BASU'S THEOREM, POINCARÉ INEQUALITIES, AND INFINITE DIVISIBILITY

by

Anirban DasGupta
Purdue University

Technical Report #02-04

Department of Statistics
Purdue University

June 2002

BASU'S THEOREM, POINCARÉ INEQUALITIES, AND INFINITE DIVISIBILITY

Anirban DasGupta

Purdue University, West Lafayette, IN

JUNE 5, 2002

ABSTRACT

We define a notion of approximate sufficiency and approximate ancillarity and show that such statistics are approximately independent pointwise under each value of the parameter, or in an average sense. We do so without mentioning the somewhat nonintuitive concept of completeness, thus providing a more transparent version of Basu's theorem. Some use of Poincaré inequalities is made in proving these results.

We also show some new types of applications of Basu's theorem in the theory of probability. The applications are to showing that large classes of random variables are infinitely divisible, and that others admit a decomposition in the form YZ, where Y is infinitely divisible, Z is not, both are nondegenerate, and Y and Z are independent.

These applications indicate that the possible spectrum of applications of Basu's theorem is much broader than has been realized.

1 Introduction

It has been nearly half a century that Basu(1955) proved what has turned out to be one of the most well known theorems of basic statistical theory. Popularly known as Basu's theorem, his result says that a boundedly complete and sufficient statistic is independent of an ancillary statistic under all values of the parameter. Although Basu's theorem is discussed in many texts, a particularly comprehensive and delightful review with many types of applications is given in Ghosh(2001). The purpose of this article is to examine Basu's theorem in certain new ways and to offer a collection of new and probably even surprising applications of Basu's theorem. We also show that there are certain technical connections between Basu's theorem and Poincaré inequalities in analysis and partial differential equations.

The statistical intuition in Basu's theorem is that a statistic which captures all the information in a sample on an unknown parameter and another which captures none should provide no information about each other, and thus should be independent. The condition of completeness is in a way a necessary technical evil; one fails to see the intuition of requiring completeness; see, however, Lehmann(1981) and Ghosh(2001). It therefore seems natural to ask whether a sufficient statistic and an ancillary statistic would be 'nearly' independent anyway, in some well formulated sense. We ask, in fact, a slightly more general question : is it the case that an approximately sufficient statistic and an approximately ancillary statistic are approximately independent under all values of the parameter ? Of course, approximate sufficiency and ancillarity would have to be defined; but that is what we show in a certain formulation of these two concepts and here is where the use of Poincaré inequalities of analysis is made. Basu's theorem itself follows as a consequence of our results when the extra condition of completeness is added in. Viewed in this manner, we think, the real intuition of Basu's theorem comes through in a more satisfactory manner because we can establish the 'near independence' without the somewhat abstract condition of completeness. Section 2 describes these new formulations of Basu's theorem.

It is well known that although Basu's theorem is a theorem in statistical inference, it can be used to find easy solutions of distributional questions in probability. Ghosh(2001) gives many examples of applications of Basu's theorem in deriving joint distributions of random variables bypassing heavy calculations that would be needed in a direct attack. However, we give some entirely new types of applications. We show that a wide variety of random variables are infinitely divisible by simultaneous use of Basu's theorem and an extended version of the well known Goldie-Steutel law of infinite divisibility (Goldie(1967), Bose, DasGupta and Rubin (2002)). The results do not follow from just the extended Goldie-Steutel law; Basu's theorem is needed. To our knowledge, Basu's theorem has never been applied prior to this in proving infinite divisibility. These results are presented in section 3.

In section 4, we present yet another new type of application of Basu's theorem. We apply Basu's theorem in factorization of random variables, a topic of some interest in probability theory. The typical result we show says that a certain random variable X can be factorized as $X = YZ$, where Y is infinitely divisible, but Z is not, and Y and Z are independent. Again, this is a new type of application of Basu's theorem.

In summary, the intention of this article is to give what we believe is a more transparent and more intuitive interpretation of Basu's theorem by avoiding the mention of completeness, and to demonstrate that the possible horizon of applications of Basu's theorem is probably much wider than has been understood so far. It seems likely that the applications to infinite divisibility and factorization will lead to further applications in the theory of probability.

2 Basu's Theorem and Poincaré Inequalities

The extensions to Basu's theorem are presented in this section. We present three results and each is given in terms of the total variation distance between a true joint distribution and the distribution that is product of the corresponding marginals. For example, for specificity, suppose T is a sufficient statistic, and U an ancillary and θ is an unknown parameter. Let P_θ denote the joint distribution of T and U and Q_θ the product measure. Since T and U are independent under each θ if T is also boundedly complete, in such a case the total variation distance between P_θ and Q_θ would be zero.We ask what kinds of bounds can we establish on the total variation distance between P_θ and Q_θ without requiring completeness of T. For some of the results, we ask the same question, only taking T to be 'approximately sufficient' and U to be 'approximately ancillary'. Choice of total variation was a conscious choice as it seems very natural, but of course similar results should be possible with other distances, such as Kullback-Leibler or Hellinger. Poincaré inequalities are used in proving two of the three results. It should be remarked that concepts of approximate sufficiency have been around for a long time; see Le Cam(1964,1974,1986), Reiss(1978), and Brown and Low(1996) for exposition and applications.

