MELLIN TRANSFORMS AND DENSITIES OF SUPREMA OF BROWNIAN PROCESSES: WITH APPLICATIONS

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ABSTRACT

We present analytical formulae for the Mellin transforms and densities of the suprema of the Brownian excursion, meander, and the reflected Brownian bridge.We show that each supremum is determined by its moments. As application, we give lower and upper bounds on the values of the Riemann ζ-function at odd and half-integer arguments, and a couple of surprising identities. We also give a probabilistic proof that Riemann's ξ-function can be computed at any real argument s by knowing its values only at the positive integer arguments $n = 1, 2, 3, \ldots$.

Another application is a sharp lower bound on the probability that a one dimensional simple symmetric random walk does not return to the origin till a given time $2n$ in terms of $\zeta(2n+1)$,$\zeta(2n+2)$, and $\zeta(2n+3)$. The plot of the density functions suggests that the suprema of the three processes satisfy a chain stochastic monotonicity property, but we could not find an analytic proof.

1 Introduction

In this article, we derive analytical formulae for the Mellin transform $E(W^s)$ and density function for the suprema of a number of Brownian processes. The processes considered are the *Brownian excursion, reflected Brownian Bridge and the Brownian Meander.* If W_t denotes the standard Brownian motion on $[0, \infty)$, τ_1 the last zero before time $t = 1$, and τ_2 the first zero after time $t = 1$, then the Brownian meander is the process $(1 - \tau_1)^{-\frac{1}{2}}|W_{\tau_1+t(1-\tau_1)}|$, and Brownian excursion is the process $(\tau_2 - \tau_1)^{-\frac{1}{2}}|W_{t\tau_2+(1-t)\tau_1}|$. Durrett,Iglehart and Miller(1977) show the convergence (in weak topology) of suitable conditioned processes to the Brownian excursion and the meander. They also derive the cdfs of the suprema in terms of certain infinite series for each of these processes. In the companion paper, Durrett and Iglehart(1977), the first moments of the suprema are derived as part of a larger body of calculations.

In section 2, we give analytical formulas for the Mellin transforms of each of the three suprema, including the case of inverse moments, as well as formulas for their densities in terms of the Jacobi Theta function and its first two derivatives. We show that each supremum is determined by its moment sequence, and as an application we give a probabilistic proof that Riemann's ξ-function can be computed at any real argument s by knowing its values only at the positive integer arguments $n = 1, 2, 3, \ldots$. As applications of the Mellin transform formulae, we derive a variety of properties of the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, which can be meromorphically defined in the complex plane with a simple pole at $s = 1$. The results present a series of inequalities on the values of the ζ-function at special arguments, as well as some *surprising identities.* A different kind of application of our results is a lower bound on the probability that the simple symmetric random walk in one dimension does not return to the origin till time $2n$. This bound uses the values of the ζ-function at the three consecutive integer arguments $2n+1, 2n+2, 2n+3$, and comparison shows that for small n, it is a better approximation than the usual local limit theorem approximation of the true probability.

The principal ingredients in our derivation of these results are schur majorization, the Gantmacher - Krein inequality, the functional equation of Riemann's ζ-function, and properties of moment matrices. To summarize, perhaps the most satisfactory results are the thin window in which we are able to place $\zeta(3)$ in equation (18), the two surprising identities in equations (8) and (9), and the random walk inequality in equation (25), seen to be more accurate than the local limit theorem approximation for small values of n.

It would be interesting to have an analytical proof of our conjecture on the chain stochastic monotonicity of the suprema of the reflected Brownian bridge, the Brownian excursion and the meander, which is indicated by the plot of their respective density functions in section 2.

