PRECISE PROPAGATIONS OF CHAOS ESTIMATES FOR FEYNMAN-KAC AND GENEALOGICAL PARTICLE MODELS

by

P. Del Moral and L. Miclo
Université Sabatier Université Paul Sabatier
F. Viens
Purdue University

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Department of Statistics
Purdue University
West Lafayette, IN USA

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# Precise Propagations of Chaos Estimates for Feynman-Kac and Genealogical Particle Models

P. Del Moral, L. Miclo[*], F. Viens [†]

**Abstract**

Strong propagations of chaos estimates for interacting particle and Feynman-Kac approximating models are studied. We use as a tool a tensor product Feynman-Kac semi-group approach with respect to time horizons and particle block sizes. Propagations of chaos estimates for Boltzmann-Gibbs measures are derived from a precise moment analysis of empirical measures and from an original transport equation relating symmetric statistic type and tensor product empirical measures. The analysis presented in this article apply to study the asymptotic behavior of genetic historical processes and their complete genealogical tree evolution yielding what seems to be the first precise propagations of chaos estimates for this type of path-particle models. Incidently this can be also be considered as an extension of the traditional asymptotic theory of $q$-symmetric statistics to interacting random sequences.

# 1 Introduction

## 1.1 Description of the models

Let $\mathcal{P}(E)$ be the set of all probability measure on a measurable space $(E, \mathcal{E})$. We consider a collection of measurable spaces $(E_n, \mathcal{E}_n)_{n \geq 0}$ and for any $0 \leq p \leq n$ we set $E_{[p,n]} = E_p \times E_{p+1} \ldots \times E_n$ and $E_{(p,n]} = E_{[p+1,n]}$. We shall use the abbreviation $\mu(f)$ for the integral of a function $f(x)$ with respect to a measure $\mu$ and $\mu Q(dy) = \int \mu(dx) Q(x, dy)$ for the integral of measure $\mu$ with respect to some bounded integral operator $Q$. We shall also use the letter $c$ to denote any universal constant whose values may vary from line to line but they do not depend on the time parameter $n$ nor on the coefficients of the models.

We denote by $G_n : E_n \to (0, \infty)$ a collection of non negative and $\mathcal{E}_n$-measurable functions such that $\sup_{x_n, y_n \in E_n} (G_n(x_n)/G_n(y_n)) < \infty$. Also let $\eta_0 \in \mathcal{P}(E_0)$ and $M_n(x_{n-1}, dx_n)$ be a sequence of Markov transitions from $E_{n-1}$ into $E_n$, $n \geq 1$. We associate to the triplet $(\eta_0, G_n, M_n)$ the

[*]Laboratoire de Statistiques et Probabilités, CNRS-UMR C5583, Université Paul Sabatier, 118, route de Narbonne, 31062 Toulouse cedex, France. delmoral{miclo}@cict.fr

[†]Department of Statistics, Purdue University, West Lafayette,

Feynman-Kac measures $\eta_n \in \mathcal{P}(E_n)$ defined for any $f_n \in \mathcal{B}_b(E_n)$ and $n \in \mathbb{N}$ by the formulae

$$\eta_n(f_n) = \gamma_n(f_n)/\gamma_n(1) \quad \text{with} \quad \gamma_n(f_n) = \mathbb{E}_{\eta_0}\left(f_n(X_n) \prod_{p=0}^{n-1} G_p(X_p)\right) \quad (1)$$

where $\mathbb{E}_{\eta_0}$ stands for the expectation with respect to the distribution of an $E_n$-valued Markov chain $X_n$ with transitions $M_n$. We recall that the distribution flow $\eta_n$ satisfies the non linear equation

$$\eta_{n+1} = \Phi_{n+1}(\eta_n)$$

where $\Phi_{n+1} : \mathcal{P}(E_n) \to \mathcal{P}(E_{n+1})$ is the one step mapping defined for any $\eta \in \mathcal{P}(E_n)$ by

$$\Phi_{n+1}(\eta) = \Psi_n(\eta) M_{n+1} \quad \text{with} \quad \Psi_n(\eta)(dx_n) = \frac{1}{\eta(G_n)}\, G_n(x_n)\, \eta(dx_n) \quad (2)$$

The genetic $N$-particle model associated to this distribution flow model is defined as non homogeneous and $E_n^N$-valued Markov chains

$$(\Omega^{(N)} = \prod_{n\geq 0} E_n^N,\ \mathcal{F}^N = (\mathcal{F}_n^N)_{n\in\mathbb{N}},\ (\xi_n)_{n\in\mathbb{N}},\ \mathbb{P}_{\eta_0}^N)$$

The initial configuration $\xi_0$ consists in $N$ independent and identically distributed random variables with common law $\eta_0$ and its elementary transitions from $E_{n-1}^N$ into $E_n^N$ are given in a symbolic integral form by

$$\mathbb{P}_{\eta_0}^N(\xi_n \in dx_n \mid \xi_{n-1}) = \prod_{p=1}^{N} \Phi_n\left(\frac{1}{N}\sum_{i=1}^{N} \delta_{\xi_{n-1}^i}\right)(dx_n^p) \quad (3)$$

where $dx_n = dx_n^1 \times \ldots \times x_n^N$ is an infinitesimal neighborhood of a point $x_n = (x_n^1, \ldots, x_n^N) \in E_n^N$. Using (2) we readily check that this particle algorithm is a simple genetic model with a two step selection/mutation transition. This class of evolutionary particle models can be interpreted as an interacting jump or as a birth and death process. Suppose the Markov chain $X_n$ in (1) is the path-historical process

$$X_n = (X_0', \ldots, X_n') \in E_n = E_{[0,n]}' \left(= E_0' \times \ldots \times E_n'\right)$$

associated to an auxiliary $E_n'$-Markov chain $X_n'$. Also suppose that the potential functions $G_n(x_0', \ldots, x_n') = G_n'(x_n')$ only depend of the terminal value of path $x_n = (x_0', \ldots, x_n')$. In this situation the $N$-path particle algorithm $\xi_n^i = (\xi_{p,n}'^i)_{0\leq p\leq n} \in E_{[0,n]}'$ represents the genealogical tree evolution of the genetic approximating model $\xi_n'$ of the Feynman-Kac measures $\eta_n'$ defined as in (1) by replacing the quantities $(E_n, X_n, G_n)$ by $(E_n', X_n', G_n')$ (cf.[2]). In this interpretation the path-particle model

$$\xi_{[0,n]}^i \quad = \quad (\xi_0^i, \xi_1^i, \ldots, \xi_n^i) \in E_0' \times E_{[0,1]}' \ldots \times E_{[0,n]}'$$

contains all historical informations on mutations and on evolution branches stopped by the selection mechanism. In this sense it represents the complete genealogical tree evolution of the genetic population model $\xi_n'$.

During the last decade the convergence of the particle empirical measures $\eta_n^N = \frac{1}{N}\sum_{i=1}^N \delta_{\xi_n^i}$ as $N$ tends to infinity towards the Feynman-Kac distribution $\eta_n$ has been the subject of many research articles. These genetic models can be thought in many different ways depending on the Feynman-Kac applications model areas we consider. There exists an extensive literature on the limiting behavior of these models and their applications in the spectral analysis of Schrödinger-Feynman-Kac semi-groups and in the development of new interacting Metropolis type models. We refer the reader to the survey paper [1] (and references therein) on theoretical aspects and advanced signal processing applications.

Strong propagations of chaos estimates measure the adequation of the law of the particles with the desired limiting distribution. They also allows to quantify the independence between the particles. Loosely speaking the initial configuration of a particle model consists in independent particles in a "complete chaos". Then they evolve and interact one each other. When the size of the system increases finite blocks consists in asymptotically independent particles. The study of propagations of chaos properties of discrete time genetic models has been started in [2]. In the latter most of the estimates depend on the regularity of the mutation transition. As a result they cannot be used to analyze propagations of chaos properties of complete genealogical tree models. The main object of this article is to study this important question combining an original tensor product semi-group technique with sharp estimations on empirical processes and Boltzmann-Gibbs transformations.