For the rest of the article, the notation d_θ will denote the total variation distance between P_θ and Q_θ, where P_θ is the joint distribution under θ of a pair of random variables T and U, and Q_θ is the corresponding product measure. The exact definition of T and U will be given in the specific context.

First, we introduce some notation to be used for the rest of the article.

With respect to some common dominating measure $\mu \otimes \nu$, let $p_\theta(t,u)$ = joint density of (T,U); $f_\theta(u,t)$ = conditional density of U given $T = t$; $g_\theta(t)$ = marginal density of T; $h_\theta(u)$ = marginal density of U; P_θ = true joint distribution of (T,U); Q_θ = product measure corresponding to the marginals of T and U; d_θ = total variation distance between P_θ and Q_θ.

Note that if T is sufficient and U is ancillary, then for a.a. $(t,u)(\mu \otimes \nu), f_\theta(u,t)$ is independent of θ; i.e., there is a function $f(u,t)$ which acts as the conditional density under each θ. For the sake of notational simplicity, we will use the Lebesgue measure for each of μ and ν.

Theorem 1. Let T, U be general statistics. Then

$$d_\theta \leq \int \sqrt{Var_\theta(f_\theta(u,T))}du.$$

Proof: By definition,

$$2d_\theta = \int |p_\theta(t,u) - g_\theta(t)h_\theta(u)|dtdu$$

$$= \int |f_\theta(u,t) - h_\theta(u)|g_\theta(t)dtdu;$$

now observe that $E_{T|\theta}(f_\theta(u,T)) = h_\theta(u)$, and hence, $\int |f_\theta(u,t) - h_\theta(u)|g_\theta(t) \leq \sqrt{Var_\theta(f_\theta(u,T))}$; integrating this inequality over u gives the inequality of the theorem.

Corollary 1. Suppose T is complete and sufficient and U is ancillary. Then under each θ, T and U are independent.

Proof : First, as we remarked before, there is a fixed function $f(u,t)$ that acts as the conditional density of U given $T = t$ for any θ. Likewise, because U is ancillary, there is a fixed function $h(u)$ which acts as the density of U for any θ.

Since $E_{T|\theta}(f(u,T)) = h(u) \forall \theta$, by virtue of the completeness of T, $f(u,t) =$

$h(u)$ for a.a. (u,t), giving $Var_\theta(f(u,T)) = 0$ for a.a. u. Hence, from Theorem 1, $d_\theta = 0$, and so under each θ, T and U are independent.

We illustrate Theorem 1 by an example.

Example 1

Suppose $X_1, X_2, \ldots, X_n$ are iid $U[0,\theta]$. Let $T = X_{(n)}$ and $U = X_{(1)}$. Note that T is sufficient (and even complete), but U is not ancillary. But intuitively, U has almost no information about θ, and so almost an ancillary. One might expect that T and U are almost independent. Let us see how Theorem 1 works out in this example.

Since T is sufficient, the conditional density of U given T is free of θ; denoting it by $f(u,t)$, by a direct calculation,

$$f(u,t) = \text{n(t-u)}^{n-1}/t^n I_{t>u}.$$

On the other hand, the density of T under θ of course is $g_\theta(t) = nt^{n-1}/\theta^n I_{0<t<\theta}$. Then on a few lines of algebra,

$$E_\theta(f(u,T)) = n^2/\theta^n[(-1)^{n-1}u^{n-1}(\log\theta - \log u) + \sum_{j=0}^{n-2}(-1)^j \begin{pmatrix} n-1 \\ j \end{pmatrix} u^j(\theta^{n-1-j} - u^{n-1-j})/(n-1-j)];$$

Similarly, $E_\theta(f(u,T)^2) = n^3/\theta^n[(-1)^{n-2} \begin{pmatrix} 2n-2 \\ n-2 \end{pmatrix} u^{n-2}(\log\theta - \log u) + \sum_{j\neq n-2}(-1)^j \begin{pmatrix} 2n-2 \\ j \end{pmatrix} u^j(\theta^{n-2-j} - u^{n-2-j})/(n-2-j)].$

These expressions provide $Var_\theta(f(u,T))$. The bound of Theorem 1 is obtained by integrating the square root over $u \in (0,\theta)$. This integral can be very easily done numerically, but not in a closed form. We provide below a few illustrative values; the bound is independent of θ.

n	Bound of Theorem 1 on d_θ
5	.168
10	.078
15	.051
20	.042

The bounds are consistent with the intuition that $X_{(n)}$ and $X_{(1)}$ should be nearly independent for large n.