2 Mellin Transform and Density Formulae

2.1 Mellin Transforms

Theorem 1 a. Let X_t^+ denote the Brownian meander and $W_M = sup\{X_t^+ : 0 \leq t \leq 1\}$. For $s > 0$,

$$E(W_M^s) = \frac{s(2^s-2)\Gamma(\frac{s}{2})\zeta(s)}{2^{\frac{s}{2}}}; \qquad (1)$$

b. Let X_t denote the Brownian Bridge, starting at 0, and $W_B = sup\{|X_t| : 0 \leq t \leq 1\}$. For $s > 0$,

$$E(W_B^s) = \frac{s(2^{s-1}-1)\Gamma(\frac{s}{2})\zeta(s)}{2^{\frac{3s}{2}-1}}; \qquad (2)$$

c. Let X_t^0 denote the Brownian excursion and $W_E = sup\{X_t^0 : 0 \leq t \leq 1\}$. For $s > 0$,

$E(W_E^s) = \frac{s(s-1)\Gamma(\frac{s}{2})\zeta(s)}{2^{\frac{s}{2}}} = 2(\frac{\pi}{2})^s \xi(s)$, where $\xi(s)$ denotes Riemann's ξ-function(see, e.g., Edwards(1974)). (3)

Proof. We will prove only part (a), as the technique is similar for all the parts. From Theorem (6.1) in Durrett, Iglehart and Miller(1977),

$$P(W_M \leq x) = 1 + 2\sum_{k=1}^{\infty}(-1)^k e^{-\frac{k^2x^2}{2}}$$

$$\Rightarrow P(W_M > x) = 2e^{-\frac{x^2}{2}} - 2\sum_{k=1}^{\infty} e^{-2k^2x^2} + 2\sum_{k=1}^{\infty} e^{-\frac{(2k+1)^2x^2}{2}},$$

on a bit of algebra.

First consider the case $s > 1$. Using the above expression for $P(W_M > x)$,

$$E(W_M^s) = -\int_0^{\infty} x^s d(2e^{-\frac{x^2}{2}} - 2\sum_{k=1}^{\infty} e^{-2k^2x^2} + 2\sum_{k=1}^{\infty} e^{-\frac{(2k+1)^2x^2}{2}})$$

$$= 2s\int_0^{\infty} x^{s-1}(e^{-\frac{x^2}{2}} - \sum_{k=1}^{\infty} e^{-2k^2x^2} + \sum_{k=1}^{\infty} e^{-\frac{(2k+1)^2x^2}{2}})dx$$

$$= s\Gamma(\tfrac{s}{2})(2^{\frac{s}{2}} - 2^{-\frac{s}{2}}\zeta(s) + 2^{-\frac{s}{2}}\sum_{k=1}^{\infty}\frac{1}{(k+\frac{1}{2})^s}) \qquad (4),$$

on change of variable and term by term integration.

In (4), use now the identity $\sum_{k=1}^{\infty}\frac{1}{(k+\frac{1}{2})^s} = \zeta(s)(2^s-1) - 2^s$ (see, e.g., Gradshteyn and Ryzhik(2002)). Substitution of this identity and a little more algebra yields the formula in part (a) of Theorem 1 for $s > 1$. But the function on the rhs of equation (1) is analytically extendable to the entire real axis, which would imply that in fact $E(W_M^s)$ exists for all real values of s, and in particular, for all $s > 0$, with the same expression as in (1) remaining valid also for $s \in (0,1]$.

Corollary 1. The first six moments of the suprema of the Brownian meander, the Brownian excursion and the reflected Brownian bridge on [0,1] are respectively :

Brownian meander $\sqrt{2\pi}\log 2, \frac{\pi^2}{3}, \frac{9\sqrt{\pi}}{2\sqrt{2}}\zeta(3), \frac{7}{45}\pi^4, \frac{225\sqrt{\pi}}{8\sqrt{2}}\zeta(5), \frac{31}{315}\pi^6;$

Brownian excursion $\sqrt{\frac{\pi}{2}}, \frac{\pi^2}{6}, \frac{3\sqrt{\pi}}{2\sqrt{2}}\zeta(3), \frac{\pi^4}{30}, \frac{15\sqrt{\pi}}{4\sqrt{2}}\zeta(5), \frac{\pi^6}{126};$

