A UNIFIED METHOD TO SUM A VARIETY OF INFINITE SERIES, WITH APPLICATIONS

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To Persi Diaconis, with admiration

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Finding closed form formulas for sums of infinite series is of use and interest in many disciplines. Standard methods often involve special tricks for special series. The purpose of this article is to present a unified method to sum a large variety of infinite series, including complicated iterated series. The unification comes through the use of two probabilistic identities, resulting in exact integral representations for the various series and hence a closed form formula. Due to the large variety of series, including iterated series, for which exact formulas are produced, the results may also have some reference and instructional value.

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1. Introduction

The purpose of this article is to present a unified method to find the sum of a large variety of infinite series by deriving exact integral representations for them. The unification comes from the nature of derivation of these integral representations; they are all derived by using one of two expectation identities. As a result we can avoid using special tricks for summing special infinite series; examples of such special tricks are use of Euler transformations and partial fraction expansions. A pleasant feature of our method is that we are able to write one dimensional exact integral representations for many types of iterated infinite series, including some rather complex iterated series. For most of the cases that we present here, fortunately, we are able to evaluate the corresponding integrals in closed form, thus resulting in an exact value for the sum of the convergent series. Failing a closed form evaluation of the needed integral in some other cases, it is still a simple task to compute the one dimensional integral numerically. This ought to be a particularly useful aspect of our method, for numerical integration of a function of one variable on a compact interval (actually [0,1]) should be faster and more reliable than numerically summing a multiple infinite series. For example, consider the quite neat formulas numbered 71, 72, 75, 76, 81, 82 in section 4.

A small selection of our personal favorite examples are:

$$\frac{1!}{1^1} + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \cdots = \int_0^1 \frac{1}{(1 + x\log x)^2} dx;$$

$$\sum\sum_{m,n \geq 1} \frac{1}{\binom{2(m+n)}{m+n}} = \frac{1}{3};$$

$$\sum\sum\sum_{m,n,p\geq 1} \frac{1}{\binom{2(m+n+p)}{m+n+p}} = \frac{2\pi}{27\sqrt{3}};$$

$$\sum\cdots\sum_{l,m,n,p,q\geq 1} \frac{1}{\binom{2(l+m+n+p+q)}{l+m+n+p+q}} = -\frac{1}{18} + \frac{10\pi}{243\sqrt{3}};$$

$$\sum\sum\sum_{m,n,p\geq 1} \frac{1}{(m+n+p)(m+n+p+1)(m+n+p+2)(m+n+p+3)} = \frac{1}{18};$$

$$\sum\sum_{m,n\geq 1} \frac{(-1)^{m+n}(m+n)!}{(m+n)^{m+n}} = 2\int_0^1 \frac{x\log\frac{1}{x}}{(1-x\log x)^3}dx;$$

$$\sum\sum\sum_{m,n,p\geq 1} \frac{1}{(m+n+p)!} = \frac{e}{2} - 1;$$

$$\sum\cdots\sum_{m,n,p,q\geq 1} \frac{1}{(m+n+p+q)!} = 1 - \frac{e}{3};$$

$$\sum\sum\sum_{m,n,p\geq 1} \frac{(-1)^{m+n+p}}{(m+n+p)!} = 1 - \frac{8}{3e};$$

$$\sum\cdots\sum_{m,n,p,q\geq 1} \frac{(-1)^{m+n+p+q}}{(m+n+p+q)!} = \frac{65}{24e} - 1.$$

In section 2, we present the two expectation identities and six representative examples. The six examples are supposed to give our reader a general understanding of how the integral representations fall out of the expectation identities. We chose to present a few illustrative examples because it seemed out of the question to give a derivation for every type of infinite series that is actually considered. There were too many. In section 3, the integral representations are used to state the sum of 27 different types of infinite series, including seven that are iterated series. We kept these 27 types as general as possible, in the sense they have parameters and particular values for the parameters lead to different particular infinite series. Such special interesting series are then presented as examples in section 4. These special examples are given in the form of a table in a way that a reader can easily figure out exactly how the sum was obtained from one of the general formulas of section 3. We believe that this table may have some reference value for general and instructional purposes.

Of course, it should be mentioned that the sums for many of the series considered are known; one would be able to locate them in a good text or one can evaluate them on symbolic software, such as Mathematica. Having said that, some of the series considered appeared to be new in the sense we did not find their explicit sums in the literature. In summary, we hope that this article has some aesthetic value due to the exact integral formulas and the unified method to derive them, as well as some practical value as a potential reference.

2. Illustrative Examples

First we state the two expectation identities. Their proofs are sketched merely for self-containedness, as they are easy to derive.

2.1 Two Expectation Identities

Lemma 1. Let $X \sim d\mu$ be a strictly positive random variable and for $t > 0$, let $\psi(t) = E(e^{-tX})$.

a. Let $n \geq 0$. If $E\left(\frac{1}{X^{n+1}}\right) < \infty$, then

$$E\left(\frac{1}{X^{n+1}}\right) = \int_0^\infty \frac{t^n}{n!}\psi(t)dt. \tag{1}$$

b. Let $n \geq 0$. If $E\left(\frac{1}{X(X+1)\cdots(X+n)}\right) < \infty$, then

$$E\left(\frac{1}{X(X+1)\cdots(X+n)}\right) = \int_0^\infty \frac{(1-e^{-t})^n}{n!}\psi(t)dt. \tag{2}$$

Sketches of proofs of (1) and (2) will be provided after the following six illustrative examples.

Example 1. In identity (2), take $n = 2$. By taking X to be degenerate at i, $i = 1, 4, 7, \cdots$, one gets

$$\frac{1}{1 \times 2 \times 3} = \frac{1}{2}\int_0^\infty (1 - e^{-t})^2 e^{-t} dt$$

$$\frac{1}{4 \times 5 \times 6} = \frac{1}{2}\int_0^\infty (1 - e^{-t})^2 e^{-4t} dt$$

$$\frac{1}{7 \times 8 \times 9} = \frac{1}{2}\int_0^\infty (1 - e^{-t})^2 e^{-7t} dt,$$

$$\cdots$$

On adding, $\frac{1}{1\times2\times3} + \frac{1}{4\times5\times6} + \frac{1}{7\times8\times9} + \cdots$

$$= \frac{1}{2}\int_0^\infty (1 - e^{-t})^2 (e^{-t} + e^{-4t} + e^{-7t} + \cdots) dt$$

$$= \frac{1}{2}\int_0^\infty (1 - e^{-t})^2 \frac{e^{-t}}{1 - e^{-3t}} dt$$

$$= \frac{1}{2}\int_0^1 \frac{(1 - x)^2}{1 - x^3} dx$$

$$= \frac{1}{2}\int_0^1 \frac{1 - x}{1 + x + x^2} dx. \tag{3}$$

Denote $R = 1 + x + x^2$; use the simple facts

$$\int \frac{1}{R} dx = \frac{2}{\sqrt{3}} \tan^{-1} \frac{1+2x}{\sqrt{3}} \tag{4}$$

$$\int \frac{x}{R} dx = \frac{1}{2} \log R - \frac{1}{2} \int \frac{1}{R} dx. \tag{5}$$

