t]$ where $L_t = (1 + 1/\alpha)t/\log^2 t$, and consider some time $T_t$ which is small compared to $t$. More information about $T_t$ will be given later. Let $N_t$ and $\bar{N}_t$ denote the number of particles located at time $t$ inside of this interval and outside of it, respectively.

LEMMA 3

$$\mathbf{P}\left(\bar{N}_{t-T_t} \leq \frac{e^{t-T_t}}{2\log t}\right) \to 1 \text{ as } t \to \infty.$$

PROOF: By the Markov inequality and the fact that the expected number of particles at time t is $e^t$, we have

$$\mathbf{P}\left(\bar{N}_{t-T_t} \geq \frac{e^{t-T_t}}{2\log t}\right) \leq 2\log t \, e^{-t+T_t}\,\mathbf{E}\bar{N}_{t-T_t}$$

$$= 2\log t \, e^{-t+T_t}\, e^{t-T_t}\,\mathbf{P}(X_{t-T_t} \notin [-L_t, L_t]) = 4\log t\,\mathbf{P}(X_{t-T_t} > L_t)$$

$$\leq 4\log t\,\mathbf{P}(\max_{0\leq s\leq t-T_t} X_s > L_t).$$

Next, we would like to show that

$$\mathbf{P}(\max_{0\leq s\leq t-T_t} X_s > L_t) \leq \exp\{-\frac{t}{8\log t}\}. \tag{3.1}$$

Then the result would follow:

$$\mathbf{P}\left(\bar{N}_{t-T_t} \geq \frac{e^{t-T_t}}{2\log t}\right) \leq 4\log t \, \exp\{-\frac{t}{8\log t}\} \to 0 \text{ as } t \to \infty.$$

To prove (3.1), we proceed as in the proof of Lemma 2. Let $\delta_t = t/\log^2 t$. The expression (2.9) for this case

$$\leq \exp\{ t^\gamma e^C - \frac{C\,\delta_t}{2} + C\,2\sqrt{d}\,t^{\frac{1}{2}+\frac{\gamma}{2}}\},$$

which for $C = \log t/2$ becomes

$$= 2\,\exp\{ t^{\gamma+\frac{1}{2}} - \frac{t}{4\log t} + \sqrt{d}\,t^{\frac{1}{2}+\frac{\gamma}{2}}\log t\}$$

$$\leq 8\,n\,\exp\{-\frac{t}{8\log t}\} \text{ for large enough } t \text{ if } \gamma < \frac{1}{2}.$$

Therefore, in the present setting the first term in (2.5) does not exceed

$$\frac{1}{2}\exp\{-\frac{t}{8\log t}\}.$$

Further,

$$V(L_t - \delta_t) = V(\frac{t}{\alpha\,\log^2 t}) > \frac{t}{2\log t}.$$

Take $\varepsilon = 1/\log t$. Then, as in (2.15), the second term in (2.5) for the problem under consideration,

$$\leq \exp\{-\frac{t}{2\log t} + \frac{t}{4\log t} + c_1 t^{1-\gamma}\log t\}$$

$$\leq \frac{1}{2}\exp\{-\frac{t}{8\log t}\} \text{ for large } t.$$

Thus, (3.1) is valid and the result holds. □

LEMMA 4

$$\mathbf{P}\left( N_{t-T_t} \geq \frac{e^{t-T_t}}{2\log t} \right) \to 1 \text{ as } t \to \infty.$$

PROOF: Denote by $\tilde{N}_t$ the total number of particles at time $t$. Compute

$$\mathbf{P}\left( \tilde{N}_{t-T_t} \geq \frac{e^{t-T_t}}{\log t} \right) = \sum_{n=e^{t-T_t}/\log t}^{\infty} e^{-t+T_t}\,(1 - e^{-t+T_t})^{n-1}$$

$$= (1 - e^{-t+T_t})^{-1+e^{t-T_t}/\log t} \to 1 \text{ as } t \to \infty.$$

This fact and the result of the previous lemma imply the assertion. □

LEMMA 5 *For any* $x \in [-L_t, L_t]$, *and for all large enough* $t$,

$$\mathbf{P}_x\left( X_{T_t} \geq \max(1, \frac{1}{\alpha})\frac{t}{\log t} \right) \geq \exp\{-t + c_t\, t\}$$

*where* $c_t = 1/(4\,\alpha\,\log t)$ *if* $\alpha < 1$, *and* $c_t = \log\log t/(2\log t)$ *if* $\alpha \geq 1$.