Reflected Brownian bridge $\sqrt{\frac{\pi}{2}}\log 2, \frac{\pi^2}{12}, \frac{9\sqrt{\pi}}{16\sqrt{2}}\zeta(3), \frac{7}{720}\pi^4, \frac{225\sqrt{\pi}}{256\sqrt{2}}\zeta(5), \frac{31}{20160}\pi^6.$

The numerical values in the respective cases are as follows :

Brownian meander 1.737,3.29,6.78,15.152,36.551,94.613;

Brownian excursion 1.253,1.645,2.26,3.247,4.873,7.63;

Reflected Brownian bridge .869,.822,.847,.947,1.142,1.478.

Remark Notice that the moments of the suprema of the meander are substantially larger than those of the excursion. As noted by Durrett,Iglehart and Miller(1977),the meander is likely to assume larger values than the excursion, and so this is consistent with our intuition.

From part a and part b of Theorem 1, one notices an interesting coincidence. On inspection of $E(W_M^s)$ and $E(W_B^s)$, one notices that $E(W_M^s) = 2^s E(W_B^s)$ for every $s > 0$. One would suspect from this relation that W_M has the same distribution as $2W_B$. This is in fact true, and we will see it again in the forms of their density functions in subsection 2.2.

Next, we will prove that each of W_M, W_B and W_E is determined by its moment sequence $E(W^n), n \geq 1$. An application of this result to Riemann's ξ-function will be provided.

Theorem 2. Each of W_M, W_B and W_E is determined by its moment sequence.

Proof. We will prove only the case of the meander; the same argument works for the other two cases. From equation (1), $c_n = E(W_M^n) = \frac{n(2^n-2)\Gamma(\frac{n}{2})\zeta(n)}{2^{\frac{n}{2}}}$. Using the obvious fact that $\zeta(n) \to 1$ as $n \to \infty$, and Stirling's approximation on $\Gamma(\frac{n}{2})$, it follows that $c_n^{\frac{1}{n}} = O(\sqrt{n})$, and hence $lim \frac{c_n^{\frac{1}{n}}}{n} = 0$ (and so $< \infty$), which would imply that the distribution of W_M is determined by its moment sequence (see, e.g., Shiryayev(1984)).

Corollary 2. Riemann's ξ-function can be computed at any real argument s by knowing only its values at $n = 1, 2, 3, \ldots$.

Proof Since W_E is determined by its moments, the sequence $\{\xi(n)\}_{n=0}^{\infty}$, by formula (3), will determine the distribution of W_E, and hence in particular $E(W_E^{\frac{s}{2}})$ for any real s (it is in fact true that $E(W_E^{\frac{s}{2}})$ is finite for all real s, and a formula for negative values of s will be provided in the next result).

Theorem 3. For $s > 0$,

a. $E(W_M^{-s}) = \frac{s(2^{s+1}-1)}{2^{\frac{s}{2}}\pi^{s+\frac{1}{2}}}\Gamma(\frac{1+s}{2})\zeta(1+s);$ (5)

b. $E(W_B^{-s}) = \frac{s(2^{s+1}-1)2^{\frac{s}{2}}}{\pi^{s+\frac{1}{2}}}\Gamma(\frac{1+s}{2})\zeta(1+s);$ (6)

c. $E(W_E^{-s}) = \frac{s(1+s)2^{\frac{s}{2}}}{\pi^{s+\frac{1}{2}}}\Gamma(\frac{1+s}{2})\zeta(1+s).$ (7)

Proof Again, we will only address the case of the meander. The formula simply uses the functional equation of the ζ-function given as :

$$\Gamma(\tfrac{1+s}{2})\pi^{-\frac{1+s}{2}}\zeta(1+s) = \Gamma(-\tfrac{s}{2})\pi^{\frac{s}{2}}\zeta(-s)$$

in the expression (1) for $E(W_M^s)$, which is actually valid $\forall$ real s. On substituting for $\Gamma(-\frac{s}{2})\zeta(-s)$ the alternative expression obtained from the above functional equation, formula (5) follows on a little algebra. We prefer formula (5) to (1) for negative values of s as it lets us avoid calculate the Γ and the ζ function for negative reals.