Plugging (4) and (5) into (3),

$$\int_0^1 \frac{1-x}{1+x+x^2} dx = \frac{3}{2} \frac{2}{\sqrt{3}} (\tan^{-1} \sqrt{3} - \tan^{-1} \frac{1}{\sqrt{3}}) - \frac{1}{2} \log 3 = \frac{\pi\sqrt{3}}{6} - \frac{1}{2} \log 3, \tag{6}$$

and so

$$\frac{1}{1 \times 2 \times 3} + \frac{1}{4 \times 5 \times 6} + \frac{1}{7 \times 8 \times 9} + \cdots = \frac{\pi\sqrt{3}}{12} - \frac{1}{4} \log 3. \tag{7}$$

Example 2. In this example, we will derive a formula for the iterated series

$$\sum_{i_1, \cdots, i_k \geq 1} \cdots \sum \frac{1}{(i_1 + i_2 + \cdots + i_k)!};$$

but to do this, we first need a one dimensional sum, as follows. In identity (1), take $n = 1$ and take X to be distributed as $\alpha +$ Poisson (1) for fixed $\alpha > 0$. Then,

$$E \frac{1}{X} = \sum_{x=0}^{\infty} \frac{1}{x+\alpha} \frac{e^{-1}}{x!} = \frac{1}{e} \sum_{x=0}^{\infty} \frac{1}{(x+\alpha)x!}. \tag{8}$$

On the other hand,

$$\psi(t) = \sum_{x=0}^{\infty} e^{-t(x+\alpha)} \frac{e^{-1}}{x!} = \frac{1}{e^{1+\alpha t}} e^{e^{-t}}. \tag{9}$$

Therefore, from identity (2),

$$\sum_{x=0}^{\infty} \frac{1}{(x+\alpha)x!} = \int_0^{\infty} e^{-\alpha t} e^{e^{-t}} dt = \frac{1}{\alpha} \int_0^1 e^{x^{\frac{1}{\alpha}}} dx = \int_0^1 x^{\alpha-1} e^x dx. \tag{10}$$

For example, taking $\alpha = \frac{1}{2}$, it follows from (10) that

$$\frac{1}{1 \times 0!} + \frac{1}{3 \times 1!} + \frac{1}{5 \times 2!} + \frac{1}{7 \times 3!} + \cdots = \int_0^1 e^{x^2} dx = 1.46265.$$

But our real interest is in the iterated series

$$\underset{i_1, \cdots, i_k \geq 1}{\Sigma \cdots \Sigma} \quad \frac{1}{(i_1 + i_2 + \cdots + i_k)!}.$$

Towards this end, note that $f_{n,k} = \#$ ways to write n as $i_1 + i_2 + \cdots + i_k$ with each of $i_1, i_2, \cdots, i_k \geq 1$ is given by $f_{n,k} = \frac{(n-1)!}{(k-1)!(n-k)!} I_{n \geq k}$

(see, e.g., Anderson (1992)). Thus,

$$S_k = \underset{i_1, \cdots, i_k \geq 1}{\Sigma \cdots \Sigma} \quad \frac{1}{(i_1 + i_2 + \cdots + i_k)!}$$

$$= \frac{1}{(k-1)!} \sum_{n=k}^{\infty} \frac{(n-1)!}{(n-k)!n!}$$

$$= \frac{1}{(k-1)!} \sum_{n=k}^{\infty} \frac{1}{n(n-k)!}$$

$$= \frac{1}{(k-1)!} \sum_{n=0}^{\infty} \frac{1}{(n+k)n!}$$

$$= \frac{1}{(k-1)!} \int_0^1 x^{k-1} e^x dx \text{ (by equation (10)).} \tag{11}$$

Therefore, from (11) by integration by parts, we immediately have the recurrence relation

$$k! S_{k+1} = e - k! S_k$$

$$\Rightarrow S_{k+1} = \frac{e}{k!} - S_k. \tag{12}$$

From (12), by induction,

$$S_{k+1} = e\sum_{j=0}^{k-2} \frac{(-1)^j}{(k-j)!} + (-1)^{k-1} S_2$$

$$= (-1)^k e \sum_{j=2}^{k} \frac{(-1)^j}{j!} + (-1)^{k-1}. \text{ (Since } S_2 = 1, \text{ directly from (11))}$$

We thus have the formula

$$\sum_{i_1, \cdots, i_k \geq 1} \cdots \sum \frac{1}{(i_1 + i_2 + \cdots + i_k)!} = (-1)^{k-1} e \sum_{j=2}^{k-1} \frac{(-1)^j}{j!} + (-1)^k. \tag{13}$$

Specifically, from (13),

$$\sum_{m,n \geq 1}\sum \frac{1}{(m+n)!} = 1;$$

$$\sum\sum_{m,n,p \geq 1}\sum \frac{1}{(m+n+p)!} = \frac{e}{2} - 1;$$

$$\sum\sum\sum_{m,n,p,q \geq 1}\sum \frac{1}{(m+n+p+q)!} = 1 - \frac{e}{3};$$

$$\sum_{l,m,n,p,q} \cdots \sum \frac{1}{(l+m+n+p+q)!} = \frac{3}{8}e - 1;$$

Example 3. This example gives an exact integral representation for the seemingly complicated iterated infinite series

$$S_k = \sum_{i_1, \cdots i_k \geq 1} \cdots \sum \frac{1}{(i_1 + \cdots + i_k)^{i_1 + \cdots + i_k}}.$$

Trivially, from identity (2) (or otherwise), for $n \geq 1$,

$$\frac{1}{n^n} = \frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-nt}dt$$

$$\Rightarrow \frac{(n-1)!}{(n-k)!n^n} = \frac{1}{(n-k)!}\int_0^\infty t^{n-1}e^{-nt}dt. \tag{14}$$