## 1.2 Outline of results

Let $(q, N)$ be a pair of integers with $1 \leq q \leq N$. Let $\langle N\rangle^{\langle q\rangle}$ be the set of all mappings from $\langle q\rangle = \{1, \dots, q\}$ into $\langle N\rangle = \{1, \dots, N\}$ and $\langle q, N\rangle \subset \langle N\rangle^{\langle q\rangle}$ the subset of all $(N)_q = N!/(N-q)!$ one to one mappings. By $\mathbb{P}_{\eta_0,n}^{(N,q)}$ we denote the distribution of the first $q$-path particles

$$\mathbb{P}_{\eta_0,n}^{(N,q)} = \text{Law}((\xi_{[0,n]}^i)_{1\leq i\leq q}) \in \mathcal{P}(E_{[0,n]}^q)$$

with $E_{[0,n]}^q = (E_{[0,n]})^q$ and $\xi_{[0,n]}^i = (\xi_0^i, \dots, \xi_n^i) \in E_{[0,n]}$. We also denote by $\mathbb{P}_{\eta_0,[n]}^{(N,q)} = \text{Law}((\xi_n^i)_{1\leq i\leq q}) \in \mathcal{P}(E_n^q)$ their $n$-th time marginals. To every path empirical measure $\Gamma_n^N = \frac{1}{N}\sum_{i=1}^N \delta_{\xi_{[0,n]}^i} \in \mathcal{P}(E_{[0,n]})$ there corresponds a pair of $q$-tensor and symmetric statistic type empirical distributions

$$\begin{aligned}(\Gamma_n^N)^{\otimes q} &= \frac{1}{N^q}\sum_{\alpha\in\langle N\rangle^{\langle q\rangle}} \delta_{(\xi_{[0,n]}^{\alpha(1)},\dots,\xi_{[0,n]}^{\alpha(q)})} \\ (\Gamma_n^N)^{\odot q} &= \frac{1}{(N)_q}\sum_{\alpha\in\langle q,N\rangle} \delta_{(\xi_{[0,n]}^{\alpha(1)},\dots,\xi_{[0,n]}^{\alpha(q)})} \qquad (4)\end{aligned}$$

In contrast to traditional $q$-symmetric statistics the $N$-random paths $\xi_{[0,n]}^i$, $1 \leq i \leq N$ are non independent but they interact with each other according to mutation and genetic selection rules. In this paper the limit behavior of these interacting $q$-tensor measures is investigated. This leads not only to propagations of chaos results for genetic and genealogical tree models but also to an extension of the classical asymptotic theory of $q$-symmetric statistics to interacting random sequences.

The first central observation is that these two types of empirical measures are connected by a Markov transport equation of the following form

$$(\Gamma_n^N)^{\otimes q} = (\Gamma_n^N)^{\odot q} R_N^{(q)} \quad \text{where} \quad R_N^{(q)} = \frac{(N)_q}{N^q}\ Id + (1 - \frac{(N)_q}{N^q})\ \tilde{R}_N^{(q)}$$

and $\tilde{R}_N^{(q)}$ a Markov transition on $E_{[0,n]}^q$. We will give the proof of this result with a precise and explicit description of $\tilde{R}_N^{(q)}$ in the subsection 4.3 of the Appendix. One easy consequence of this formula is that

$$\|(\Gamma_n^N)^{\otimes q} - (\Gamma_n^N)^{\odot q}\|_{tv} \;\; \leq \;\; (1 - (N)_q/N^q) \; \leq \; (q-1)^2/N \qquad (5)$$

By symmetry arguments we observe that for any $F \in \mathcal{B}_b(E_{[0,n]}^q)$ we have

$$\mathbb{P}_{\eta_0,n}^{(N,q)}(F) = \mathbb{E}_{\eta_0}^N\left(F((\xi_{[0,n]}^i)_{1\leq i\leq q})\right) = \mathbb{E}_{\eta_0}^N\left((\Gamma_n^N)^{\odot q}(F)\right)$$

As mentioned in the introduction to analyze precisely the limiting behavior of the path-space distributions $(\Gamma_n^N)^{\odot q}$ we develop an original approach based on $q$-tensor product and path space Feynman-Kac semi-groups. This strategy enters in a natural way the dynamical structure of interactions in the study of the propagations of chaos properties. It allows to use the stability properties of the limiting system to derive precise and uniform estimates with respect to the time parameter. The systematic investigation of Feynman-Kac particle models using semi-group techniques has been initiated by two of the authors in [1] and to derive central limit theorems and empirical processes convergence results.

In section 2 we extend this technique to tensor product semi-groups with respect to particle block sizes and time horizons. We express precise strong propagations of chaos estimates in terms of the Dobrushin's ergodic coefficient associated to a Markovian and Feynman-Kac type transition on a product space. For a precise definition of the Dobrushin's ergodic coefficient and its applications in the context of Feynman-Kac and particle models we refer the reader to the articles [1] and references therein.

To describe precisely our first main result we let $Q_{p,n}$, respectively $Q_{p,n}^{(q)}$, be the linear semi-group associated to the un-normalized Feynman-Kac distributions $\gamma_n$ and respectively $\gamma_n^{\otimes q}$. Notice that

$$Q_{p,n}(f_n) = G_{p,n}\ P_{p,n}(f_n)$$

with the potential function $G_{p,n}$ and the Markov transition $P_{p,n}$

$$G_{p,n} = Q_{p,n}(1) \quad \text{and} \quad P_{p,n}(f_n) = Q_{p,n}(f_n)/Q_{p,n}(1)$$

Let $(G_{p,n}^{(q)}, P_{p,n}^{(q)})$ be the corresponding pair potential and Markov transition associated to the semi-group $Q_{p,n}^{(q)}$.

Our first main result is

**Theorem 1.1** *For any $N \geq q \geq 1$ we have*

$$\|\mathbb{P}^{(N,q)}_{\eta_0,[n]} - \eta_n^{\otimes q}\|_{tv} \leq c\ \frac{q^2}{N}\ \left(1+\sum_{p=0}^{n} \beta(P^{(q)}_{p,n})\ [1+e_{p,n}\left(2q^2/N\right)]\right) \quad (6)$$

*where $\beta(P^{(q)}_{p,n}) \in [0,1]$ represents the Dobrushin ergodic coefficient associated to the Markov transition $P^{(q)}_{p,n}$ and $e_{p,n} : (0,\infty) \to (0,\infty)$, is the collection of mappings defined by*

$$\begin{aligned} e_{p,n}(u) &= (r_{p,n}-1)^2(1+(r_{p,n}-1)\ \sqrt{u})\ \exp\left((r_{p,n}-1)^2\ u\right) \quad (7)\\ r_{p,n} &= \sup_{x_p,y_p\in E_p} (G_{p,n}(x_p)/G_{p,n}(y_p)) \end{aligned}$$

The estimate (6) holds true for a fairly general and abstract class of Feynman-Kac models. It can be used to analyze the strong propagations of chaos properties of genetic particle systems as well as those of the corresponding genealogical tree models. To illustrate another impact of this result in practice we present hereafter two easily derived consequences of theorem 1.1. For simplicity we further assume that the Feynman-Kac model (1) is time homogeneous $(E_n, G_n, M_n) = (E, G, M)$ and the following regularity condition is met for any $x, y \in E$ and for some $m \geq 1$ and $\epsilon(G), \epsilon(M) \in (0,1)$

$$(G,M)\ : \qquad G(x) \geq \epsilon(G)\ G(y) \quad \text{and} \quad M^m(x,.) \geq \epsilon(M)\ M^m(y,.)$$

In this situation combining theorem 1.1 with some well known results on the stability of Feynman-Kac semi-group we will prove the following increasing propagations of chaos properties: let $n(N)$ and $q(N)$, $N \geq 1$, be respectively a non decreasing sequence of time horizons and particle block sizes such that $\lim_{N\to\infty} n(N)q^2(N)/N = 0$. In this situation we have

$$\overline{\lim}_{N\to\infty} \frac{N}{q^2(N)n(N)} \|\mathbb{P}^{(N,q(N))}_{\eta_0,[n(N)]} - \eta_{n(N)}^{\otimes q(N)}\|_{tv} \leq c\ /(\epsilon^m(G)\epsilon(M))^2$$

Theorem 1.1 does not apply to study the asymptotic behavior of the complete $N$-genealogical particle model $\xi_{[0,n]}$.

Our second main result is

**Theorem 1.2** *For any $n, q, N \geq 1$ such that $(n+1)q \leq N$ we have*

$$\|\mathbb{P}^{(N,q)}_{\eta_0,n} - (\eta_0 \otimes \ldots \otimes \eta_n)^q\|_{tv} \leq c\ \frac{q^2}{N}(n+1)^3\ [1+e_n(\frac{2(q(n+1))^2}{N})] \quad (8)$$

*with the mapping $e_n(u)$ defined as in (7) by replacing the constants $r_{p,n}$ by $r_n = \sup_{p\leq n} r_{p,n}$.*

This second estimate readily implies the following increasing propagations of chaos property: If we have $\lim_{N\to\infty} q^2(N)/N = 0$ then for any $n \in \mathbb{N}$

$$\overline{\lim}_{N\to\infty} \frac{N}{q^2(N)} \left\|\mathbb{P}^{(N,q(N))}_{\eta_0,n} - (\eta_0 \otimes \ldots \otimes \eta_n)^{\otimes q(N)}\right\|_{tv} \leq C(n)$$

with $C(n) \leq c\ (n+1)^3(1+(r_n-1)^2)$. In the case of time homogeneous models satisfying condition $(G, M)$ for some $m \geq 1$ and $\epsilon(G), \epsilon(M) \in (0,1)$ then we shall also prove that

$$C(n) \leq c\ (n+1)^3/(\epsilon^m(G)\epsilon(M))^2$$

In section 3 we measure the propagations of chaos properties of Boltzmann-Gibbs transformations. This section contains several central key estimates including a sharp complement of Burkholder's type inequality for sequences of independent and identically distributed random variables. The complete proofs of theorem 1.1 and theorem 1.2 are housed in section 4.