Corollary 3. $E(W_M^{-1}) = \frac{\sqrt{\pi}}{2\sqrt{2}}, E(W_B^{-1}) = \sqrt{\frac{\pi}{2}}, E(W_E^{-1}) = \frac{\sqrt{2\pi}}{3}$.

The main reason for stating these values of the first inverse moment is the following 'surprising' identity.

Corollary 4. $E(W_B^{-1}) = E(W_E)$. (8)

In fact, it is worth stating another surprising identity (identity (9) below) that follows from the moment formula in Theorem 1 and the inverse moment formula in Theorem 3. It is the assertion of the next corollary.

Corollary 5. a.$E(W_E^2)E(W_E^{-2}) = \zeta(3)$; (9)

b. For every $n \geq 1, E(W_E^{2n})E(W_E^{-2n}) = 2n(4n^2-1)|B_{2n}|\zeta(2n+1)$, (10)

where B_{2n} denotes the $2n$th Bernoulli number.

In particular, $\forall n \geq 1, E(W_E^{2n})E(W_E^{-2n})/\zeta(2n+1)$ is a rational number.

Proof part a is a consequence of the more general identity in part b. The formula in part b follows from the two moment formulae in Theorem 1 and

Theorem 3, after substitution of $\frac{(2\pi)^{2n}|B_{2n}|}{2(2n)!}$ for $\zeta(2n)$(see, e.g., Gradshteyn and Ryzhik(2002)). Part c follows from the fact that B_{2n} is a rational number for every $n \geq 1$.

We end this section with a stochastic domination result.

Proposition 1. $W_M \stackrel{\mathcal{L}}{=} 2W_B$, and hence, $W_M \succ W_B$, where $\succ$ denotes stochastically larger.

Proof Since W_M, W_B are each determined by their moment sequence by Theorem 2, and $E(W_M^n) = 2^n E(W_B^n)$ by Theorem 1, it follows that $W_M \stackrel{\mathcal{L}}{=} 2W_B$, and hence obviously $W_M \succ W_B$.

Remark It will be seen in the plots of the density functions of W_M and W_B in the next section that the two densities cross only once, as one might expect from the above stochastic dominance result.

2.2 Density Functions

We will next provide the density functions of the suprema in each of these three cases. The formulae will permit us to plot the densities. Moreover, it seems it would be good to have the density formulae for the sake of completeness.

Theorem 4 Let$\Theta(x) = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 x}$ denote the *Jacobi Theta function.* Then the density functions of the supremum of the Brownian meander, reflected Brownian bridge and the Brownian excursion are, respectively :

a. $f(x) = \frac{8x}{\pi}\Theta'(\frac{2x^2}{\pi}) - \frac{x}{\pi}\Theta'(\frac{x^2}{2\pi});$ (11)

b.$f(x) = \frac{32x}{\pi}\Theta'(\frac{8x^2}{\pi}) - \frac{4x}{\pi}\Theta'(\frac{2x^2}{\pi});$ (12)

c. $f(x) = \frac{12x}{\pi}\Theta'(\frac{2x^2}{\pi}) + \frac{16x^3}{\pi^2}\Theta''(\frac{2x^2}{\pi}).$ (13)

Proof. We will describe the proof only for the case of the Brownian meander, and merely sketch it for the other two cases.