PROOF: First consider the case $\alpha < 1$. An individual trajectory in our model satisfies the equation $\dot{X} = -X^{\alpha} + \dot{\xi}$. Suppose, the process $X$ makes jumps of size $t^{\alpha}$ and reaches the level $t/(\alpha\log t)$. Therefore, $\Delta\,\xi = \Delta\,X + X^{\alpha} = t^{\alpha} + (\frac{t}{\alpha\,\log t} - x)^{\alpha} = k$, say. Now, we choose $T_t$ to satisfy $T_t\,k\,\log k = t$. Since $-(1+\frac{1}{\alpha})\,\frac{t}{\log^2 t} \leq x \leq (1+\frac{1}{\alpha})\frac{t}{\log^2 t}$, we estimate $k$ by

$$t^{\alpha} < t^{\alpha}\left[1 + \left(\frac{1}{\alpha\,\log t} - (1+\frac{1}{\alpha})\frac{1}{\log^2 t}\right)^{\alpha}\right] \leq k$$

$$\leq t^{\alpha}\left[1 + \left(\frac{1}{\alpha\,\log t} + (1+\frac{1}{\alpha})\frac{1}{\log^2 t}\right)^{\alpha}\right] < t^{2\alpha}.$$

Hence,

$$T_t k = \frac{t}{\log k} > \frac{t}{2\,\alpha\,\log t}, \quad T_t < \frac{t^{1-\alpha}}{\alpha\,\log t}, \quad \text{and} \quad T_t \log k < t^{1-\alpha}.$$

Therefore, we obtain for all sufficiently large $t$,

$$\mathbf{P}_x\Big(X_{T_t} \geq \frac{t}{\alpha\,\log t}\Big)$$

$$\geq \mathbf{P}_x\Big(X \text{ makes jumps of size } t^\alpha \text{ on } T_t \text{ unit intervals}\Big)$$

$$= \mathbf{P}_x\Big(\xi \text{ makes jumps of size } k \text{ on } T_t \text{ unit intervals}\Big) = \prod_{i=1}^{T_t} \frac{1^k}{k!}\,e^{-1}$$

which, by Stirling's formula,

$$= \exp\{-T_t + T_t k - T_t k \log k - \frac{\log 2\pi}{2} T_t - \frac{T_t}{2}\log k + o(\log k)\}$$

$$> \exp\{-\frac{3\,t^{1-\alpha}}{\alpha\,\log t} + \frac{t}{2\,\alpha\,\log t} - t - \frac{t^{1-\alpha}}{2} + o(\log t)\} > \exp\{-t + \frac{t}{4\,\alpha\,\log t}\}.$$

In case $\alpha \geq 1, T_t = 1$, and a trajectory reaches the desired level in one jump. We write

$$\mathbf{P}_x\Big(X_1 \geq \frac{t}{\log t}\Big)$$

$$\geq \mathbf{P}\Big(\xi \text{ makes one jump of size } \frac{t}{\log t} - x \text{ on the unit interval}\Big)$$

$$= \frac{e^{-1}}{(\frac{t}{\log t} - x)!} = \exp\{-1 + \frac{t}{\log t} - x - (\frac{t}{\log t} - x)\log(\frac{t}{\log t} - x) + o(\frac{t}{\log t})\}$$

$$> \exp\{-1 - t + \frac{t}{\log t}(1 + \frac{1}{\log t})(\log\log t - \log(1 + \frac{1}{\log t}))\}$$

$$> \exp\{-t + \frac{t\,\log\log t}{2\,\log t}\}. \quad \square$$

THEOREM 2

$$\mathbf{P}\left( R_t \geq \max(1, \frac{1}{\alpha})\frac{t}{\log t}\right) \to 1 \text{ as } t \to \infty.$$

PROOF: Let $H_t = \max(1, 1/\alpha)t/\log t$. Then, by Lemmas 4 and 5, we have

$$\mathbf{P}(R_t \geq H_t) \geq \mathbf{P}\left( R_t \geq H_t \,\middle|\, N_{t-T_t} \geq \frac{e^{t-T_t}}{2\log t}\right) \times$$

$$\times\, \mathbf{P}\left( N_{t-T_t} \geq \frac{e^{t-T_t}}{2\log t}\right)$$

$$\geq \mathbf{P}_x(\text{at least one of the particles reaches } H_t \text{ in time } T_t) \times$$

$$\times\, \mathbf{P}\left( N_{t-T_t} \geq \frac{e^{t-T_t}}{2\log t}\right)$$

$$\geq \left[1 - \left(1 - e^{-t+c_t t}\right)^{\frac{e^{t-T_t}}{2\log t}}\right] \mathbf{P}\left( N_{t-T_t} \geq \frac{e^{t-T_t}}{2\log t}\right) \to 1 \text{ as } t \to \infty. \quad \square$$

## REFERENCES


[1] J.D. Biggins. How fast does a general branching random walk spread? In K.B. Athreya and P. Jagers, editors, *Classical and Modern Branching Processes*. Springer, New York, 1996.

[2] M.I. Freidlin. *Functional integration and partial differential equations.* Annals of Mathematics Studies, 109. Princeton University Press, Princeton, NJ, 1985.

[3] M.I. Freidlin and A.D. Wentzell. *Random perturbations of dynamical systems.* Springer-Verlag, New York, 1998. Translated from the 1979 Russian original.

[4] I.I. Gikhman and A.V. Skorokhod. *Introduction to the theory of random processes.* Dover Publications, Inc., Mineola, NY, 1996. Translated from the 1965 Russian original. Reprint of the 1969 English translation.

[5] A.P. Korostelev and O.A. Korosteleva. Limit theorem for the spread of a branching diffusion with a stabilizing drift. *(Submitted).*

[6] B.A. Sevast'yanov. Branching stochastic processes for particles diffusing in a bounded domain with absorbing boundaries. *Th. Prob. Appl.*, 3:111–126, 1958.