Using now again the fact that the number of ways to write n as $i_1 + \cdots + i_k$ with $i_1, \cdots, i_k \geq 1$ is $\frac{(n-1)!}{(k-1)!(n-k)!}I_{n\geq k}$, we evidently have

$$S_k = \frac{1}{(k-1)!}\sum_{n=k}^\infty \frac{(n-1)!}{(n-k)!n^n}$$

$$= \frac{1}{(k-1)!}\sum_{n=k}^\infty \int_0^\infty \frac{t^{n-1}e^{-nt}}{(n-k)!}dt \text{ (from (14))}$$

$$= \frac{1}{(k-1)!}\int_0^\infty t^{k-1}e^{-kt}\ e^{te^{-t}}dt$$

$$= \frac{1}{(k-1)!}\int_0^1 (x\log\frac{1}{x})^{k-1} x^{-x}dx \quad \text{(substituting } e^{-t} = x\text{)}. \tag{15}$$

Thus, in particular,

$$\sum_{n=1}^\infty \frac{1}{n^n} = \int_1^0 x^{-x}dx = 1.29129$$

$$\sum\sum_{m,n\geq 1} \frac{1}{(m+n)^{m+n}} = \int_0^1 (x\log\frac{1}{x})x^{-x}dx = .33719$$

$$\sum\sum\sum_{m,n,p\geq 1} \frac{1}{(m+n+p)^{m+n+p}} = \frac{1}{2}\int_0^1 \left(x\log\frac{1}{x}\right)^2 x^{-x}dx = .05091$$

$$\sum\sum\sum\sum_{l,m,n,p\geq 1} \frac{1}{(l+m+n+p)^{l+m+n+p}} = \frac{1}{6}\int_0^1 \left(x\log\frac{1}{x}\right)^3 x^{-x}dx = .00543$$

Example 4. In identity (2), take $n = 3$, and let X be degenerate at

$$\frac{1}{m}, \frac{1}{m}+1, \frac{1}{m}+2, \cdots,$$

for some fixed $m \geq 1$. Then we will have

$$\frac{1}{\frac{1}{m}(\frac{1}{m}+1)(\frac{1}{m}+2)(\frac{1}{m}+3)} = \frac{1}{3!}\int_0^\infty (1-e^{-t})^3 e^{-\frac{t}{m}}dt$$

$$\frac{1}{(\frac{1}{m}+1)(\frac{1}{m}+2)(\frac{1}{m}+3)(\frac{1}{m}+4)} = \frac{1}{3!}\int_0^\infty (1-e^{-t})^3 e^{-\frac{t}{m}-t}dt,$$

$$\cdots$$

On adding,

$$\frac{1}{1\times(m+1)\times(2m+1)\times(3m+1)} + \frac{1}{(m+1)\times(2m+1)\times(3m+1)\times(4m+1)} + \cdots$$

$$= \frac{1}{6m^4}\int_0^\infty e^{-\frac{t}{m}}(1-e^{-t})^3\frac{1}{1-e^{-t}}dt$$

$$= \frac{1}{6m^4}\int_0^\infty e^{-\frac{t}{m}}(1-e^{-t})^2 dt$$

$$= \frac{1}{6m^4} \int_0^1 x^{-\frac{1}{m}-1}(1-x)^2 dx$$

$$= \frac{1}{3m^4} \frac{\Gamma(\frac{1}{m})}{\Gamma(3+\frac{1}{m})} = \frac{1}{3m(m+1)(2m+1)}. \tag{16}$$

For instance, in particular,

$$\frac{1}{1\times 2\times 3\times 4} + \frac{1}{2\times 3\times 4\times 5} + \frac{1}{3\times 4\times 5\times 6} + \cdots = \frac{1}{18}$$

$$\frac{1}{1\times 3\times 5\times 7} + \frac{1}{3\times 5\times 7\times 9} + \frac{1}{5\times 7\times 9\times 11} + \cdots = \frac{1}{90}$$

$$\frac{1}{1\times 4\times 7\times 10} + \frac{1}{4\times 7\times 10\times 13} + \frac{1}{7\times 10\times 13\times 16} + \cdots = \frac{1}{252}.$$

Example 5. This is another example of an iterated series. Suppose we want to find the value of $S_{k,m} = \sum\cdots\sum_{i_1,\cdots,i_k\geq 1} \frac{1}{\binom{m(i_1+\cdots+i_k)}{i_1+\cdots+i_k}}$ for some given $m \geq 2$.

In identity (2), by taking X to be degenerate at $(m-1)n$, one gets

$$\frac{n!}{(m-1)n\Big((m-1)n+1\Big)\cdots\Big((m-1)n+n\Big)} = \int_0^\infty (1-e^{-t})^n e^{-t(m-1)n} dt$$

$$\Rightarrow \frac{n!\times 1\times 2\times\cdots\times\Big((m-1)n-1\Big)}{(mn)!} = \int_0^\infty (1-e^{-t})^n e^{-t(m-1)n} dt$$

$$\Rightarrow \frac{n!(mn-n)!}{(mn)!} = (mn-n)\int_0^\infty (1-e^{-t})^n e^{-t(m-1)n} dt$$

$$= (m-1)\int_0^\infty n(1-e^{-t})^n e^{-t(m-1)n}dt$$

$$\Rightarrow \frac{1}{\binom{mn}{n}} = (m-1)\int_0^\infty n(1-e^{-t})^n e^{-t(m-1)n}dt \tag{17}$$

Therefore, as in Example 3,

$$S_{k,m} = \frac{1}{(k-1)!}\sum_{n=k}^\infty \frac{(n-1)!}{(n-k)!}(m-1)\int_0^\infty n(1-e^{-t})^n e^{-t(m-1)n}dt$$

$$= \frac{(m-1)}{(k-1)!}\int_0^\infty \sum_{n=k}^\infty n(n-1)\cdots(n-k+1)\Big((1-e^{-t})e^{-t(m-1)}\Big)^n dt$$

$$= \frac{(m-1)}{(k-1)!}\int_0^\infty \Big((1-e^{-t})e^{-t(m-1)}\Big)^k \sum_{n=k}^\infty n(n-1)\cdots(n-k+1)\Big((1-e^{-t})e^{-t(m-1)}\Big)^{n-k} dt$$

$$= \frac{(m-1)k!}{(k-1)!}\int_0^\infty \frac{\Big((1-e^{-t})e^{-t(m-1)}\Big)^k}{(1-(1-e^{-t})e^{-t(m-1)})^{k+1}}dt$$

$$= k(m-1)\int_0^1 \frac{(1-x)^k x^{k(m-1)-1}}{\Big(1-x^{m-1}(1-x)\Big)^{k+1}}dx. \tag{18}$$