First recall that the cdf of the supremum of the meander is given by

$$F(x) = 1 + 2(\sum_{n=1}^{\infty} e^{-2n^2x^2} - \sum_{n=1}^{\infty} e^{-\frac{(2n-1)^2x^2}{2}}). \quad (14)$$

Using the identity $\sum_{n=1}^{\infty} e^{-\frac{n^2x^2}{2}} = \sum_{n=1}^{\infty} e^{-2n^2x^2} + \sum_{n=1}^{\infty} e^{-\frac{(2n-1)^2x^2}{2}}$, and the definition of the Jacobi Theta function, we will get, from (14),

$$F(x) = 1 + 2(2\frac{\Theta(\frac{2x^2}{\pi})-1}{2} - \frac{\Theta(\frac{x^2}{2\pi})-1}{2})$$

$$= 2\Theta(\frac{2x^2}{\pi}) - \Theta(\frac{x^2}{2\pi}). \quad (15)$$

Differentiation of (15) yields the density function formula in (11).

As regards the other two cases, one can show, after calculation, that the cdfs have, respectively, the following formulae in terms of the Jacobi Theta function :

For the reflected bridge, $F(x) = 2\Theta(\frac{8x^2}{\pi}) - \Theta(\frac{2x^2}{\pi})$; (16)

For the excursion, $F(x) = \Theta(\frac{2x^2}{\pi}) + \frac{4x^2}{\pi}\Theta'(\frac{2x^2}{\pi})$. (17).

Differentiation of (16) and (17) will give the density formulae in (12) and (13).

The formulae of Theorem 4 are used to plot the three densities below. Note that the density of the supremum of the meander cuts that of the excursion *once from below.* Thus, the plot would suggest that the supremum of the meander is *stochastically larger* than that of the excursion. Durrett, Iglehart and Miller(1977) comment on this from their plots of the cdfs, but an analytical proof was not given. Note that similar density crossings are seen in the other cases as well, and so stochastic dominance seems to be occuring between all pairs. An analytical proof of the unimodality in all three cases would also be interesting.

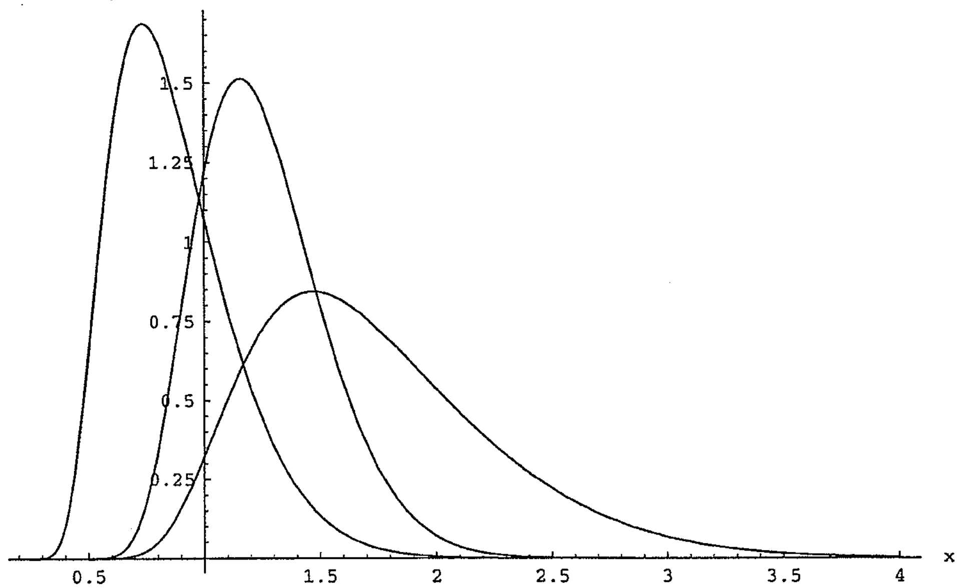
Conjecture Let W_B, W_E, W_M, respectively, denote the suprema of the reflected Brownian bridge, Brownian excursion, and the Brownian meander. Then $W_B \prec W_E \prec W_M$, where $\prec$ denotes *stochastically smaller.* A "proof by picture" is that for each pair, one density function cuts the other one only once and from below. If this were actually *proved analytically,* the stochastic dominance will follow.