# 2 Feynman-Kac semi-groups

We let $Q_{p,n}$ and $\Phi_{p,n}$, $p \leq n$, be the semi-groups associated respectively to the Feynman-Kac distribution flows $\gamma_n$ and $\eta_n$ defined in (1),

$$Q_{p,n} = Q_{p+1} \ldots Q_{n-1} Q_n \quad \text{and} \quad \Phi_{p,n} = \Phi_n \circ \Phi_{n-1} \circ \ldots \circ \Phi_{p+1}$$

with $Q_n(x_{n-1}, dx_n) = G_{n-1}(x_{n-1}) M_n(x_{n-1}, dx_n)$. We use the convention $Q_{n,n} = Id$ and $\Phi_{n,n} = Id$ for $p = n$. To analyze propagations of chaos properties in path space it is convenient to consider the Feynman-Kac tensor product distributions on path space

$$\Gamma_n = \eta_0 \otimes \ldots \otimes \eta_n \in \mathcal{P}(E_{[0,n]})$$

By definition of $\Phi_{p,n}$ we have for any $p \leq n$

$$\Gamma_n = \Omega_{p,n}(\Gamma_p)$$

with the (non linear) semi-group $\Omega_{p,n} : \mathcal{P}(E_{[0,p]}) \longrightarrow \mathcal{P}(E_{[0,n]})$ defined for any $\mu \in \mathcal{P}(E_{[0,p]})$ by

$$\Omega_{p,n}(\mu) = \mu \otimes \Phi_{p,p+1}(\mu_p) \otimes \Phi_{p,p+2}(\mu_p) \otimes \ldots \otimes \Phi_{p,n}(\mu_p) \qquad (9)$$

In the above display $\mu_p \in \mathcal{P}(E_p)$ stands for the $p$-th time marginal of $\mu$ defined for any $\varphi_p \in \mathcal{B}_b(E_p)$ by

$$\mu_p(\varphi_p) = \mu(\underbrace{1 \otimes \ldots \otimes 1}_{p-\text{times}} \otimes \varphi_p)$$

Again we use the convention $\Omega_{n,n} = Id$ for $p = n$. To check that $\Omega_{p,n}$ is a well defined semi-group we observe that for any $\mu \in \mathcal{P}(E_{[0,p]})$ we have

$$\Omega_{p,p+1}(\mu) = \mu \otimes \Phi_{p,p+1}(\mu_p)$$

It follows that

$$\begin{aligned}
&\Omega_{p+1,n}(\Omega_{p,p+1}(\mu)) \\
&= \Omega_{p+1,n}(\mu \otimes \Phi_{p,p+1}(\mu_p)) \\
&= \mu \otimes \Phi_{p,p+1}(\mu_p) \otimes \Phi_{p+1,p+2}(\Phi_{p,p+1}(\mu_p)) \otimes \ldots \otimes \Phi_{p+1,n}(\Phi_{p,p+1}(\mu_p)) \\
&= \mu \otimes \Phi_{p,p+1}(\mu_p) \otimes \Phi_{p,p+2}(\mu_p) \otimes \ldots \otimes \Phi_{p,n}(\mu_p) = \Omega_{p,n}(\mu)
\end{aligned}$$

In the forthcoming development of this section we fix a positive integer $q \geq 1$ and we denote by $\Gamma_n^{(q)}$ the $q$-tensor product Feynman-Kac measures defined by

$$\Gamma_n^{(q)} = \eta_0^{\otimes q} \otimes \ldots \otimes \eta_n^{\otimes q} \in \mathcal{P}(E_{[0,n]}^{(q)}) \quad \text{with} \quad E_{[0,n]}^{(q)} = E_0^q \times \ldots \times E_n^q$$

Notice that

$$\Gamma_n^{(q)} = \Gamma_n^{\otimes q} \circ (\Theta_n^q)^{-1}$$

with the mapping $\Theta_n^q : E_{[0,n]}^{(q)} \longrightarrow E_{[0,n]}^q$ defined by

$$\Theta_n^q[(x_0^i)_{1\leq i\leq q}, \ldots, (x_n^i)_{1\leq i\leq q}] = (x_0^i, \ldots, x_n^i)_{1\leq i\leq q} \tag{10}$$

The next two subsections are devoted respectively to the study of the dynamical structure of the tensor product distributions $\eta_n^{\otimes q}$ and $\Gamma_n^{(q)}$.

## 2.1 Marginal models

We observe that $\eta_n^{\otimes q}$ can alternatively be defined for any $f \in \mathcal{B}_b(E_n)$ by the Feynman-Kac formulae

$$\eta_n^{\otimes q}(f) = \gamma_n^{\otimes q}(f)/\gamma_n^{\otimes q}(1) \quad \text{with} \quad \gamma_n^{\otimes q}(f) = \mathbb{E}_{\eta_0^{\otimes q}}^{(q)}(f(X_n^{(q)}) \prod_{p=0}^{n-1} G_p^{(q)}(X_p^{(q)}))$$

where

- $\mathbb{E}_{\eta_0^{\otimes q}}^{(q)}(.)$ represents the integration with respect to the law $\mathbb{P}_{\eta_0^{\otimes q}}^{(q)}$ of $q$ independent copies

  $$X_n^{(q)} = (X_n^{(q,1)}, X_n^{(q,2)}, \ldots, X_n^{(q,q)}) \in E_n^q$$

  of a Markov chain with initial distribution $\eta_0 \in \mathcal{P}(E_0)$ and Markov transitions $M_n$. In other words $X_n^{(q)}$ is a non homogeneous and $E_n^q$-valued Markov chain with transitions

  $$M_n^{(q)}((x_{n-1}^1, \ldots, x_{n-1}^q), d(x_n^1, \ldots, x_n^q)) = \prod_{i=1}^q M_n(x_{n-1}^i, dx_n^i)$$

- $G_n^{(q)} : E_n^q \to (0,\infty)$, $n \geq 0$, is the sequence of $q$-tensor product potential functions defined for any $(x_n^1, \ldots, x_n^q) \in E_n^q$ by

  $$G_n^{(q)}(x_n^1, \ldots, x_n^q) = \prod_{i=1}^q G_n(x_n^i)$$

This rather simple representation indicates that the sequence of distribution flows $\eta_n^{\otimes q}$ and $\gamma_n^{\otimes q}$, $q \geq 1$, have exactly the same semi-group structure. Let $Q_{n+1}^{(q)}$ and respectively $\Phi_{n+1}^{(q)}$ be the bounded integral operator from $E_n^q$ into $E_{n+1}^q$ and the mapping from $\mathcal{P}(E_n^q)$ into $\mathcal{P}(E_{n+1}^q)$ defined for any $(\eta, f) \in \mathcal{P}(E_n^q) \times \mathcal{B}_b(E_{n+1}^q)$ by

$$Q_{n+1}^{(q)}(f) = G_n^{(q)}\ M_{n+1}^{(q)}(f) \quad \text{and} \quad \Phi_{n+1}^{(q)}(\eta) = \Psi_n^{(q)}(\eta) M_{n+1}^{(q)}$$

with the Boltzmann-Gibbs transformations $\Psi_n^{(q)}$ on $\mathcal{P}(E_n^q)$ given by

$$\Psi_n^{(q)}(\eta)(dx_n) = \frac{1}{\eta(G_n^{(q)})}\, G_n^{(q)}(x_n)\, \eta(dx_n)$$

By the Markov property and the multiplicative form of the Feynman-Kac models we prove that the distribution flows $\gamma_n^{\otimes q}$ and $\eta_n^{\otimes q}$ satisfy the recursions

$$\gamma_{n+1}^{\otimes q} = \gamma_n^{\otimes q} Q_{n+1}^{(q)} \quad \text{and} \quad \eta_{n+1}^{\otimes q} = \Phi_{n+1}^{(q)}(\eta_n^{\otimes q})$$

We let $Q_{p,n}^{(q)}$ and $\Phi_{p,n}^{(q)}$, $p \leq n$, be the semi-groups associated respectively to $\gamma_n^{\otimes q}$ and $\eta_n^{\otimes q}$. That is we have that

$$Q_{p,n}^{(q)} = Q_{p+1}^{(q)} \cdots Q_{n-1}^{(q)} Q_n^{(q)} \quad \text{and} \quad \Phi_{p,n}^{(q)} = \Phi_n^{(q)} \circ \Phi_{n-1}^{(q)} \circ \ldots \circ \Phi_{p+1}^{(q)}$$