In particular, on taking $k=1$ and $m=2$,

$$\sum_{n=1}^\infty \frac{1}{\binom{2n}{n}} = \int_0^1 \frac{1-x}{(1-x+x^2)^2}dx = \frac{9+2\sqrt{3}\pi}{27}; \tag{19}$$

(the closed form integration is possible and we did it on Mathematica) on taking $k = 2$ and $m = 2$,

$$\sum\sum_{m,n \geq} \frac{1}{\binom{2(m+n)}{m+n}} = \frac{1}{3}; \tag{20}$$

(closed form integration was possible)

on taking $k = 3$ and $m = 2$,

$$\sum\sum\sum_{m,n,p \geq 1} \frac{1}{\binom{2(m+n+p)}{m+n+p}} = \frac{2\pi}{27\sqrt{3}}; \tag{21}$$

(again, closed form integration possible)

on taking $k = 2$ and $m = 3$,

$$\sum\sum_{m,n \geq 1} \frac{1}{\binom{3(m+n)}{m+n}} = .09820; \tag{22}$$

(numerical integration was done)

Formula (20) and (21) seem to be pretty and interesting.

Example 6. In identity (2), by taking X to be degenerate at $n + 1$,

$$\frac{n!}{(n+1)^{n+1}} = \int_0^\infty t^n e^{-nt} e^{-t} dt$$

$$\Rightarrow \frac{(n+1)!}{(n+1)^{n+1}} = \int_0^\infty (n+1)(te^{-t})^n e^{-t} dt, \quad n \geq 0$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{n!}{n^n} = \int_0^\infty \sum_{n=1}^{\infty} n(te^{-t})^{n-1} e^{-t} dt$$

$$= \int_0^\infty \frac{1}{(1-te^{-t})^2} e^{-t} dt$$

$$= \int_0^1 \frac{1}{(1+x\log x)^2} dx. \tag{23}$$

Similarly,

$$\sum_{n=1}^\infty (-1)^n \frac{n!}{n^n} = -\int_0^1 \frac{1}{(1-x\log x)^2} dx. \tag{24}$$

Using the two exact integral representations, we get

$$\frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \cdots = \int_0^1 \frac{1}{(1+x\log x)^2} dx - 1 = .87986$$

$$\frac{2!}{2^2} - \frac{3!}{3^3} + \frac{4!}{4^4} - \cdots = 1 - \int_0^1 \frac{1}{(1-x\log x)^2} dx = .34417$$

Proof of Lemma 1.

a . Actually, a more general identity holds. Let $s \geq 0$. Then,

$$\int_s^\infty t\psi(t)dt = \int_s^\infty t(\int e^{-tx} d\mu(x))dt$$

$$= \int (\int_s^\infty te^{-tx} dt) d\mu(x)$$

$$= \int (-\frac{d}{dx} \int_s^\infty e^{-tx} dt) d\mu(x)$$

$$= -\int \frac{d}{dx} (\frac{e^{-sx}}{x}) d\mu(x)$$

$$= s\int \frac{e^{-sx}}{x} d\mu(x) + \int \frac{e^{-sx}}{x^2} d\mu(x) \tag{25}$$

By induction,

$$\int_s^\infty t^k \psi(t)dt = (-1)^k \int (\frac{d^k}{dx^k} \frac{e^{-sx}}{x}) d\mu(x)$$

$$= \sum_{i=0}^k \frac{k!}{i!} s^i \int \frac{e^{-sx}}{x^{k-i+1}} d\mu(x). \tag{26}$$

By taking, $s = 0$,

$$\int_0^\infty t^k \psi(t)dt = k!E\left(\frac{1}{X^{k+1}}\right),$$

proving the identity.

b . The simplest way to prove identity (2) is to use the following pretty identity (see Molter (1985) for a proof): for any $x > 0, n \geq 0$,

$$\frac{\binom{n}{0}}{x} - \frac{\binom{n}{1}}{x+1} + \frac{\binom{n}{2}}{x+2} - \cdots + (-1)^n \frac{\binom{n}{n}}{x+n} = \frac{n!}{x(x+1)\cdots(x+n)}. \tag{27}$$

From (27),

$$n!E\left(\frac{1}{X(X+1)\cdots(X+n)}\right) = \sum_{k=0}^n (-1)^k \binom{n}{k} E\frac{1}{X+k}$$

$$= \sum_{k=0}^n (-1)^k \binom{n}{k} \int_0^\infty e^{-tk}\psi(t)dt$$

$$= \int_0^\infty (1-e^{-t})^n \psi(t)dt,$$

proving the identity.

3. General Applications

Identities (1) and (2) are used in this section to give formulas for sums of a large variety of infinite series (including iterated series). We have tried to keep these series as general as possible. Specific interesting special cases of these general series are presented as examples in the next section (section 4). For easy reference, first we state all the formulas.

Theorem 1. The following formulas each follow from Lemma 1:

1) For $\alpha > 0, n \geq 0$,

$$\sum_{k=0}^{\infty} \frac{1}{(\alpha+k)(\alpha+k+1)\cdots(\alpha+k+n)} = \frac{1}{n\alpha(\alpha+1)\cdots(\alpha+n-1)}. \tag{28}$$

2) 2) For $\alpha > 0, n \geq 0$,

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(\alpha+k)(\alpha+k+1)\cdots(\alpha+k+n)} = \frac{1}{\alpha(\alpha+1)\cdots(\alpha+n)} \quad {}_2F_1(1,\alpha;\alpha+n+1;-1). \tag{29}$$

3) For $n \geq 0$,

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)2k\cdots(2k+n)} = \frac{2^n}{(n+1)!}\left[\log 2 + \sum_{j=1}^{n} (-1)^j \binom{n}{j} \frac{2^j-1}{j2^j}\right]. \tag{30}$$

4) For $n \geq 0, m \geq 1$,

$$\sum_{k=1}^{\infty} \frac{1}{(mk-m+1)(mk-m+2)\cdots(mk+n)} = \frac{1}{(m+n-1)!}\int_0^1 \frac{(1-x)^{m+n-1}}{1-x^m}dx. \tag{31}$$

5) For $n \geq 0$,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k(k+1)\cdots(k+n)} = \frac{2^n}{n!}\left[\log 2 + \sum_{j=1}^{n}(-1)^j \binom{n}{j} \frac{2^j-1}{j2^j}\right]. \tag{32}$$

6) For $n \geq 0, m \geq 1$,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(mk-m+1)(mk-m+2)\cdots(mk+n)} = \frac{1}{(m+n-1)!}\int_0^1 \frac{(1-x)^{m+n-1}}{1+x^m}dx. \tag{33}$$