3 Inequalities on the Riemann ζ-function and a Random Walk Probability

3.1 Inequalities on the Riemann ζ-function

The Mellin transform formulae of Theorem 1 are used in this section to provide bounds, both lower and upper, on the values of the ζ-function at odd and mid-integer arguments. Some of the bounds we give are quite sharp. We also give a bound on the probability that a simple symmetric random walk in one dimension does not return to the origin upto the $2n$th time by using the values of the ζ-function at the consecutive integer arguments $2n+1, 2n+2, 2n+3$. As we remarked before, analytical bounds on the ζ-

Density of the supremum of Reflected Brownian Bridge, Excursion, and the Meander, Left to Right



function at odd and mid-integer arguments are usually interesting, because not much is known about them. For example, it is not known if $\frac{\zeta(2n+1)}{\pi^{2n+1}}$ is rational or not, and it was proved as recently as 1979(Apery(1979)) that $\zeta(3)$ is irrational. The results here use two main facts; one is a result on Schur-convexity, and another the Gantmacher-Krein inequality on determinants of moment matrices. For easy reference, we state them as lemmas.

Lemma 1 Let $c(s)$ be the Mellin transform of a nonnegative random variable. Let $f(s_1, s_2, \ldots, s_n) = \prod_{i=1}^{n} c(s_i)$. Then f is a Schur-convex function.

Lemma 2 Let W be a nonnegative random variable and $c_i = E(W^i), i = 0, 1, 2, \ldots$, assumed to be finite. For given integers $n \geq 0, p \geq 1$, let $M_{p+1 \times p+1}$ denote the matrix with (i, j)th element equal to $c_{2n+i+j-2}$. Then,

$$detM \leq detM[\{1, 2, \ldots, k\}, \{1, 2, \ldots, k\}].$$

$$detM[\{k+1, k+2, \ldots, p+1\}, \{k+1, k+2, \ldots, p+1\}],$$

where the notation $detM[\{1, 2, \ldots, k\}, \{1, 2, \ldots, k\}]$ means the submatrix of elements in the rows $1, 2, \ldots, k$ and the columns $1, 2, \ldots, k$ of M.

The proofs of Lemma 1 and 2 can be seen in Marshall and Olkin(1979) and Gantmacher(1959); see also Fischer(1908).

First we give two examples to illustrate the possible use of the general inequalities we prove a little later.

Example 1 Consider the case of the Brownian excursion. From Corollary 1, the first three moments of the supremum are $\sqrt{\frac{\pi}{2}}, \zeta(2) = \frac{\pi^2}{6}$, and $\frac{3\sqrt{\pi}}{2\sqrt{2}}\zeta(3)$. Hence, by the Schur-convexity result in Lemma 1, $c(1)c(2)c(3) \geq c(2)^3$, which on a small amount of algebra reduces to the bound $\zeta(3) \geq \frac{\pi^3}{27}$. It may be added that the numerical value of $\zeta(3)$ is very close to $\frac{\pi^3}{26}$. **Example 2** Consider again the case of the Brownian excursion. If we let $p = 2$ and $n = 0$ in Lemma 2, then directly one gets $c_3^2 - 2c_1c_2c_3 + c_2^3 \geq 0$. Equivalently,

$$c_3 \leq \frac{c_2^3+c_3^2}{2c_1c_2}$$

$$\leq \frac{c_2^3+c_2c_4}{2c_1c_2}$$

(by the inequality $c_3^2 \leq c_2c_4$)

$$= \frac{c_2^2+c_4}{2c_1}.$$

Now if we plug the values of c_1, c_2, c_3 and c_4 for the Brownian excursion case, then on a little more algebra we get the *upper* bound $\zeta(3) \leq \frac{11}{270}\pi^3$.