As usually we use the convention $Q_{n,n}^{(q)} = Id$ and $\Phi_{n,n}^{(q)} = Id$ for $p = n$. Our final objective is to provide a Boltzmann-Gibbs representation of the semi-group $\Phi_{p,n}^{(q)}$. To this end we let $G_{p,n}^{(q)} : E_p^q \to (0, \infty)$ and $P_{p,n}^{(q)}$ be respectively the potential function and the Markov transition from $E_p^q$ into $E_n^q$ defined for any $f \in \mathcal{B}_b(E_n^q)$ by the formulae

$$G_{p,n}^{(q)} = Q_{p,n}^{(q)}(1) \quad \text{and} \quad P_{p,n}^{(q)}(f_n) = Q_{p,n}^{(q)}(f_n)/Q_{p,n}^{(q)}(1)$$

If we set $G_{p,n} = Q_{p,n}(1)$ then we find that for any $(x_p^1, \ldots, x_p^q) \in E_p^q$

$$\begin{aligned} G_{p,n}^{(q)}(x_p^1, \ldots, x_p^q) &= Q_{p,n}(1)(x_p^1) \ldots Q_{p,n}(1)(x_p^q) \\ &= G_{p,n}(x_p^1) \ldots G_{p,n}(x_p^q) \end{aligned}$$

From previous considerations we readily see that for any $\mu \in \mathcal{P}(E_p^q)$ we have

$$\Phi_{p,n}^{(q)}(\mu) = \Psi_{p,n}^{(q)}(\mu) P_{p,n}^{(q)} \tag{11}$$

with the Boltzmann-Gibbs transformations $\Psi_{p,n}^{(q)}$ on $\mathcal{P}(E_p^q)$ associated to the potential function $G_{p,n}^{(q)}$ and defined for any $(\mu, f) \in \mathcal{P}(E_p^q) \times \mathcal{B}_b(E_p^q)$ by $\Psi_{p,n}^{(q)}(\mu)(f) = \mu(G_{p,n}^{(q)}\, f)/\mu(G_{p,n}^{(q)})$.

## 2.2 Path space models

To describe the dynamical structure of the semi-groups $\Omega_{p,n}$ introduced in (9) we first observe that for $\eta \in \mathcal{P}(E_p)$ and $F \in \mathcal{B}_b(E_{(p,n]})$ we have

$$\begin{aligned} &(\Phi_{p,p+1}(\eta) \otimes \Phi_{p,p+2}(\eta) \otimes \ldots \otimes \Phi_{p,n}(\eta))\,(F) \\ &= \left[\prod_{k=1}^{n} \frac{1}{\eta Q_{p,p+k}(1)}\right] \ (\eta Q_{p,p+1} \otimes \ldots \otimes \eta Q_{p,n})\,(F) = \frac{\eta^{\otimes(n-p)}(T_{p,n}(F))}{\eta^{\otimes(n-p)}(T_{p,n}(1))} \end{aligned}$$

with the bounded operator $T_{p,n}$ from $E_p^{(n-p)}$ into $E_{(p,n]}$ defined for any $(x_p^1, \ldots, x_p^{(n-p)}) \in E_p^{(n-p)}$ by

$$T_{p,n}(F)(x_p^1, \ldots, x_p^{(n-p)}) = \int_{E_{(p,n]}} \prod_{k=1}^{n-p} Q_{p,p+k}(x_p^k, dx_{p+k})\ F(x_{p+1}, \ldots, x_n)$$

Also observe that the mapping $T_{p,n}(1)$ coincide with the $(n-q)$-tensor product potential function

$$T_{p,n}(1)(x_p^1, \ldots, x_p^{(n-p)}) = \prod_{k=1}^{n-p} Q_{p,p+k}(1)(x_p^k)$$

In other words in terms of the potential functions $G_{p,n} = Q_{p,n}(1)$ we have that

$$T_{p,n}(1) = G_{p,p+1} \otimes G_{p,p+2} \otimes \ldots \otimes G_{p,n} \tag{12}$$

In these notations (9) can be rewritten for any $\mu \in \mathcal{P}(E_{[0,p]})$ as follows

$$\Omega_{p,n}(\mu)(.) = \mu \otimes \frac{\mu_p^{\otimes(n-p)} T_{p,n}(.)}{\mu_p^{\otimes(n-p)} T_{p,n}(1)} = \mu \otimes (B_{p,n}[\mu_p^{\otimes(n-p)}] U_{p,n})$$

with

- the $p$-th time marginal distribution $\mu_p \in \mathcal{P}(E_p)$ of $\mu \in \mathcal{P}(E_{[0,p]})$
- the Boltzmann-Gibbs transformation $B_{p,n}$ on $\mathcal{P}(E_p^{(n-p)})$ and the Markov transition $U_{p,n}$ from $E_p^{(n-p)}$ into $E_{(p,n]}$ defined for any pair $(\nu, f) \in (\mathcal{P}(E_p^{(n-p)}) \times \mathcal{B}_b(E_p^{(n-p)}))$ and $F \in \mathcal{B}_b(E_{(p,n]})$ by

$$B_{p,n}(\nu)(f) = \frac{\nu(T_{p,n}(1)\ f)}{\nu(T_{p,n}(1))} \quad \text{and} \quad U_{p,n}(F) = \frac{T_{p,n}(F)}{T_{p,n}(1)}$$

This updating-prediction type representation of the semi-group $\Omega_{p,n}$ provides a precise description of the dependence of $\Omega_{p,n}(\nu)$ with respect to the measure $\nu$. Next we present a formula which emphasize the role of the one step mappings $\Phi_p$ in the dynamical structure of these transformations.

**Lemma 2.1** *For any $p \geq 1$ and $\eta \in \mathcal{P}(E_{p-1})$ we have*

$$B_{p-1,n}[\eta^{\otimes(n-p+1)}] U_{p-1,n} = \Phi_p(\eta) \otimes (B_{p,n}[\Phi_p\,(\eta)^{\otimes(n-p)}] U_{p,n})$$

**Proof:**
By definition of the operator $T_{p,n}$ we have

$$\begin{aligned}
&\eta^{\otimes(n-p+1)} T_{p-1,n} \\
&= [\eta Q_{p-1,p}] \otimes [\eta Q_{p-1,p+1}] \otimes \ldots \otimes [\eta Q_{p-1,n}] \\
&= (\eta Q_p) \otimes [(\eta Q_p) Q_{p,p+1}] \otimes \ldots \otimes [(\eta Q_p) Q_{p,n}] \\
&= (\eta Q_p) \otimes [(\eta Q_p)^{\otimes(n-p)} T_{p,n}]
\end{aligned}$$

This implies that

$$\eta^{\otimes(n-p+1)} T_{p-1,n}(1) = \eta Q_p(1) \ \times \ [(\eta Q_p)^{\otimes(n-p)} T_{p,n}(1)]$$

On the other hand, for any $\varphi_1 \in \mathcal{B}_b(E_p)$ and $\varphi_2 \in \mathcal{B}_b(E(p,n])$ we have $\Phi_p(\eta)(\varphi_1) = \eta Q_p(\varphi_1)/\eta Q_p(1)$ and

$$\begin{aligned}
\frac{(\eta Q_p)^{\otimes(n-p)} T_{p,n}(\varphi_2)}{(\eta Q_p)^{\otimes(n-p)} T_{p,n}(1)} &= \frac{\Phi_p(\eta)^{\otimes(n-p)} T_{p,n}(\varphi_2)}{\Phi_p(\eta)^{\otimes(n-p)} T_{p,n}(1)} \\
&= B_{p,n}(\Phi_p(\eta)^{\otimes(n-p)}) U_{p,n}(\varphi_2)
\end{aligned}$$

From these observations we find that for any $f \in \mathcal{B}_b(E_p \times E_{(p,n]})$

$$\begin{aligned}
B_{p-1,n}(\eta^{\otimes(n-p+1)}) U_{p-1,n} &= \frac{\eta^{\otimes(n-p+1)} T_{p-1,n}(f)}{\eta^{\otimes(n-p+1)} T_{p-1,n}(1)} \\
&= \Phi_p(\eta) \otimes [B_{p,n}(\Phi_p(\eta)^{\otimes(n-p)}) U_{p,n}]
\end{aligned}$$