7) For $\alpha > 1$,

$$\sum_{k=1}^{\infty} \frac{k!}{(\alpha+1)\cdots(\alpha+k)} = \frac{1}{(\alpha-1)}. \tag{34}$$

8) For $\alpha > 0$,

$$\sum_{k=1}^{\infty} \frac{(-1)^k k!}{(\alpha+1)\cdots(\alpha+k)} = \alpha \int_1^2 \frac{(2-x)^{\alpha-1}(1-x)}{x} dx. \tag{35}$$

9) For $m, n \geq 1$,

$$\sum_{k=0}^{\infty} \frac{1}{(mk+1)(mk+m+1)\cdots(mk+nm+1)} = \frac{\Gamma(\frac{1}{m})}{nm^{n+1}\Gamma(n+\frac{1}{m})} \tag{36}$$

10) For $m, n \geq 1$,

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(mk+1)(mk+m+1)\cdots(mk+nm+1)}$$

$$= \frac{\Gamma(\frac{1}{m})}{m^{n+1}\Gamma(n+1+\frac{1}{m})} \;\; {}_2F_1(1, \frac{1}{m}; \frac{1}{m}+n+1; -1). \tag{37}$$

11) For $m \geq 2$,

$$\sum_{k=0}^{\infty} \frac{1}{\binom{mk}{k}} = (m-1) \int_0^1 \frac{x^{m-2}(1-x)}{\left(1-x^{m-1}(1-x)\right)^2} dx. \tag{38}$$

12) For $m \geq 2$,

$$\sum_{k=1}^{\infty} \frac{1}{k\binom{mk}{m}} = (m-1) \int_0^1 \frac{x^{m-2}(1-x)}{1-x^{m-1}(1-x)} dx. \tag{39}$$

13) For $m \geq 1$,

$$\sum_{k=1}^{\infty} \frac{1}{k^2\binom{mk}{m}} = -(m-1) \int_0^1 \frac{\log\left(1-x^{m-1}(1-x)\right)}{x} dx. \tag{40}$$

14) For $n \geq 1$,

$$\sum_{k=1}^{\infty} \frac{(k+n-1)!}{(2k+n)!} = e^{\frac{1}{4}} \int_0^1 x^{n-1} e^{-(x-\frac{1}{2})^2} dx. \tag{41}$$

15) For $m, n \geq 1$,

$$\sum_{k=1}^{\infty} \frac{(mk+n-1)!}{((m+1)k+n)!} = \int_0^1 x^{n-1} e^{x^m(1-x)} dx. \tag{42}$$

16) For $m \geq 1, \alpha < m$, and $\beta > 0$,

$$\sum_{k=1}^{\infty} \frac{1}{(mk-\alpha)(mk+\beta)}$$

$$= \frac{1}{\beta(\alpha+\beta)} + \frac{1}{m(\alpha+\beta)} \left(\psi\left(\frac{\beta}{m}\right) - \psi\left(\frac{\alpha}{m}\right) \right) - \frac{\pi}{m(\alpha+\beta)} \cot\left(\frac{\pi\alpha}{m}\right), \tag{43}$$

where $\psi\ (\cdot)$ is the diGamma function.

17) For any real α, and $\beta > 0, \beta + \gamma > 0$,

$$\sum_{n=1}^{\infty} \frac{\alpha^n}{(\beta n+\gamma)^n} = \frac{\alpha}{\beta} \int_0^1 x^{\frac{\gamma-\alpha x}{\beta}} dx. \tag{44}$$

18) For α, β and $\frac{\beta}{2}+\gamma > 0$,

$$\sum_{n=1}^{\infty} \frac{\alpha^n}{(\beta n+\gamma)^{2n}} = \frac{\sqrt{\alpha}}{2} \int_0^1 x^{\frac{\beta}{2}+\gamma-1} (x^{-\sqrt{\alpha}x^{\frac{\beta}{2}}} - x^{\sqrt{\alpha}x^{\frac{\beta}{2}}}) dx. \tag{45}$$

19) For $m \geq 1$, α real and $\gamma > 0$,

$$\sum_{n=0}^{\infty} \frac{\alpha^n}{(n+\gamma)^m n!} = \frac{1}{(m-1)!} \int_0^1 (\log \frac{1}{x})^{m-1} x^{\gamma-1} e^{\alpha x} dx. \tag{46}$$

20) For any real $\alpha, \beta > 0, \gamma \geq 0$, and $|\alpha| < \beta e$,

$$\sum_{n=0}^{\infty} \frac{\alpha^n n!}{(\beta n+\gamma)^n} = \alpha\beta \int_0^1 \frac{x^{\frac{\gamma}{\beta}}}{(\beta+\alpha x \log x)^2} dx. \tag{47}$$

21) For any real $\alpha, \beta > 0, \beta + \gamma > 0$ and $k \geq 1$,

$$\sum_{i_1,\cdots,i_k \geq 1} \cdots \sum \frac{\alpha^{i_1+i_2+\cdots+i_k}}{\Big(\beta(i_1+\cdots+i_k)+\gamma\Big)^{i_1+\cdots+i_k}} = \frac{\alpha^k}{\beta^k(k-1)!}\int_0^1 \left(x\log\frac{1}{x}\right)^{k-1} x^{\frac{\gamma-\alpha x}{\beta}}\,dx. \tag{48}$$

22) For any real $\alpha, \beta, \gamma > 0$ and $|\alpha| < \beta e$,

$$\sum_{i_1,\cdots,i_k \geq 1} \cdots \sum \frac{\alpha^{i_1+\cdots+i_k}(i_1+\cdots+i_k)!}{\Big(\beta(i_1+\cdots+i_k)+\gamma\Big)^{i_1+\cdots+i_k}} = k\alpha^k\beta\int_0^1 \frac{x^{\frac{\gamma}{\beta}}(x\log\frac{1}{x})^{k-1}}{(\beta+\alpha x\log x)^{k+1}}\,dx. \tag{49}$$

23) For any $m \geq 2$,

$$\sum_{i_1,\cdots,i_k \geq 1} \cdots \sum \frac{1}{\binom{m(i_1+\cdots+i_k)}{i_1+\cdots+i_k}} = k(m-1)\int_0^1 \frac{(1-x)^k x^{km-k-1}}{\Big(1-x^{m-1}(1-x)\Big)^{k+1}}\,dx. \tag{50}$$