Next we give two results that show that an approximately sufficient statistic and an approximately ancillary statistic are approximately independent. There is no mention of completeness in these results. The theorems differ somewhat in their approach in that one of them talks about independence under each θ, while the other talks about independence on an *average* (see the exact statement of Theorem 3). First we specify a notion of approximate sufficiency and approximate ancillarity.

Definition 1 A statistic T is δ - sufficient with respect to another statistic U if $|\partial f_\theta(u,t)/\partial\theta| \leq \delta$ for a.a. (u,t) under each θ.

Definition 2A statistic U is called ϵ - ancillary if $|\partial h_\theta(u)/\partial\theta| \leq \epsilon$ for a.a. u under each θ.

Remark. Obviously, it is a part of the definitions that the stated partial derivatives exist. If T is δ - sufficient with $\delta = 0$, then the conditional density of U given T does not depend on θ, which would be true if T is sufficient.

Thus δ sufficiency is a notion of approximate sufficiency, but only locally, in the sense it is with respect to the specified statistic U. Analogously, ϵ ancillarity with $\epsilon = 0$ would mean ancillarity. Thus, the notions of approximate sufficiency and ancillarity given above are weaker than their usual meanings. However, we will see in the next two theorems that one can obtain results in the spirit of Basu's theorem with these weaker notions.

We now provide the theorems and their proofs. Two different Poincaré inequalities are used in the proofs of these two theorems. We state them first before giving the theorems.

Lemma 1 Let u be once continuously differentiable on the interval [0,1] and suppose $\lim_{x\to 0} u(x) = 0$. Then, pointwise, $|u(x)| \leq \sqrt{\int_0^1 (\partial u(x)/\partial x)^2 dx}$.

Lemma 2 Let u be once continuously differentiable on the interval [0,1] and suppose $\lim_{x\to 0(1)} u(x) = 0$. Then, $\int_0^1 u^2(x)dx \leq 4/\pi^2 \int_0^1 (\partial u(x)/\partial x)^2 dx$.

See Flavin and Rinero(1996) and Bullen(1998) for these and a variety of other Poincaré inequalities in one and many dimensions.

Theorem 2. Let U be ϵ - ancillary and T δ - sufficient with respect to U. Suppose U and θ are bounded, taking values in, say, [0,1]. Assume $f_\theta(u,t), g_\theta(t)$ and $h_\theta(u)$ are once continuously differentiable in θ for a.a. (u,t) and that $\lim_{\theta\to 0}(f_\theta(u,t) - h_\theta(u))\sqrt{g_\theta(t)} = 0$ for a.a. (u,t). Assume also that $\exists$ constants k_1, k_2 such that $f_\theta(u,t) \leq k_1$, and $h_\theta(u) \leq k_2$. Then,

$$d_\theta \leq 1/\sqrt{2}\sqrt{2(\delta^2 + \epsilon^2) + max^2(k_1,k_2) \int I_g(\theta) d\theta/4},$$

where $I_g(\theta)$ denotes the Fisher information in T about θ.

Remark. The interpretation of Theorem 2 is that if T is approximately sufficient and U approximately ancillary, then provided that T and U are independent under a 'degenerate' value of θ, they are approximately independent under all θ. There is no mention of completeness here. The conditions that U and θ belong to [0,1] can be relaxed to the conditions that they belong to bounded intervals. However, due to our use of an unweighted Poincaré inequality in the proof, the boundedness condition is necessary.

Proof: By definition, $2d_\theta = \int_0^1 \int |f_\theta(u,t) - h_\theta(u)| g_\theta(t) dt du$

$\leq \sqrt{\int_0^1 (\int |f_\theta(u,t) - h_\theta(u)| g_\theta(t) dt)^2 du}$

(by Schwartz's inequality applied to U)

$\leq \sqrt{\int_0^1 \int (f_\theta(u,t) - h_\theta(u))^2 g_\theta(t) dt du}$

(by Schwartz's inequality applied to T).

Consider now the function $u = u(\theta) = (f_\theta(u,t) - h_\theta(u))\sqrt{g_\theta(t)}$. By the pointwise Poincaré inequality (Lemma 1),

$u(\theta) \leq \sqrt{\int_0^1 (\partial/\partial\theta u(\theta))^2 d\theta}$; but,

$\partial/\partial\theta u(\theta)$

$= (\partial/\partial\theta f_\theta - \partial/\partial\theta h_\theta)\sqrt{g_\theta} + (f_\theta - h_\theta)\partial/\partial\theta g_\theta/(2\sqrt{g_\theta})$

$\Rightarrow (\partial/\partial\theta u(\theta))^2$

$\leq 2[2(\delta^2 + \epsilon^2) g_\theta + max^2(k_1, k_2)(\partial/\partial\theta g_\theta)^2/(4 g_\theta)]$;

(by the simple Holder inequality $(a+b)^2 \leq 2(a^2+b^2)$ and by the hypotheses of δ - sufficiency and ϵ - ancillarity)

Thus, $u^2(\theta) \leq 2\int_0^1[2(\delta^2+\epsilon^2)g_\theta + max^2(k_1,k_2)(\partial/\partial\theta g_\theta)^2/(4g_\theta)]d\theta$; note the important point that the bound on $u^2(\theta)$ is a *uniform* bound - it does not depend on θ.