Actually, both bounds in Example 1 and 2 are strict; thus, the two examples together show the rather pleasing inequalities :

$$\frac{10}{270}\pi^3 < \zeta(3) < \frac{11}{270}\pi^3. \qquad (18)$$

Theorem 5 a. For any $n \geq 2$,

$$\zeta(\tfrac{n+1}{2}) < \frac{4(n!(n-1)!\sqrt{\pi})^{\frac{1}{n}}(\prod_{k=2}^{n}\Gamma(\frac{k}{2})\zeta(k))^{\frac{1}{n}}}{(n^2-1)\Gamma(\frac{n+1}{4})}; \qquad (19)$$

b. For any $n \geq 1$,

$$\zeta(2n+1) < \frac{\sqrt{(2n-1)|B_{2n}B_{2n+2}|2^{4n-1}}}{n(2n+1)(2n-1)!!)^2}\pi^{2n+\frac{1}{2}}; \qquad (20)$$

here, B_k denotes the k-th Bernoulli number and the !! notation denotes, as usual, the *skipped* factorial.

c. For any $n \geq 1$,

$$\zeta(n+1) \leq \frac{2[\prod_{k=1}^{n}(k!\zeta(2k))]^{\frac{1}{n}}(2n-1)!!}{n(n+1)\Gamma(\frac{n+1}{2})}. \quad (21)$$

Proof We will prove only part b, as the idea of the proof is similar for the other parts.

Towards this end, consider the function f of Lemma 1 and use the three arguments $2n, 2n+1, 2n+2$. By the Schur-convexity of f,

$$f(2n, 2n+1, 2n+2) \geq f^3(2n+1)$$

$$\Rightarrow c_{2n}c_{2n+2} \geq c_{2n+1}^2. \quad (22)$$

Use now the formulae for the Mellin transform of the supremum of the Brownian excursion in Theorem 1 to get :

$$c_{2n} = \frac{2n(2n-1)(n-1)!\zeta(2n)}{2^n}$$

$$= \frac{(2n-1)|B_{2n}|\pi^{2n}}{(2n-1)!!}, \quad (23)$$

on using the well known formula $\zeta(2n) = \frac{2^{2n}\pi^{2n}|B_{2n}|}{2(2n)!}$;

and, $c_{2n+1} = \frac{2n(2n+1)\Gamma(n+\frac{1}{2})\zeta(2n+1)}{2^{n+\frac{1}{2}}}$

$$= \frac{n(2n+1)!!\sqrt{\pi}\zeta(2n+1)}{2^{2n-\frac{1}{2}}}, \quad (24)$$

on using the *duplication formula* for the Γ function; for both of these, see Gradshteyn and Ryzhik(2001).

The inequality in part b will follow from substituting these expressions for c_{2n}, c_{2n+1} and c_{2n+2} into the inequality (22). We omit the algebra.

For proving part a, use the Schur-convexity inequality $f(1, 2, \ldots, n) \geq f(\frac{n+1}{2}, \frac{n+1}{2}, \ldots, \frac{n+1}{2})$, and for part c, use the inequality $f(2, 4, \ldots, 2n) \geq$

$f(n+1, n+1, \ldots, n+1)$.

We provide a few examples to illustrate the kinds of bounds we can get from an application of Theorem 5.

Example 3 Use the upper bound in part b of Theorem 3 with $n = 1, 2$ as examples. Then, on calculation, the following bounds are obtained; the exact numerical values show that the analytical bounds are quite sharp.

$\zeta(3) = 1.20206 < \frac{\sqrt{\frac{2}{5}}}{9}\pi^{\frac{5}{2}} = 1.22931;$

$\zeta(5) = 1.03693 < \frac{2\sqrt{\frac{2}{105}}}{45}\pi^{\frac{9}{2}} = 1.05904.$

Example 4 This example gives a sharp analytical bound on the value of the ζ-function at the half integer argument $\frac{3}{2}$.