This ends the proof of the lemma. ■

From the Feynman-Kac representation of $q$-tensor marginal distributions given in section 2.1 we see that the semi-group structure of the $q$-tensor product measures on path space

$$\Gamma_n^{(q)} = \eta_0^{\otimes q} \otimes \ldots \otimes \eta_n^{\otimes q} \in \mathcal{P}(E_{[0,n]}^{(q)})$$

can be studied using the same lines of arguments as above by replacing the pair semi-groups $(Q_{p,n}, \Phi_{p,n})$ by the $q$-tensor product semi-groups $(Q_{p,n}^{(q)}, \Phi_{p,n}^{(q)})$. We will use the superscript $(.)^{(q)}$ to define the corresponding mathematical quantities. To be more precise let $\Omega_{p,n}^{(q)}$, $0 \leq p \leq n$, be the (non linear) semi-group associated to the distributions flow $\Gamma_n^{(q)}$ and given by

$$\Gamma_n^{(q)} = \Omega_{p,n}^{(q)}(\Gamma_p^{(q)})$$

From the preceding construction we check $\Omega_{p,n}^{(q)}$ can be described for any $\mu \in \mathcal{P}(E_0^q \times \ldots E_p^q)$ by the following formula

$$\Omega_{p,n}^{(q)}(\mu) = \mu \otimes B_{p,n}^{(q)}(\mu_p^{\otimes(n-p)}) U_{p,n}^{(q)}$$

where

- $\mu_p^{\otimes(n-p)} \in \mathcal{P}(E_p^{q(n-p)})$ is the $(n-p)$-tensor product distribution of the $p$-th time marginal $\mu_p \in \mathcal{P}(E_p^q)$ of $\mu$.
- $U_{p,n}^{(q)}$ is the Markov transition from $E_p^{q(n-p)}$ into

  $$E_{(p,n]}^q = E_{p+1}^q \times \ldots \times E_n^q$$

  and defined for any $F \in \mathcal{B}_b(E_{p+1}^q \times \ldots \times E_n^q)$ by

  $$U_{p,n}^{(q)}(F) = T_{p,n}^{(q)}(F)/T_{p,n}^{(q)}(1)$$

  with

  $$\begin{aligned}
  &T_{p,n}^{(q)}(F)(x_p^1, \ldots, x_p^{(n-p)}) \\
  &= \int_{E_{(p,n]}^{(q)}} \prod_{k=1}^{n-p} Q_{p,p+k}^{(q)}(x_p^k, dx_{p+k})\ F(x_{p+1}, \ldots, x_n)
  \end{aligned}$$

- $B^{(q)}_{p,n}$ is the Boltzmann-Gibbs transformation on $\mathcal{P}(E_p^{q(n-p)})$ defined for any pair $(\nu, f) \in (\mathcal{P}(E_p^{q(n-p)}) \times \mathcal{B}_b(E_p^{q(n-p)}))$ by

$$B^{(q)}_{p,n}(\nu)(f) = \nu(T^{(q)}_{p,n}(1)\ f)/\nu(T^{(q)}_{p,n}(1))$$

As in (12) we notice that the non homogeneous potential functions $T^{(q)}_{p,n}(1)$ are given by

$$T^{(q)}_{p,n}(1) \quad = \quad G^{(q)}_{p,p+1} \otimes G^{(q)}_{p,p+2} \otimes \ldots \otimes G^{(q)}_{p,n}$$

In other words in terms of the potential functions $G_{p,n}$ for any

$$(x_p^1, \ldots, x_p^{(n-p)}) \in E_p^{q(n-p)} \quad \text{with} \quad x_p^k = (x^{k,1}, \ldots, x^{k,q}) \in E_p^q$$

for each $1 \leq k \leq (n-p)$ we have

$$T^{(q)}_{p,n}(1)(x_p^1, \ldots, x_p^{(n-p)}) = \prod_{k=1}^{(n-p)} \prod_{i=1}^{q} G_{p,p+k}(x_p^{k,i})$$

We end this section with the version of lemma 2.1 in the context of $q$-tensor product semi-groups.

**Lemma 2.2** *For any $q, p \geq 1$ and $\eta \in \mathcal{P}(E^q_{p-1})$ we have*

$$B^{(q)}_{p-1,n}[\eta^{\otimes(n-p+1)}]U^{(q)}_{p-1,n} = \Phi^{(q)}_p(\eta) \otimes (B^{(q)}_{p,n}[\Phi^{(q)}_p(\eta)^{\otimes(n-p)}]U^{(q)}_{p,n})$$

# 3 Asymptotic properties of Boltzmann-Gibbs distributions

Let $\mu$ be a probability measure on a given measurable state space $(E, \mathcal{E})$. During the further development of this section we fix an integer $N \geq 1$ and we denote by

$$m(X) = \frac{1}{N}\sum_{i=1}^{N} \delta_{X^i}$$

the $N$-empirical measure associated to a collection of independent and identically distributed random variables $X = (X^i)_{i\geq 1}$, with common law $\mu$. We denote by $m(X)^{\otimes q}$ and $m(X)^{\odot q}$, $q \leq N$, the random distributions on $E^q$ defined by

$$\begin{aligned} m(X)^{\otimes q} &= \frac{1}{N^q} \sum_{\alpha \in \langle N \rangle^{\langle q \rangle}} \delta_{(X^{\alpha(1)}, \ldots, X^{\alpha(q)})} \\ m(X)^{\odot q} &= \frac{1}{(N)_q} \sum_{\alpha \in \langle q, N \rangle} \delta_{(X^{\alpha(1)}, \ldots, X^{\alpha(q)})} \end{aligned}$$

Let $g = (g_i)_{i\geq 1}$ be a collection of $\mathcal{E}$-measurable and non negative functions on $E$ such that $\mu(g_i) \in (0, \infty)$, for each $i \geq 1$. For any fixed integer $q \geq 1$ we denote by $g^{(q)}$ the $q$-tensor product function on $E^q$ defined by

$$g^{(q)} = g_1 \otimes \ldots \otimes g_q : (x^1, \ldots, x^q) \in E^q \longrightarrow g_1(x_1) \ldots g_q(x^q) \in (0, \infty)$$

In these notations we notice that

$$m(X)^{\otimes q}(g^{(q)}) = \prod_{i=1}^{q} m(X)(g_i) \quad \text{and} \quad \mu^{\otimes q}(g^{(q)}) = \prod_{i=1}^{q} m(X)(g_i)$$

It is also convenient to introduce the mapping

$$e_{\mu,g} : u \in (0,\infty) \to e_{\mu,g}(u) \in (0,\infty)$$

defined by

$$e_{\mu,g}(u) = \mathrm{osc}_{\mu}^2(g)(1 + \mathrm{osc}_{\mu}(g)\ \sqrt{u})\ \ \exp\left(\mathrm{osc}_{\mu}^2(g)\ u\right)$$

with $\mathrm{osc}_{\mu}(g) = \sup_{i\geq 1} \mathrm{osc}(g_i/\mu(g_i))$. When the potential functions $g$ are chosen such that $\mu(g_i) = 1$ for any $i \geq 1$ we simplify notations and we write $e_g$ instead of $e_{\mu,g}$ to emphasize that the function does not depend on $\mu$. We associate to the pair $(g,q)$ the Boltzmann-Gibbs transformation $\Psi^{(q)} : \mathcal{P}(E^q) \to \mathcal{P}(E^q)$ defined for any $(\eta, f) \in \mathcal{P}(E^q) \times \mathcal{B}_b(E^q)$ by the formula

$$\Psi^{(q)}(\eta)(f) = \eta(g^{(q)}f)/\eta(g^{(q)})$$

The main object of this section is to analyze the asymptotic properties of the random distributions $\Psi^{(q)}(m(X)^{\otimes q})$ as the pair parameter $(q, N)$ tends to infinity. Our main result is

**Theorem 3.1** *Let $(g_i)_{i\geq 1}$ be a collection of measurable functions $g_i$ with uniformly bounded oscillations $\mathrm{osc}(g) = \sup_{i\geq 1} \mathrm{osc}(g_i) < \infty$. For any $N \geq q \geq 1$ and $f \in \mathcal{B}_b(E^q)$ with $\mathrm{osc}(f) \leq 1$ we have*

$$|\mathbb{E}(\Psi^{(q)}(m(X)^{\otimes q})(f)) - \Psi^{(q)}(\mu^{\otimes q})(f)| \leq c\ \frac{q^2}{N}\ [1 + e_{\mu,g}\left(\frac{2q^2}{N}\right)] \tag{13}$$

*and for any $n \geq 1$*

$$\mathbb{E}([\Psi^{(q)}(m(X)^{\otimes q})(f) - \Psi^{(q)}(\mu^{\otimes q})(f)]^{2n}) \leq c\ 2^{4n}\ \frac{(nq)^2}{N}\ \ [1 + e_{\mu,g}\left(\frac{2(nq)^2}{N}\right)] \tag{14}$$

Theorem 3.1 will be proved in the end of the section. In order to prepare for its proof we first present three technical lemmas of separate interest.