Now, since we have already proved that $2d_\theta \leq \sqrt{\int_0^1 \int u^2(\theta)dtdu}$, the theorem follows on integrating the above pointwise inequality on $u^2 = u^2(\theta)$ with respect to t, u on performing the integrations in the order u, t, θ.

Theorem 2 gives an upper bound on d_θ, pointwise for every θ. Thus it talks about approximate indpendence of T and U under each θ. In contrast, the next theorem is about the average of d_θ over θ.

Theorem 3. Let U be ϵ - ancillary and T δ - sufficient with respect to U, and suppose T and U are both bounded random variables, taking values (say) in [0,1]. Suppose $f_\theta(u,t), g_\theta(t)$ and $h_\theta(u)$ are once continuously differentiable in θ for a.a. (u,t) and that $\lim_{\theta\to 0(1)}(f_\theta(u,t) - h_\theta(u))g_\theta(t) = 0$ for a.a. (u,t). Assume also that $\exists$ constants k_1, k_2 such that $f_\theta(u,t) \leq k_1$, and $h_\theta(u) \leq k_2$. Then,

$$\int_0^1 d_\theta d\theta \leq \sqrt{2}/\pi\sqrt{\int_0^1\int_0^1[2(\delta^2+\epsilon^2)g_\theta^2(t) + max^2(k_1,k_2)(\partial/\partial\theta g_\theta(t))^2]d\theta dt}.$$

Remark. Note that if T is sufficient and U is ancillary, then the first term inside the integral sign completely drops out. The second term $max^2(k_1,k_2)(\partial/\partial\theta g_\theta(t))^2$ stays. The interpretation is that if T is sufficient and U is ancillary, then without mentioning completeness, T and U should still be approximately independent *on the average* over all values of θ. Note that thus Theorem 3 comments about independence averaged over θ, while Theorem 2 takes the more traditional approach of independence under all θ.

Proof of Theorem 3 : By definition, $2\int d_\theta d\theta = \int\int\int |f_\theta(u,t) - h_\theta(u)| g_\theta(t) d\theta dt du$

$\leq \int\int \sqrt{\int (f_\theta(u,t) - h_\theta(u))^2 g_\theta^2(t) d\theta} dtdu$

(since $\theta \in [0,1]$, and by Schwartz's inequality).

Let us now work with the θ integral in this expression. By Lemma 2,

$\int (f_\theta(u,t) - h_\theta(u))^2 g_\theta^2(t) d\theta$

$\leq 4/\pi^2 \int \{\partial/\partial\theta[(f_\theta(u,t) - h_\theta(u)) g_\theta(t)]\}^2 d\theta$

$= 4/\pi^2 \int \{g_\theta(t)\partial/\partial\theta(f_\theta(u,t) - h_\theta(u)) + (f_\theta(u,t) - h_\theta(u))\partial/\partial\theta g_\theta(t)\}^2 d\theta$

$\leq 8/\pi^2 \int \{2g_\theta^2(t)(\delta^2 + \epsilon^2) + max^2(k_1, k_2)(\partial/\partial\theta g_\theta(t))^2\} d\theta.$

Hence, $2\int d_\theta d\theta$

$\leq \int\int \sqrt{8/\pi^2 \int \{2g_\theta^2(t)(\delta^2 + \epsilon^2) + max^2(k_1, k_2)(\partial/\partial\theta g_\theta(t))^2\} d\theta} dtdu$

$\leq 2\sqrt{2}/\pi \sqrt{\int\int\int \{2g_\theta^2(t)(\delta^2 + \epsilon^2) + max^2(k_1, k_2)(\partial/\partial\theta g_\theta(t))^2\} d\theta dtdu}$

(by Jensen's inequality and on using that t,u $\in [0,1]$), proving the result stated in the Theorem.

3 Basu's Theorem and Infinite Divisibility

In this section and the next, we will show various applications of Basu's theorem to infinite divisibility and factorization of random variables. Some of the examples are known; but others are new. More than the new examples, the interesting thing is that Basu's theorem is useful in these kinds of problems.

The results and the applications of the results depend on two facts. See Steutel(1970,1979), and Bose, DasGupta and Rubin(2002). First we state these two facts as lemmas.

Lemma 3. Let $V \sim Exp(1)$ and W independent of V. Then the product VW is id (infinitely divisible).

Remark. If W is nonnegative, then Lemma 3 is the Goldie-Steutel law which says that scale mixtures of Exponential densities are id. Essentially the same proof handles the case of a general W as well.