24) For $\alpha \geq 0, k \geq 1, m > k-1$,

$$\sum_{i_1,\cdots,i_k \geq 1} \cdots \sum \frac{1}{(\alpha+i_1+\cdots+i_k)(\alpha+i_1+\cdots+i_k+1)\cdots(\alpha+i_1+\cdots i_k+m)}$$

$$= \frac{\Gamma(\alpha+k)}{m(m-1)\cdots(m-k+1)\Gamma(\alpha+m+1)}. \tag{51}$$

25) For $k \geq 1$, $\alpha > k$,

$$\sum_{i_1,\ldots,i_k \geq 1} \cdots \sum \frac{(i_1+\cdots+i_k)!}{(\alpha+1)(\alpha+2)\cdots(\alpha+i_1+\cdots+i_k)} = \frac{k!}{(\alpha-1)\cdots(\alpha-k)}. \tag{52}$$

26) For $k \geq 2$,

$$\sum_{i_1,\ldots,i_k \geq 1} \cdots \sum \frac{1}{(i_1+i_2+\cdots+i_k)!} = (-1)^{k-1}e\sum_{j=2}^{k-1}\frac{(-1)^j}{j!} + (-1)^k. \tag{53}$$

27) For $k \geq 2$,

$$\sum_{i_1,\ldots,i_k \geq 1} \cdots \sum \frac{(-1)^{i_1+\cdots+i_k}}{(i_1+\cdots+i_k)!} = \frac{(-1)^k}{e}(1+\sum_{j=1}^{k}\frac{1}{j!}) + (-1)^{k-1} \tag{54}$$

4. Specific Examples

In this final section, we present interesting special cases of the general formulas of section 3 and give explicit values for the sums of a variety of infinite series. We present these examples so that it would be easy for an interested reader to verify the reported values of the sums. The examples of this section may also be of some value as a useful single location summary for instructors and researchers.

	Series	Sum	Follows From Formula #	On using
28)	$\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\cdots$	1	1	$\alpha=1, n=1$
29)	$\frac{1}{1\times2\times3}+\frac{1}{2\times3\times4}+\frac{1}{3\times4\times5}+\cdots$	$\frac{1}{4}$	1	$\alpha=1, n=2$
30)	$\frac{1}{1\times2\times3\times4}+\frac{1}{2\times3\times4\times5}+\frac{1}{3\times4\times5\times6}+\cdots$	$\frac{1}{18}$	1	$\alpha=1, n=3$
31)	$\frac{1}{1\times2}-\frac{1}{2\times3}+\frac{1}{3\times4}-\cdots$	$2\log2-1$	2	$\alpha=1, n=1$
32)	$\frac{1}{1\times2\times3}-\frac{1}{2\times3\times4}+\frac{1}{3\times4\times5}-\cdots$	$2\log2-\frac{5}{4}$	2	$\alpha=1, n=2$
33)	$\frac{1}{1\times2\times3\times4}-\frac{1}{2\times3\times4\times5}+\frac{1}{3\times4\times5\times6}-\cdots$	$\frac{4}{3}\log2-\frac{8}{9}$	2	$\alpha=1, n=3$
34)	$\frac{1}{1\times2}+\frac{1}{3\times4}+\frac{1}{5\times6}+\cdots$	$\log2$	3	$n=0$
35)	$\frac{1}{1\times2\times3}+\frac{1}{3\times4\times5}+\frac{1}{5\times6\times7}$	$\log2-\frac{1}{2}$	3	$n=1$
36)	$\frac{1}{1\times2\times3\times4}+\frac{1}{3\times4\times5\times6}+\frac{1}{5\times6\times7\times8}+\cdots$	$\frac{2}{3}\log2-\frac{5}{12}$	3	$n=2$
37)	$\frac{1}{1\times2}-\frac{1}{3\times4}+\frac{1}{5\times6}-\cdots$	$\frac{\pi}{4}-\frac{\log2}{2}$	6	$m=2, n=0$
38)	$\frac{1}{1\times2\times3}-\frac{1}{3\times4\times5}+\frac{1}{5\times6\times7}-\cdots$	$\frac{1}{2}-\frac{\log2}{2}$	6	$m=2, n=1$
39)	$\frac{1}{1\times2\times3\times4}-\frac{1}{3\times4\times5\times6}+\frac{1}{5\times6\times7\times8}-\cdots$	$\frac{5}{12}-\frac{\pi}{12}-\frac{\log2}{6}$	6	$m=2, n=2$
40)	$\frac{1}{1\times2\times3}+\frac{1}{4\times5\times6}+\frac{1}{7\times8\times9}+\cdots$	$\frac{\pi\sqrt{3}}{12}-\frac{1}{4}\log3$	4	$m=3, n=0$
41)	$\frac{1}{1\times2\times3\times4}+\frac{1}{4\times5\times6\times7}+\frac{1}{7\times8\times9\times10}+\cdots$	$\frac{1}{6}+\frac{\pi}{12\sqrt{3}}-\frac{1}{4}\log3$	4	$m=3, n=1$

	Series	Sum	Follows From Formula #	On using
42)	$\frac{1}{1\times2\times3\times4}+\frac{1}{5\times6\times7\times8}+\frac{1}{9\times10\times11\times12}+\cdots$	$\frac{1}{4}\log 2-\frac{\pi}{24}$	4	$m=4$, $n=0$
43)	$\frac{1}{1\times2\times3}-\frac{1}{4\times5\times6}+\frac{1}{7\times8\times9}-\cdots$	$\frac{2}{3}\log 2-\frac{\pi\sqrt{3}}{18}$	6	$m=3$, $n=0$
44)	$\frac{1}{1\times2\times3\times4}-\frac{1}{4\times5\times6\times7}+\frac{1}{7\times8\times9\times10}-\cdots$	$\frac{4}{9}\log 2-\frac{1}{6}-\frac{\pi\sqrt{3}}{54}$	6	$m=3$, $n=1$
45)	$\frac{1}{\binom{2}{1}}+\frac{1}{\binom{4}{2}}+\frac{1}{\binom{6}{3}}+\cdots$	$\frac{1}{3}+\frac{2\sqrt{3}\pi}{27}$	11	$m=2$
46)	$\frac{1}{1\times\binom{2}{1}}+\frac{1}{2\times\binom{4}{2}}+\frac{1}{3\times\binom{6}{3}}+\cdots$	$\frac{\pi}{3\sqrt{3}}$	12	$m=2$
47)	$\frac{1}{1\times\binom{2}{1}}+\frac{1}{4\times\binom{4}{2}}+\frac{1}{9\times\binom{6}{3}}+\cdots$	$\frac{\pi^2}{18}$	13	$m=2$
48)	$\frac{1}{\binom{3}{1}}+\frac{1}{\binom{6}{2}}+\frac{1}{\binom{9}{3}}+\cdots$	${}_3F_2(1,\frac{3}{2},2;\frac{4}{3},\frac{5}{3};\frac{4}{27})/3$ $=.41432$	11	$m=3$
49)	$\frac{1}{1\times\binom{3}{1}}+\frac{1}{2\times\binom{6}{2}}+\frac{1}{3\times\binom{9}{3}}+\cdots$	${}_3F_2(1,1,\frac{3}{2};\frac{4}{3},\frac{5}{3};\frac{4}{27})/3$ $=.37122$	12	$m=3$
50)	$\frac{1}{1\times\binom{3}{1}}+\frac{1}{4\times\binom{6}{2}}+\frac{1}{9\times\binom{9}{3}}+\cdots$	${}_4F_3(1,1,1,\frac{3}{2};\frac{4}{3},\frac{5}{3},2;\frac{4}{27})/3$ $=.35146$	13	$m=3$
51)	$\frac{1!}{2!}+\frac{2!}{4!}+\frac{3!}{6!}+\cdots$	$e^{\frac{1}{4}}\sqrt{\pi}(\Phi(\frac{1}{\sqrt{2}})-\frac{1}{2})$ ($\Phi=N(0,1)$ CDF)	14	$n=2$
52)	$\frac{1!}{3!}+\frac{2!}{5!}+\frac{3!}{7!}+\cdots$	$e^{\frac{1}{4}}\sqrt{\pi}(2\Phi(\frac{1}{\sqrt{2}})-1)$	14	$n=1$
53)	$\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\cdots$	$\frac{1}{2}$	16	$m=2$, $\alpha=1,\beta=1$