**Lemma 3.1** *For any $1 \leq q \leq N$ there exists a Markov transition $\tilde{R}_N^{(q)}$ from $E^q$ into itself such that for any $E$-valued sequence $x = (x^i)_{i\geq 1}$ we have*

$$m(x)^{\otimes q} = m(x)^{\odot q} R_N^{(q)} \quad \textit{with} \quad R_N^{(q)} = \frac{(N)_q}{N^q}\ Id + (1 - \frac{(N)_q}{N^q})\ \tilde{R}_N^{(q)}$$

*and where*

$$\begin{aligned} m(x)^{\otimes q} &= \frac{1}{N^q} \sum_{\alpha \in \langle N \rangle^{\langle q \rangle}} \delta_{(x^{\alpha(1)},\ldots,x^{\alpha(q)})} \\ m(x)^{\odot q} &= \frac{1}{(N)_q} \sum_{\alpha \in \langle q, N \rangle} \delta_{(x^{\alpha(1)},\ldots,x^{\alpha(q)})} \end{aligned}$$

**Lemma 3.2** *The following assertions are satisfied for any $\mathcal{E}$-measurable function $h$ such that $\mu(h) = 0$.*

- *If $h$ has finite oscillations* $\mathrm{osc}(h) < \infty$ *then for any $n \geq 1$ we have*

$$\begin{aligned} N^n \mathbb{E}(m(X)(h)^{2n}) &\leq (2n)_n\ 2^{-n}\ \mathrm{osc}(h)^{2n} \qquad (15) \\ N^{n-1/2}\ \mathbb{E}(|m(X)(h)|^{2n-1}) &\leq \frac{(2n-1)_n}{\sqrt{n-1/2}}\ 2^{-(n-1/2)}\ \mathrm{osc}(h)^{(2n-1)} \end{aligned}$$

- *If we have $\mu(h^{2n}) < \infty$ for some $n \geq 1$ then*

$$\begin{aligned} N^n\ \mathbb{E}(m(X)(h)^{2n}) &\leq (2n)_n\ 2^n\ \mu(h^{2n}) \\ N^{n-1/2}\ \mathbb{E}(|m(X)(h)|^{2n-1}) &\leq \frac{(2n-1)_n}{\sqrt{n-1/2}}\ 2^{n-1/2}\ \mu(h^{2n})^{1-\frac{1}{2n}} \end{aligned}$$

**Lemma 3.3** *Let $(g_i)_{i\geq 1}$ be a collection of measurable functions $g_i$ with uniformly bounded oscillations* $\mathrm{osc}(g) = \sup_{i\geq 1} \mathrm{osc}(g_i) < \infty$ *and such that $\mu(g_i) = 1$ for any $i \geq 1$. Then, for any $n \geq 1$ we have*

$$|\ \mathbb{E}(\ [m(X)^{\otimes q}(g^{(q)}) - 1]^n\ )\ |\ \leq\ 2^{n-1}\ \frac{(nq)^2}{N}\ \ e_g\left(\frac{(nq)^2}{2N}\right) \qquad (16)$$

At this stage it is convenient to pause for a while and to make a couple of remarks:

The first lemma 3.1 connects the $q$-tensor product measures $m(X)^{\otimes q}$ with the $q$-symmetric statistic type distributions $m(X)^{\odot q}$. This connection is expressed in terms of an abstract Markov transport equation. Its proof relies on purely combinatorial techniques and it is housed in the subsection 4.3 of the Appendix. Lemma 3.2 and lemma 3.3 provide some precise $L_p$-type mean error estimates. Their proofs rely on symmetrization and combinatorial techniques and they are presented in subsections 4.1 and 4.2 of the Appendix. There is a number of significant and related estimates in the literature on martingales which apply to our context. For instance using Burkholder's inequality (cf. for instance [3]) we would find that

$$N^n\ \mathbb{E}(m(X)(h)^{2n})\ \leq\ (18B_{2n})^{2n}\ \mathrm{osc}(h)^{2n}$$

with $(2n) \leq B_{2n} = 2n\sqrt{n/(n-1/2)} \leq \sqrt{2}\ (2n)$. This would lead to the estimate

$$N^n\ \mathbb{E}(m(X)(h)^{2n}) \leq 2^n\ 18^{2n}\ (2n)^{2n}\ \mathrm{osc}(h)^{2n}$$

Next inequality gives a quick and simple way to measure the improvements obtained in lemma 3.2

$$\frac{2^{-n}\ (2n)_n}{2^n\ 18^{2n}\ (2n)^{2n}} = \frac{1}{6^{4n}\ (2n)^n} \prod_{p=1}^{n-1}\left(1 - \frac{p}{2n}\right) \leq \frac{1}{6^{4n}\ (2n)^n}$$

On the other hand, by the central limit theorem, we have the following asymptotic result

$$\left(\sqrt{N}\ m(X)[h/\|h\|_{2,\mu}]\right)^{2n} \xrightarrow{\ d\ } W^{2n}$$

where $W$ is a centered and Gaussian random variable with $\mathbb{E}(W^2) = 1$ and the superscript $\xrightarrow{d}$ stands for the convergence in distribution as $N$ tends to infinity. In this connection if we have $\mu(h^{2n}) < \infty$ for some integer $n \geq 1$ then it is well known that

$$\lim_{N\to\infty} N^n \; \mathbb{E}\left(m(X)[h/\|h\|_{2,\mu}]\right)^{2n} = \mathbb{E}(W^{2n}) = \; (2n)_n \; 2^{-n}$$

This asymptotic result already indicates that in this sense the estimates presented in lemma 3.2 are sharp. As we already mentioned these estimates will be used in the further development of section 4 to derive increasing propagations of chaos properties for Feynman-Kac interacting particle approximating models. In this context the use of Burkholder's type estimates will lead to different conclusions and much coarse properties. The proof of theorem 3.1 will be easily established using the following

**Proposition 3.1** *Let $(g_i)_{i\geq 1}$ be a collection of measurable functions $g_i$ with uniformly bounded oscillations $\mathrm{osc}(g) = \sup_{i\geq 1} \mathrm{osc}(g_i) < \infty$ and such that $\mu(g_i) = 1$ for any $i \geq 1$. For any $n \geq 1$, $N \geq q \geq 1$ and $f \in \mathcal{B}_b(E^q)$ with $\|f\| \leq 1$ and $\mathrm{osc}(f) \leq 1$ we have*

$$|\mathbb{E}([m(X)^{\otimes q}(g^{(q)}f) - \mu^{\otimes q}(g^{(q)}f)]^n)| \leq 2^{n+1} \; \frac{(nq)^2}{N} \;\; [1 + e_g\left(\frac{(nq)^2}{2N}\right)] \tag{17}$$

**Proof:**
From proposition 4.1 we have the Markovian transport equation

$$m(X)^{\otimes q} = m(X)^{\odot q} R_N^{(q)} \quad \text{with} \quad R_N^{(q)} = \frac{(N)_q}{N^q} \; Id + (1 - \frac{(N)_q}{N^q}) \; \tilde{R}_N^{(q)}$$

for some Markov kernel $\tilde{R}_N^{(q)}$ on $E^q$ and for any $q \leq N$. Since

$$\left(R_N^{(q)} - Id\right) = (1 - (N)_q/N^q) \; \left(\tilde{R}_N^{(q)} - Id\right)$$

and recalling that $\mathbb{E}(m(X)^{\odot q}(g^{(q)}f)) = \mu^{\otimes q}(g^{(q)}f)$ we readily prove that

$$\begin{aligned}
\mathbb{E}(m(X)^{\otimes q}(g^{(q)}f)) - \mu^{\otimes q}(g^{(q)}f) &= \mathbb{E}\left(m(X)^{\odot q}[R_N^{(q)} - Id](g^{(q)}f)\right) \\
&= (1 - (N)_q/N^q) \; \mu^{\otimes q}[\tilde{R}_N^{(q)} - Id](g^{(q)}f)
\end{aligned}$$

To estimate the r.h.s. term in the above display we use the decomposition

$$\mu^{\otimes q}[\tilde{R}_N^{(q)} - Id](g^{(q)}f) = I_1 + I_2$$

with

$$\begin{aligned}
I_1 &= \mu^{\otimes q}\left(\tilde{R}_N^{(q)}(g^{(q)}) \left[\frac{\tilde{R}_N^{(q)}(g^{(q)}f)}{\tilde{R}_N^{(q)}(g^{(q)})} - \mu^{\otimes q}(g^{(q)}f)\right]\right) \\
I_2 &= \mu^{\otimes q}(g^{(q)}f) \; [\mu^{\otimes q}\tilde{R}_N^{(q)}(g^{(q)}) - 1]
\end{aligned}$$