Lemma 4. Let $V \sim Exp(1)$ and suppose $\alpha > 0$. Then V^α admits the representation $V^\alpha = UW$, where $U \sim Exp(1)$, and W is independent of U.

We now state and prove our main theorems of this section.

Theorem 4. Let f be any homogeneous function of two variables, i.e., suppose $f(cx, cy) = c^2 f(x, y) \forall x, y$ and $\forall c > 0$. Let Z_1, Z_2 be iid $N(0,1)$ random variables and $Z_3, Z_4, \ldots, Z_m$ any other random variables such that $(Z_3, Z_4, \ldots, Z_m)$ is independent of (Z_1, Z_2). Then for any positive integer k, and an arbitrary measurable function g, $f^k(Z_1, Z_2)g(Z_3, Z_4, \ldots, Z_m)$ is infinitely divisible.

Due to the fact that f can be any homogeneous function and k and g are completely arbitrary, in principle, Theorem 4 has a very broad range of applications. Before giving a proof of Theorem 4, we state a corollary of this theorem and show a fairly large number of examples as applications of Theorem 4. The examples establish a large variety of random variables as being infinitely divisible.

Corollary 2. a) Let $f(x, y)$ be any of the functions $xy, x^2 + y^2, |x|^\alpha |y|^\beta$ where $\alpha, \beta \geq 0$ and $\alpha + \beta = 2$, $\sqrt{x^4 + y^4}, (x^n + y^n)/(x^{n-2} + y^{n-2})$ where $n \geq 2$.

Then for iid $N(0,1)$ random variables Z_1, Z_2, and $Z_3, Z_4, \ldots, Z_m$ any other random variables such that $(Z_3, Z_4, \ldots, Z_m)$ is independent of (Z_1, Z_2), any positive integer k, and an arbitrary measurable function g, $f^k(Z_1, Z_2)g(Z_3, Z_4, \ldots, Z_m)$ is infinitely divisible.

b)Let $n \geq 1$ and $X_1, X_2, \ldots, X_n$ independent random variables, each a *scale mixture* of zero mean normal distributions. Then for any $k \geq 1$, $(X_1X_2\ldots X_n)^k$ is infinitely divisible.

c)Let $m, n \geq 1, m < n$ and $X_1, X_2, \ldots, X_n$ independent random variables, each a *scale mixture* of zero mean normal distributions. Then for any $p, q \geq 1$, the ratio $R = (X_1X_2\ldots X_m)^p/(X_{m+1}\ldots X_n)^q$ is infinitely divisible.

Proof of Corollary 2 : We will first prove part a); parts b) and c) essentially follow from part a).

The only thing to observe is that each function f mentioned in part a) is a homogeneous function as is easily verified. Thus, directly from Theorem 4, part a) follows as a corollary.

To see the result of part b), write $X_1X_2\ldots X_n = (\sigma_1 Z_1)(\sigma_2 Z_2)\ldots(\sigma_n Z_n)$, where the $\{Z_i\}$ are $N(0,1)$ random variables and the $\{Z_i\}$ and the $\{\sigma_i\}$ are all mutually independent. Therefore, $X_1X_2\ldots X_n = Z_1Z_2(Z_3\ldots Z_n\sigma_1\sigma_2\ldots\sigma_n)$, and so it follows from part a) that $X_1X_2\ldots X_n$ is infinitely divisible by taking f to be xy, k as 1, and g as the product function in the corresponding space. The proof for a general k is the same.

Part c) follows exactly similarly and so we will just omit the proof.

We will now give, as specific illustrative examples, density functions that correspond to infinitely divisible distributions. It should be remarked that some are known, and others new. Densities (i), (iii), (v) and (viii) in the example below are simple functions and it is nice to know that they are infinitely divisible.

Example 2. Each of the following density functions on $(-\infty, \infty)$ correspond to infinitely divisible distributions.

(i) $f(x) = 1/\pi \;\; K_0(|x|)$, where K_0 denotes the Bessel K_0 function;

(ii) $f(x) = -\exp(x^2/2)Ei(-x^2/2)/(\sqrt{2}\pi^{3/2})$, where $Ei(.)$ denotes the Exponential integral;

(iii) $f(x) = K_0(2\sqrt{|x|})$;

(iv) f(x) = $[\sin(|x|)(\pi - 2si(|x|)) - 2\cos(x)ci(|x|)]/(2\pi)$, where si(.) and ci(.) stand for the sine and the cosine integral;

(v) $f(x) = 2\log(|x|)/(\pi^2(x^2-1))$;

(vi) f(x) $=[|x| - \sqrt{2\pi}\exp(1/(2x^2))\Phi(-1/|x|)]/(\sqrt{2\pi}|x|^3)$, where $\Phi(.)$ denotes the $N(0,1)$ CDF;

(vii) $f(x) = (1 - \sqrt{2\pi}|x|\exp(x^2/2)\Phi(-|x|))/\sqrt{2\pi}$;

(viii) f(x) $= 1/(2(1+|x|)^2)$;

(ix) f(x) $= \exp(1/(2x^2))\Gamma(0, 1/(2x^2))/(\sqrt{2}\pi^{3/2})$,where $\Gamma(.,.)$ denotes the incomplete Gamma function;

(x) $f(x) = [\sin(1/|x|)(\pi - 2si(1/|x|)) - 2\cos(1/|x|)ci(1/|x|)]/(2\pi x^2)$.