	Series	Sum	Follows From Formula #	On using
54)	$\frac{1}{2\times 3}+\frac{1}{4\times 5}+\frac{1}{6\times 7}+\cdots$	$1-\log 2$	16	$m=2$, $\alpha=0,\beta=1$
55)	$\frac{1}{2\times 5}+\frac{1}{5\times 8}+\frac{1}{8\times 11}+\cdots$	$\frac{1}{6}$	16	$m=3$, $\alpha=1,\beta=1$
56)	$\frac{1}{3\times 7}+\frac{1}{7\times 11}+\frac{1}{11\times 15}+\cdots$	$\frac{1}{12}$	16	$m=4$, $\alpha=1,\beta=3$
57)	$\frac{1}{1^1}+\frac{1}{2^2}+\frac{1}{3^3}+\cdots$	$\int_0^1 x^{-x}dx$ $=1.29129$	17	$\alpha=1$, $\beta=1,\gamma=0$
58)	$\frac{1}{2^1}+\frac{1}{3^2}+\frac{1}{4^3}+$	$\int_0^1 x^{1-x}dx$ $=.62848$	17	$\alpha=1$, $\beta=1,\gamma=1$
59)	$\frac{1}{1^1}+\frac{1}{3^2}+\frac{1}{5^3}+\frac{1}{7^4}+\cdots$	$\frac{1}{2}\int_0^1 x^{-\frac{1+x}{2}}dx$ $=1.11955$	17	$\alpha=1$, $\beta=2,\gamma=-1$
60)	$\frac{1}{3^1}+\frac{1}{5^2}+\frac{1}{7^3}+\frac{1}{9^4}+\cdots$	$\frac{1}{2}\int_0^1 x^{\frac{1-x}{2}}dx$ $=.37641$	17	$\alpha=1$, $\beta=2,\gamma=1$
61)	$\frac{1}{1^1}-\frac{2}{2^2}+\frac{1}{3^3}-\frac{1}{4^4}+\cdots$	$\int_0^1 x^{x}dx$ $=.78343$	17	$\alpha=-1$, $\beta=1,\gamma=0$
62)	$\frac{1}{2^1}-\frac{1}{3^2}+\frac{1}{4^3}-\cdots$	$\int_0^1 x^{1+x}dx$ $=.40343$	17	$\alpha=-1$, $\beta=1,\gamma=1$
63)	$\frac{1}{1^1}-\frac{1}{3^2}+\frac{1}{5^3}-\frac{1}{7^4}+\cdots$	$\frac{1}{2}\int_0^1 x^{\frac{x-1}{2}}dx$ $=.89649$	17	$\alpha=-1$, $\beta=2,\gamma=-1$
64)	$\frac{1}{3^1}-\frac{1}{5^2}+\frac{1}{7^3}-\frac{1}{9^4}+\cdots$	$\frac{1}{2}\int_0^1 x^{\frac{1+x}{2}}dx$ $=.37641$	17	$\alpha=-1$, $\beta=2,\gamma=1$

	Series	Sum	Follows From Formula #	On using
65)	$\frac{1!}{1^1} + \frac{2!}{2^2} + \frac{3!}{3^3} + \cdots$	$\int_0^1 \frac{1}{(1+x\log x)^2} dx = 1.87985$	20	$\alpha = 1,$ $\beta = 1, \gamma = 0$
66)	$\frac{1!}{2^1} + \frac{2!}{3^2} + \frac{3!}{4^3} + \cdots$	$\int_0^1 \frac{x}{(1+\log x)^2} dx = .87985$	20	$\alpha = 1,$ $\beta = 1, \gamma = 1$
67)	$\frac{1!}{1^1} + \frac{2!}{3^2} + \frac{3!}{5^3} + \frac{4!}{7^4} + \cdots$	$2\int_0^1 \frac{1}{\sqrt{x}(1+x\log x)^2} dx = 1.28276$	20	$\alpha = 1,$ $\beta = 2, \gamma = -1$
68)	$\frac{1!}{3^1} + \frac{2!}{5^2} + \frac{3!}{7^3} + \frac{4!}{9^4} + \cdots$	$2\int_0^1 \frac{\sqrt{x}}{(1+x\log x)^2} dx = .37641$	20	$\alpha = 1,$ $\beta = 2, \gamma = 1$
69)	$\sum\sum_{m,n\geq 1} \frac{1}{(m+n)^{m+n}}$	$\int_0^1 (x\log\frac{1}{x}) x^{-x} dx = .33719$	21	$\alpha = 1, \beta = 1,$ $\gamma = 0, k = 2$
70)	$\sum\sum_{m,n\geq 1} \frac{(-1)^{m+n}}{(m+n)^{m+n}}$	$\int_0^1 (x\log\frac{1}{x}) x^{x} dx = .18647$	21	$\alpha = -1, \beta = 1,$ $\gamma = 0, k = 2$
71)	$\sum\sum\sum_{m,n,p\geq 1} \frac{1}{(m+n+p)^{m+n+p}}$	$\frac{1}{2}\int_0^1 (x\log\frac{1}{x})^2 x^{-x} dx = .05091$	21	$\alpha = 1, \beta = 1,$ $\gamma = 0, k = 3$
72)	$\sum\sum\sum_{m,n,p\geq 1} \frac{(-1)^{m+n+p}}{(m+n+p)^{m+n+p}}$	$-\frac{1}{2}\int_0^1 (x\log\frac{1}{x})^2 x^{x} dx = -.02704$	21	$\alpha = -1, \beta = 1,$ $\gamma = 0, k = 3$
73)	$\sum\sum_{m,n\geq 1} \frac{(m+n)!}{(m+n)^{m+n}}$	$2\int_0^1 \frac{x\log\frac{1}{x}}{(1+x\log x)^3} dx = 1.52317$	22	$\alpha = 1, \beta = 1,$ $\gamma = 0, k = 2$
74)	$\sum\sum_{m,n\geq 1} \frac{(-1)^{m+n}(m+n)!}{(m+n)^{m+n}}$	$2\int_0^1 \frac{x\log\frac{1}{x}}{(1-x\log x)^3} dx = .23526$	22	$\alpha = -1, \beta = 1,$ $\gamma = 0, k = 2$
75)	$\sum\sum\sum_{m,n,p\geq 1} \frac{(m+n+p)!}{(m+n+p)^{m+n+p}}$	$3\int_0^1 \frac{(x\log\frac{1}{x})^2}{(1+x\log x)^4} dx = 1.03795$	22	$\alpha = 1, \beta = 1,$ $\gamma = 0, k = 3$