We observe that

$$\begin{aligned}
|I_2| &\leq |\mu^{\otimes q}\tilde{R}_N^{(q)}(g^{(q)}) - 1| = |\mu^{\otimes q}[\tilde{R}_N^{(q)} - Id](g^{(q)})| \\
|I_1| &\leq \mu^{\otimes q}\tilde{R}_N^{(q)}(g^{(q)}) \leq 1 + |\mu^{\otimes q}[\tilde{R}_N^{(q)} - Id](g^{(q)})|
\end{aligned}$$

From these estimates we find that

$$|\mathbb{E}(m(X)^{\otimes q}(g^{(q)}f) - \mu^{\otimes q}(g^{(q)}f)|$$

$$\leq\ (1-(N)_q/N^q)\ [1+2|\mu^{\otimes q}[\tilde{R}_N^{(q)} - Id](g^{(q)})|]$$

$$= (1-(N)_q/N^q) + 2\ |\mu^{\otimes q}[R_N^{(q)} - Id](g^{(q)})|$$

Consequently we have

$$|\mathbb{E}(m(X)^{\otimes q}(g^{(q)}f)) - \mu^{\otimes q}(g^{(q)}f)| \leq (1-(N)_q/N^q) + 2|\mathbb{E}(m(X)^{\otimes q}(g^{(q)})) - 1|$$

and by lemma 3.3 this implies that

$$\begin{aligned}|\mathbb{E}(m(X)^{\otimes q}(g^{(q)}f) - \mu^{\otimes q}(g^{(q)}f)| &\leq (1-(N)_q/N^q) + \frac{2q^2}{N}\ e_g\left(\frac{q^2}{2N}\right)\\ &\leq \frac{2q^2}{N}\ [1+e_g\left(\frac{q^2}{2N}\right)]\end{aligned}$$

Using the same lines of reasoning as in the end of the proof of lemma 3.3 (cf. pp. 25, subsection 4.2 of the Appendix) we also prove that for any $n \geq 1$

$$|\mathbb{E}(\ [m(X)^{\otimes q}(g^{(q)}f) - \mu^{\otimes q}(g^{(q)}f)]^n\ )| \leq 2^{n+1}\ \frac{(nq)^2}{N}\ \ [1+e_g\left(\frac{(nq)^2}{2N}\right)]$$

This ends the proof of the proposition. ∎

**Proof of theorem 3.1:**

By definition of $\Psi^{(q)}$ no generality is lost and much convenience is gained by supposing (as it will be done) that we have $\mu(g_i) = 1$, for each $i \geq 1$. To prove (14) we use the decomposition

$$\begin{aligned}\Psi^{(q)}(m(X)^{\otimes q})(f) - \Psi^{(q)}(\mu^{\otimes q})(f) &= \Psi^{(q)}(m(X)^{\otimes q})(f - \mu^{\otimes q}(g^{(q)}f))\\ &= I_1 + I_2 \end{aligned} \tag{18}$$

with

$$\begin{aligned}I_1 &= m(X)^{\otimes q}(g^{(q)}(f - \mu^{\otimes q}(g^{(q)}f))\\ I_2 &= \Psi^{(q)}(m(X)^{\otimes q})(f - \mu^{\otimes q}(g^{(q)}f))\ \left(1 - m(X)^{\otimes q}(g^{(q)})\right)\end{aligned}$$

It is now convenient to observe that

$$\begin{aligned}\mu^{\otimes q}(g^{(q)}(f - \mu^{\otimes q}(g^{(q)}f)) &= 0\\ \|f - \mu^{\otimes q}(g^{(q)}f)\| &\leq \operatorname{osc}(f) = \operatorname{osc}(f - \mu^{\otimes q}(g^{(q)}f)) \leq 1\end{aligned}$$

and for any $n \geq 1$ we have

$$\mathbb{E}([\Psi^{(q)}(m(X)^{\otimes q})(f) - \Psi^{(q)}(\mu^{\otimes q})(f)]^{2n}) \leq 2^{2n-1}\ \ (\mathbb{E}(I_1^{2n}) + \mathbb{E}(I_2^{2n}))$$

Therefore using proposition 3.1 and lemma 3.3 we check that

$$\mathbb{E}([\Psi^{(q)}(m(X)^{\otimes q})(f) - \Psi^{(q)}(\mu^{\otimes q})(f)]^{2n}) \leq c\, 2^{4n}\ \frac{(nq)^2}{N}\ \ [1 + e_g\left(\frac{2(nq)^2}{N}\right)]$$

This ends the proof of (14). To prove (13) we use again the decomposition (18). By (17) we find that

$$|\mathbb{E}(I_1)| \leq c\ \frac{q^2}{N}\ \ [1 + e_g\left(\frac{q^2}{2N}\right)]$$

To estimate the mean value of $I_2$ we first use Cauchy-Schwartz's inequality to check that

$$|\mathbb{E}(I_2)|^2 \leq \mathbb{E}([\Psi^{(q)}(m(X)^{\otimes q})(f - \mu^{\otimes q}(g^{(q)}f))]^2)\ \mathbb{E}([1 - m(X)^{\otimes q}(g^{(q)})]^2)$$

Via (16) and (14) this implies that

$$|\mathbb{E}(I_2)| \leq c\ \frac{q^2}{N}\ \ [1 + e_g\left(\frac{2q^2}{N}\right)]$$

from which we conclude that

$$|\mathbb{E}(\Psi^{(q)}(m(X)^{\otimes q})(f)) - \Psi^{(q)}(\mu^{\otimes q})(f)| \leq c\ \frac{q^2}{N}\ [1 + e_g\left(\frac{2q^2}{N}\right)]$$

This ends the proof of the theorem. ■

# 4 Propagations of chaos estimates

This section is mainly concerned with the proofs of the theorem 1.1 and theorem 1.2 stated in section 1.2.

**Proof of Theorem 1.1:** We use the decomposition

$$(\eta_n^N)^{\otimes q} - \eta_n^{\otimes q} = \sum_{p=0}^{n} \left[\Phi_{p,n}^{(q)}((\eta_p^N)^{\otimes q}) - \Phi_{p,n}^{(q)}(\Phi_{p-1,p}^{(q)}((\eta_{p-1}^N)^{\otimes q}))\right] \tag{19}$$

with the convention $\Phi_{-1,0}((\eta_{-1}^N)^{\otimes q}) = \eta_0^{\otimes q}$ for $p = 0$. Our next objective is to estimate the differences of measures

$$I_{p,n}^{(q)} =_{def.} \left[\Phi_{p,n}^{(q)}((\eta_p^N)^{\otimes q}) - \Phi_{p,n}^{(q)}(\Phi_{p-1,p}^{(q)}((\eta_{p-1}^N)^{\otimes q}))\right]$$

Using (11) we first observe that $\Phi_{p-1,p}^{(q)}((\eta_{p-1}^N)^{\otimes q}) = \Phi_p(\eta_{p-1}^N)^{\otimes q}$ and for any $f \in \mathcal{B}_b(E_n^q)$

$$I_{p,n}^{(q)}(f) = \left[\Psi_{p,n}^{(q)}((\eta_p^N)^{\otimes q}) - \Psi_{p,n}^{(q)}(\Phi_p(\eta_{p-1}^N)^{\otimes q})\right] P_{p,n}^{(q)}(f)$$

The conclusion now follows from theorem 3.1. First we notice that for any $\mu \in \mathcal{P}(E_p)$ we have

$$\mathrm{osc}_\mu(G_{p,n}) = \frac{\mathrm{osc}(G_{p,n})}{\mu(G_{p,n})} \leq (r_{p,n} - 1) \quad \text{with} \quad r_{p,n} = \sup_{x_p, y_p \in E_p} \frac{G_{p,n}(x_p)}{G_{p,n}(y_p)} \tag{20}$$

Therefore recalling that $\eta_p^N = \frac{1}{N}\sum_{i=1}^N \delta_{\xi_p^i}$ is the empirical measure associated to a collection of $N$ conditionally independent and identically distributed random variables $\xi_p = (\xi_p^i)_{1\leq i\leq N}$ with common law $\Phi_p(\eta_{p-1}^N)$ we find from (13) the $\mathbb{P}_{\eta_0}^N$-almost sure estimate

$$|\mathbb{E}_{\eta_0}^N(I_{p,n}^{(q)}(f)|F_{p-1}^N)| \leq c\ \frac{q^2}{N}\ [1+e_{p,n}\left(\frac{2q^2}{N}\right)]\ \mathrm{osc}(P_{p,n}^{(q)}(f))$$

We recall that for any Markov transition $M$ from a measurable space $(E,\mathcal{E})$ into a (possibly different) measurable space $(E',\mathcal{E}')$ and for any $f\in\mathcal{B}_b(E')$ we have the inequality

$$\mathrm{osc}(M(f))\ \leq\ \beta(M)\ \mathrm{osc}(f)$$

From this property we conclude that for any $f\in\mathcal{B}_b(E_n^q)$ with $\mathrm{osc}(f)\leq 1$ we have

$$|\mathbb{E}_{\eta_0}^N(I_{p,n}^{(q)}(f)|F_{p-1}^N)| \leq c\ \frac{q^2}{N}\ \beta(P_{p,n}^{(q)})[1+e_{p,n}\left(\frac{2q^2}{N}\right)] \qquad \mathbb{P}_{\eta_0}^N - \text{a.s.}$$