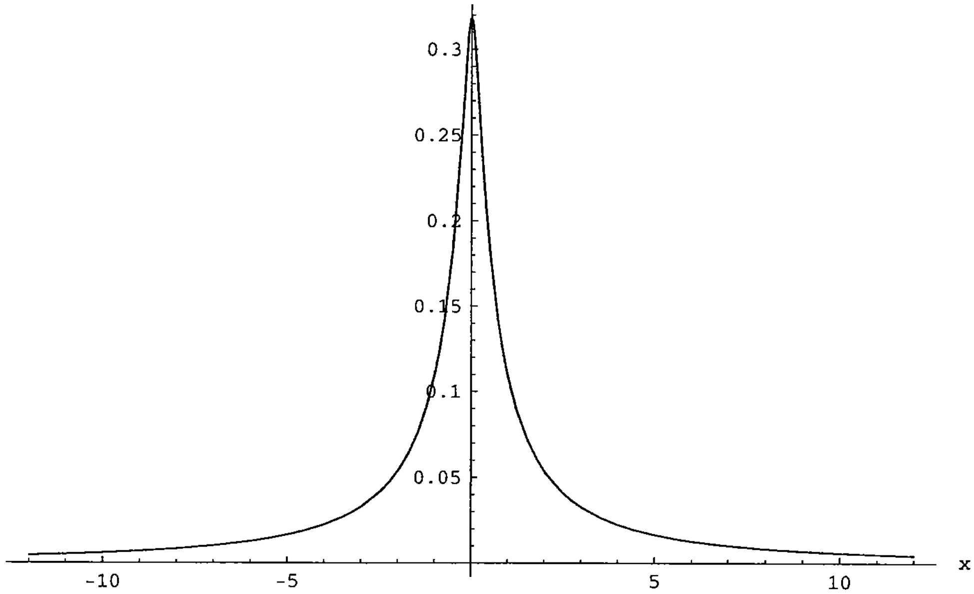
Proof: It follows as a special case of parts (b) and (c) of Corollary 2 that the densities stated in Example 2 are all infinitely divisible. For the purpose of this example, we will denote a standard normal, a standard Double Exponential,and a standard Cauchy random variable as N, D, and C respectively. Then the densities (i) - (x) are respectively the density functions of NN, NC, DD, CD, CC, N/D, D/N, D/D, C/N and C/D, where the notation NN means the product of two independent standard normals, etc. Therefore, the infinite divisibility of each one directly follows from parts (b) and (c) of Corollary 2.

Remark. Note that the density of N/C is the same as that of NC, and the density of D/C is the same as that of CD. Thus they are not separately mentioned in the example. And of course, N/N is the same as C, and therefore not mentioned either. The densities of NC and C/D are plotted below; the plots are provided only because they have interesting shapes, especially the cusps, and there could be some interest in seeing them.

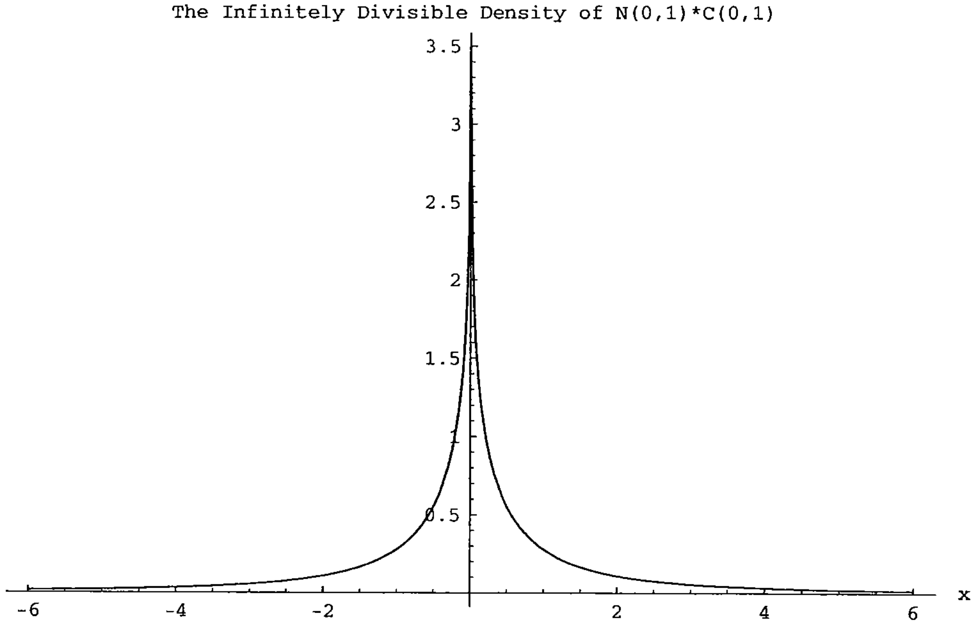
Proof of Theorem 4: Let Z_1, Z_2 be iid $N(0,1)$ and $(Z_3, \ldots, Z_m)$ independent of (Z_1, Z_2). Consider any homogeneous function $f(Z_1, Z_2)$ and write it as $f(Z_1, Z_2) = (Z_1^2 + Z_2^2)f(Z_1, Z_2)/(Z_1^2 + Z_2^2)$. Basu's theorem will now be used. Introduce, just for the sake of the proof, a parameter σ^2 and take the more general case where Z_1, Z_2 are iid $N(0, \sigma^2)$. Then $Z_1^2 + Z_2^2$ is complete and sufficient and $f(Z_1, Z_2)/(Z_1^2 + Z_2^2)$ is ancillary because f is homogeneous. Therefore $f(Z_1, Z_2)/(Z_1^2 + Z_2^2)$ and $Z_1^2 + Z_2^2$ are independent under *any* σ, and so in particular, under $\sigma = 1$. Therefore, $f(Z_1, Z_2)$ can be written as VW, where $V \sim Exp(2)$ and W is independent of V.