Using $n = 2$ in part a of Theorem 5, and using $\zeta(2) = \frac{\pi^2}{6}$, one would obtain the analytical bound $\zeta(\frac{3}{2}) = 2.612 < \frac{4}{3\sqrt{3}\Gamma(\frac{3}{4})}\pi^{\frac{5}{4}} = 2.627$.

The next example shows that *combining two different inequalities given in Theorem 5 can produce aesthethically nice and quite sharp bounds.*

Example 5 If we let n = 4 in part a (i.e., equation (19)) of Theorem 5, then on doing the necessary algebra, we will get $\zeta(\frac{5}{2}) < \frac{16}{15\Gamma(\frac{1}{4})}(\frac{2\zeta(3)}{15})^{\frac{1}{4}}\pi^{\frac{7}{4}}$. Now use the previously obtained bound $\zeta(3) < \frac{11}{270}\pi^3$ in equation (18). Substituting this upper bound for $\zeta(3)$, upon some more simplification, we get the bound :

$\zeta(\frac{5}{2}) < \frac{16(11)^{\frac{1}{4}}}{45\sqrt{5}\Gamma(\frac{1}{4})}\pi^{\frac{5}{2}} = .0799\pi^{\frac{5}{2}}$, hence giving the final bound:

$$\zeta(\tfrac{5}{2}) = 1.3415 < \tfrac{2}{25}\pi^{\frac{5}{2}} = 1.3995.$$

3.2 Application to One Dimensional Random Walk

The final result is a lower bound on the probability that a simple symmetric random walk in one dimension does not return to the origin till time $2n$ in terms of the values of the ζ-function at the three consecutive integer arguments $2n+1, 2n+2, 2n+3$. The bound is of the correct asymptotic order, as will be obvious from the result below. Further, the leading term in the bound is the expression one gets from the usual local limit theorem, and the rest is asymptotically $O(1)$. We will see that for small n, the bound in the next theorem produces a better approximation to the true probability than does the local limit theorem. The bound is the following.

Theorem 6 Consider the simple symmetric random walk S_n in one dimension, and let u_{2n} denote the probability $P(S_1 \neq 0, S_2 \neq 0, \ldots, S_{2n} \neq 0)$. Then,

$$u_{2n} > \frac{1}{\sqrt{n\pi}}\sqrt{\frac{2n+2}{2n+3}}\frac{\zeta(2n+2)}{\sqrt{\zeta(2n+1)\zeta(2n+3)}}. \tag{25}$$

Proof Consider the function f of Lemma 1 and use the Schur-convexity inequality $f(2n+1, 2n+2, 2n+3) \geq f(2n+2, 2n+2, 2n+2) \Leftrightarrow c_{2n+1}c_{2n+3} \geq c_{2n+2}^2$. Now use the formulae of c_{2n} and c_{2n+1} previously given in equations (23) and (24), but *keeping* $\zeta(2n)$ as such, i.e., without writing it in terms of the Bernoulli number B_{2n}. Half page of algebra then produces the inequality : $\frac{(2n)!}{n!^2 2^{2n}} > \frac{1}{\sqrt{n\pi}}\sqrt{\frac{2n+2}{2n+3}}\frac{\zeta(2n+2)}{\sqrt{\zeta(2n+1)\zeta(2n+3)}}$; but $u_{2n} = P(S_1 \neq 0, S_2 \neq 0, \ldots, S_{2n} \neq 0) = \frac{(2n)!}{n!^2 2^{2n}}$ (see Feller(1966)), and hence the bound (25) is established.

Example 6 A small numerical table is given to illustrate the accuracy of the bound (25).

Table 1

n	u_{2n}	Bound (25)	Local limit Thm
2	.375	.3675	.3989
5	.2461	.2424	.2523
8	.1964	.1942	.1995
20	.1254	.1247	.1262
50	.0796	.0794	.0798

Thus, the local limit theorem approximation overestimates the true probability while the bound (25) of Theorem 6 is by construction an underestimate, but for small n, the bound in (25) is a better approximation than the local limit theorem.

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