	Series	Sum	Follows From Formula #	On using
76)	$\sum\sum\sum_{m,n,p\geq 1} \frac{(-1)^{m+n+p}(m+n+p)!}{(m+n+p)^{m+n+p}}$	$-3\int_0^1 \frac{(x\log\frac{1}{x})^2}{(1-x\log x)^4}dx$ $= -.07389$	22	$\alpha=-1,\beta=1,$ $\gamma=0,k=3$
77)	$\sum\sum_{m,n\geq 1} \frac{1}{\binom{2(m+n)}{m+n}}$	$\frac{1}{3}$	23	$k=2,m=2$
78)	$\sum\sum\sum_{m,n,p\geq 1} \frac{1}{\binom{2(m+n+p)}{m+n+p}}$	$\frac{2\pi}{27\sqrt{3}}$	23	$k=3,m=2$
79)	$\sum\sum\sum\sum_{\ell,m,n,p\geq 1} \frac{1}{\binom{2(\ell+m+n+p)}{\ell+m+n+p}}$	$\frac{1}{9}-\frac{8\pi}{243\sqrt{3}}$	23	$k=4,m=2$
80)	$\sum\sum\sum\sum_{\ell,m,n,p,q\geq 1} \frac{1}{\binom{2(\ell+m+n+p+q)}{\ell+m+n+p+q}}$	$-\frac{1}{18}+\frac{10\pi}{243\sqrt{3}}$	23	$k=5,m=2$
81)	$\sum\sum_{m,n\geq 1} \frac{1}{\binom{3(m+n)}{m+n}}$	$4\int_0^1 \frac{(1-x)^2x^3}{(1-x^2(1-x))^3}dx$ $= .09820$	23	$k=2,m=3$
82)	$\sum\sum\sum_{m,n,p\geq 1} \frac{1}{\binom{3(m+n+p)}{m+n+p}}$	$6\int_0^1 \frac{(1-x)^3x^5}{(1-x^2(1-x))^4}dx$ $= .020667$	23	$k=3,m=3$
83)	$\sum\sum_{m,n\geq 1} \frac{1}{(m+n)(m+n+1)(m+n+2)}$	$\frac{1}{4}$	24	$\alpha=0,k=2,$ $m=2$
84)	$\sum\sum_{m,n\geq 1} \frac{1}{(m+n)(m+n+1)(m+n+2)(m+n+3)}$	$\frac{1}{36}$	24	$\alpha=0,k=2,$ $m=3$
85)	$\sum\sum\sum_{m,n,p\geq 1} \frac{1}{(m+n+p)(m+n+p+1)(m+n+p+2)(m+n+p+3)}$	$\frac{1}{18}$	24	$\alpha=0,k=3,$ $m=3$
86)	$\sum\cdots\sum_{m,n,p,q\geq 1} \frac{1}{(m+n+p+q)\cdots(m+n+p+q+4)}$	$\frac{1}{96}$	24	$\alpha=0,k=4,$ $m=4$
87)	$\sum\sum_{m,n\geq 1} \frac{1}{(m+n)!}$	1	26	$k=2$
88)	$\sum\sum\sum_{m,n,p\geq 1} \frac{1}{(m+n+p)!}$	$\frac{e}{2}-1$	26	$k=3$

	Series	Sum	Follows From Formula #	On using
89)	$\sum\cdots\sum_{m,n,p,q\geq 1} \frac{1}{(m+n+p+q)!}$	$1-\frac{e}{3}$	26	$k=4$
90)	$\sum\cdots\sum_{\ell,m,n,p,q\geq 1} \frac{1}{(\ell+m+n+p+q)!}$	$\frac{3}{8}e-1$	26	$k=5$
91)	$\sum\cdots\sum_{\ell,m,n,p,q,r\geq 1} \frac{1}{(\ell+m+n+p+q+r)!}$	$1-\frac{11}{30}e$	26	$k=6$
92)	$\sum\sum\sum_{m,n,\geq 1} \frac{(-1)^{m+n}}{(m+n)!}$	$\frac{5}{2e}-1$	27	$k=2$
93)	$\sum\sum\sum_{m,n,p\geq 1} \frac{(-1)^{m+n+p}}{(m+n+p)!}$	$1-\frac{8}{3e}$	27	$k=3$
94)	$\sum\cdots\sum_{m,n,p,q\geq 1} \frac{(-1)^{m+n+p+q}}{(m+n+p+q)!}$	$\frac{65}{24e}-1$	27	$k=4$
95)	$\sum\cdots\sum_{\ell,m,n,p,q\geq 1} \frac{(-1)^{(\ell+m+n+p+q)}}{(\ell+m+n+p+q)!}$	$1-\frac{163}{60e}$	27	$k=5$
96)	$\sum\cdots\sum_{\ell,m,n,p,q,r\geq 1} \frac{(-1)^{\ell+m+n+p+q+r}}{(\ell+m+n+p+q+r)!}$	$\frac{1957}{720e}-1$	27	$k=6$

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