Hence, $f^k(Z_1, Z_2)g(Z_3, \ldots, Z_m) = V^k W^k g(Z_3, \ldots, Z_m)$. Now apply Lemma 4. By Lemma 4, $f^k(Z_1, Z_2)g(Z_3, \ldots, Z_m) = V^k W^k g(Z_3, \ldots, Z_m) = UW_* W^k$ g($Z_3, \ldots, Z_m$), where $U \sim Exp(1)$ and the rest are independent of U, and therefore by the extended Goldie-Steutel Law (Lemma 3), $f^k(Z_1, Z_2)g(Z_3, \ldots, Z_m)$

The Infinitely Divisible Density of C(0,1)/DExp(0,1)



The Infinitely Divisible Density of N(0,1)*C(0,1)



is infinitely divisible.

4 Factorization of Random Variables

In this final section, we will show that Basu's theorem can be used to decompose various functions of iid $N(0,1)$ random variables in the form YZ where Y is infinitely divisible, Z is not, both are nondegenerate, and Y and Z are independent. Although they are all functions of iid $N(0,1)$ random variables, the class of functions that admit such a decomposition is large, as we see in the following theorem.

Theorem 5. Let $X_1, X_2, \ldots, X_n$ be iid $N(0,1)$ random variables. Suppose $h_i(x_1, x_2, \ldots, x_n), 1 \leq i \leq n$, are scale invariant functions, i.e., $h_i(cx_1, cx_2, \ldots, cx_n) = h_i(x_1, x_2, \ldots, x_n) \forall c > 0$, and f is any continuous homogeneous function in the n-space, i.e., $f(cx_1, cx_2, \ldots, cx_n) = c^n f(x_1, x_2, \ldots, x_n) \forall c > 0$.

Define $g(X_1, X_2, \ldots, X_n) = f(||X||h_1(X_1, X_2, \ldots, X_n), \ldots, ||X||h_n(X_1, X_2, \ldots, X_n))$, where $||\mathbf{X}||$ denotes the Euclidean norm of the vector $(X_1, X_2, \ldots, X_n)$.

Then $g(X_1, X_2, \ldots, X_n)$ admits the representation $g(X_1, X_2, \ldots, X_n) = YZ$, where Y is infinitely divisible, Z is not, both are nondegenerate, and Y and Z are independent.

Before giving a proof of Theorem 5, we give a few interesting examples of such a decomposition that would follow from Theorem 5.

Example 3. Suppose $X_1, X_2, \ldots, X_n$ are iid $N(0,1)$; then each of the following random variables can be decomposed as YZ, with Y and Z as in Theorem 5 :

(i) $W = X_1 X_2 \dots X_n$;

(ii) $\mathrm{W} = \mathrm{X}_1^n + X_2^n + \dots + X_n^n$;

(iii) $\mathrm{W} = \mathrm{X}_1^2 X_2^2 \dots X_n^2/(X_1^2 + X_2^2 + \dots + X_n^2)^{n/2}$

(iv) $\mathrm{W} = (\mathrm{X}_1^{2n} + X_2^{2n} + \dots + X_n^{2n})/(X_1^2 + X_2^2 + \dots + X_n^2)^{n/2}$.

Of these four specific examples, it is particularly nice that (i) and (ii) have the decomposition we are discussing because they have a particular special form. That each of these four random variables has the stated decomposition can be seen by proving two more general facts. Let us do that now.

Consider the scale invariant functions $h_i(x_1, x_2, \dots, x_n) = x_i^m/(x_1^2 + x_2^2 + \dots + x_n^2)^{m/2}$, and the continuous homogeneous function $f(x_1, x_2, \dots, x_n) = x_1^n + x_2^n + \dots + x_n^n$.

Then $f(||x||h_1(x_1, x_2, \dots, x_n), \dots, ||x||h_n(x_1, x_2, \dots, x_n))$

$= ||x||^n \sum_{i=1}^n h_i^n(x_1, x_2, \dots, x_n)$

$= ||x||^n \sum_{i=1}^n x_i^{mn}/||x||^{mn}$

$= \sum_{i=1}^n x_i^{mn}/||x||^{n(m-1)}$;

The special value m= 1 gives the random variable in (ii), and the special value m = 2 gives the random variable in (iv).

For the other two examples, consider the same scale invariant functions h_i, but change the function f to $f(x_1, x_2, \dots, x_n) = x_1 x_2 \dots x_n$.

Then $f(||x||h_1(x_1, x_2, \dots, x_n), \dots, ||x||h_n(x_1, x_2, \dots, x_n)$

$= ||x||^n (x_1 x_2 \dots x_n)^m/(x_1^2 + x_2^2 + \dots + x_n^2)^{(nm/2)}$.

The special value m = 1 gives the random variable in (i), while m = 2 gives the random variable in (iii).

Of course, numerous other examples can be worked out by simply choosing other functions h_i and f.

We will finish by giving a proof of Theorem 5.

Proof of Theorem 5: By definition of g, and from the homogeneity of f, we have that $g(X_1, X_2, \ldots, X_n) = ||X||^n f(h_1(X_1, X_2, \ldots, X_n), \ldots,$
$\mathrm{h}_n(X_1, X_2, \ldots, X_n))$.

Of these, $||X||^n$ is a power of a chi-square and hence infinitely divisible (this is well known). As regards the second factor, if we introduce as before an artificial parameter σ^2 and let $X_1, X_2, \ldots, X_n$ be iid $N(0, \sigma^2)$, then $||X||$ is complete and sufficient, and the vector of functions $U =$
$(\mathrm{h}_1(X_1, X_2, \ldots, X_n), \ldots, h_n(X_1, X_2, \ldots, X_n))$ is ancillary because by hypothesis each h_i is scale invariant. *Hence*, by Basu's theorem $||X||$ and U are independent, under any σ, and so in particular under $\sigma = 1$. So if we let $Y = ||X||^n$, and $Z = f(U)$, then Y and Z are independent. To see that Z cannot be infinitely divisible, note that by its scale invariance, f is determined by its values on the unit disk and therefore must be bounded as f was also assumed to be continuous. As a bounded random variable cannot be infinitely divisible, it now follows that $g(X_1, X_2, \ldots, X_n)$ has the decomposition stated in the theorem.

Acknowledgement One day, like on most other days, Herman Rubin walked into my office, sat down, and without further introduction stated and recited a proof of Lemma 4. I am really glad and grateful that he did so.

Bibliography

Basu, D. (1955). On statistics independent of a complete sufficient statistic, Sankhyā, 15, 377-380.

Bose, A., DasGupta, A. and Rubin, H. (2002). A contemporary review and bibliography of infinitely divisible distributions and processes, Invited Article for the Basu Memorial Volume, Sankhyā.

Brown, L. and Low, M. (1996). Asymptotic equivalence of nonparametric regression and white noise, Ann. Stat., 24, 6, 2384-2398.

Bullen, P. (1998). A Dictionary of Inequalities, Addison Wesley Longman Limited, Essex, England

Flavin, J.N. and Rinero, S. (1996). Qualitative Estimates for Partial Differential Equations, CRC Press, New York.

Ghosh, M. (2001). Basu's Theorem with applications : a personalistic review, Invited Article for the Basu Memorial Volume, Sankhyā (To appear).

Goldie, C. (1967). A class of infinitely divisible random variables, Proc. Cambridge Philos. Soc., 63, 1141-1143.

Le Cam, L. (1964). Sufficiency and approximate sufficiency, Ann. Math. Stat., 35, 1419-1455

Le Cam, L. (1974). Construction of asymptotically sufficient estimates in some nonGaussian situations, Proc. Prague Symp. on Asymp. Stat., 1, 179-200.

Le Cam, L. (1986). Asymptotic Methods in Statistical Decision Theory, *Springer-Verlag, New York.*

Lehmann,E.L.(1981). An interpretation of completeness and Basu's theorem, Jour. Amer. Stat. Assoc., 76,335-340.

Reiss,R.(1978). Approximate Distributions of Order Statistics, Springer - Verlag, New York.

Steutel,F.W.(1970). Preservation of Infinite Divisibility Under Mixing and Related Topics, Math Centre Tracts,33,Mathematisch Centrum,Amsterdam.

Steutel,F.W.(1979).Infinite divisibility in theory and practice, Scand.J. Stat.,6,2,57-64.