By (19) it follows that for any $f\in\mathcal{B}_b(E_n^q)$ with $\mathrm{osc}(f)\leq 1$ we have

$$|\mathbb{E}_{\eta_0}^N((\eta_n^N)^{\otimes q}(f)) - \eta_n^{\otimes q}(f)| = c\ \frac{q^2}{N}\ \sum_{p=0}^n\ \beta(P_{p,n}^{(q)})[1+e_{p,n}\left(\frac{2q^2}{N}\right)]$$

Taking into account that

$$\mathbb{P}_{\eta_0,[n]}^{(N,q)}(f) = \mathbb{E}_{\eta_0}^N(f(\xi_n^1,\cdots,\xi_n^q)) = \mathbb{E}_{\eta_0}^N(m(\xi_n)^{\odot q})$$

lemma 3.1 ensures that for any $f\in\mathcal{B}_b(E_n^q)$ with $\mathrm{osc}(f)\leq 1$ we have

$$\begin{aligned}
&|\mathbb{P}_{\eta_0,[n]}^{(N,q)}(f) - \eta_n^{\otimes q}(f)|\\
&\leq \frac{(q-1)^2}{N} + c\ \frac{q^2}{N}\ \sum_{p=0}^n\ \beta(P_{p,n}^{(q)})[1+e_{p,n}\left(\frac{2q^2}{N}\right)]\\
&\leq c\ \frac{q^2}{N}\ \left(1+\sum_{p=0}^n\ \beta(P_{p,n}^{(q)})[1+e_{p,n}\left(\frac{2q^2}{N}\right)]\right)
\end{aligned}$$

This ends the proof of theorem 1.1. ■

To illustrate the impact of theorem 1.1 we present hereafter some easily derived strong and uniform propagations of chaos estimates.

**Corollary 4.1** *Let us suppose that the triplet $(E_n,G_n,M_n)$ is time homogeneous and the following regularity conditions are met*

$$G(x)\geq\epsilon(G)\ G(y)\quad\text{and}\quad M^m(x,\,.)\geq\epsilon(M)\ M^m(y,\,.) \tag{21}$$

*for some $\epsilon(G)$, $\epsilon(M)>0$, $m\geq 1$ and for any $x,y\in E$. Then we have*

$$\|\mathbb{P}_{\eta_0,[n]}^{(N,q)} - \eta_n^{\otimes q}\|_{tv} \leq c\ \frac{q^2}{N}\ \left(1+\ d_m^{(\epsilon)}(q,n)\ [1+e_m^{(\epsilon)}(2q^2/N)]\right)$$

*where*

$$d_m^{(\epsilon)}(q,n) \;=\; \sum_{p=0}^{n}(1-\epsilon_m^q(G,M))^{[p/m]} \leq (n+1)\wedge\left(m\ \epsilon_m^{-q}(G,M)\right)$$

*with* $\epsilon_m(G,M)=\epsilon^{(m-1)}(G)\ \epsilon(M)$ *and* $e_m^{(\epsilon)}(u)$ *is the mapping defined as* $e_{p,n}(u)$ *(cf. (7)) by replacing the constant* $r_{p,n}$ *by* $r_m^{(\epsilon)}=\epsilon^{-m}(G)\epsilon^{-1}(M)$.

**Proof:**
When the regularity conditions (21) are met we recall that for any $0\leq p+m\leq n$ we have the uniform estimate

$$r_{p,n}\leq\epsilon^{-m}(G)\epsilon^{-1}(M)$$

and for any $x_p,y_p\in E_p^q$ and any non negative function $\varphi$ on $E_{p+m}^q$

$$\frac{Q_{p,p+m}^{(q)}(\varphi)(x_p)}{Q_{p,p+m}^{(q)}(1)(x_p)}\ \geq\ \epsilon_m^q(G,M)\ \frac{M_{p,p+m}^{(q)}(\varphi)(y_p)}{M_{p,p+m}^{(q)}(1)(y_p)}$$

By definition of the Dobrushin's ergodic coefficient this yields that

$$\beta(P_{p,n}^{(q)})\ \leq\ (1-\epsilon_m^q(G,M))^{[(n-p)/m]}$$

Recalling that $r_{p,n}\leq\epsilon^{-(n-p)}(G)$ we observe that for any $p\leq n$

$$r_{p,n}\leq\epsilon^{-m}(G)(\epsilon^{-1}(M)\vee 1)=\epsilon^{-m}(G)\epsilon^{-1}(M)$$

and consequently $\sup_{p\leq n}e_{p,n}(u)\ \leq\ e_m^{(\epsilon)}(u)$. From previous calculations we easily find that

$$\|\mathbb{P}_{\eta_0,[n]}^{(N,q)}-\eta_n^{\otimes q}\|_{tv}\quad\leq\quad c\ \frac{q^2}{N}\ \left(1+[1+e_m^{(\epsilon)}(2q^2/N)]\ d_m^{(\epsilon)}(q,n)\right)$$

This ends the proof of the corollary. ∎

**Corollary 4.2** *Assume that the regularity assumptions stated in corollary 4.1 are met for some* $\epsilon(G)$, $\epsilon(M)>0$ *and* $m\geq 1$. *Then using the same notations as in there we have the uniform propagations of chaos estimate*

$$\sup_{n\geq 0}\left\|\mathbb{P}_{\eta_0,[n]}^{(N,q)}-\eta_n^{\otimes q}\right\|_{tv}\ \leq\ c\ \frac{q^2}{N}\ \left(1+\ m\ \epsilon_m^{-q}(G,M)\ [1+e_m^{(\epsilon)}(2q^2/N)]\right)$$

*In addition for any non decreasing sequence of time horizons* $n(N)$ *and particle block sizes* $q(N)$ *such that* $\lim_{N\to\infty}n(N)q^2(N)/N=0$ *we have the increasing propagation of chaos property*

$$\limsup_{N\to\infty}\frac{N}{q^2(N)n(N)}\|\mathbb{P}_{\eta_0,[n(N)]}^{(N,q(N))}-\eta_{n(N)}^{\otimes q(N)}\|_{tv}\leq c/(\epsilon^m(G)\epsilon(M))^2$$

The end of this section is concerned with the proof of theorem 1.2. Our first task is to better connect the distributions

$$\mathbb{P}^{(N,q)}_{\eta_0,n} = \mathrm{Law}((\xi^i_{[0,n]})_{1\leq i\leq q}) \in \mathcal{P}(E^q_{[0,n]})$$

with the Markovian structure of the interacting particle model defined in (3). Notice that for each $0 \leq p \leq n$ and $1 \leq q \leq N$ the state space

$$E^{(q)}_{[p,n]} = E^q_p \times \ldots \times E^q_n$$

represents the set of the first $q$ paths from time $p$ to time $n$ of the Markov particle model while the product space

$$E^q_{[p,n]} = (E_p \times \ldots \times E_n)^q$$

represents the state space of the paths of each of the first $q$ elementary particle from time $p$ up to time $n$. We shall also use the notation $E^{(q)}_{(p,n]} = E^{(q)}_{[p+1,n]}$. We recall that $E^{(q)}_{[0,n]}$ and $E^q_{[0,n]}$ are connected by the mapping $\Theta^q_n : E^{(q)}_{[0,n]} \longrightarrow E^q_{[0,n]}$ defined in (10). For instance for $q = N$ we have $\Theta^N_n(\xi_0, \ldots, \xi_n) = \xi_{[0,n]}$ and $\mathbb{P}^{(N)}_{\eta_0,n} = \mathbb{P}^N_{\eta_0,n} \circ (\Theta^N_n)^{-1}$ where

$$\mathbb{P}^N_{\eta_0,n} = \mathbb{P}^N_{\eta_0} \circ (\xi_0, \ldots, \xi_n)^{-1} \in \mathcal{P}(E^{(N)}_{[0,n]})$$

In this connection it is also convenient to associate to the pair of path measures $((\Gamma^N_n)^{\otimes q}, (\Gamma^N_n)^{\odot q})$ defined in (4) the distributions

$$\Gamma^{(N,q)}_n = (\Gamma^N_n)^{\otimes q} \circ (\Theta^q_n)^{-1} \quad \text{and} \quad \Gamma^{\langle N,q\rangle}_n = (\Gamma^N_n)^{\odot q} \circ (\Theta^q_n)^{-1} \in \mathcal{P}(E^{(q)}_{[0,n]})$$

In other words we have with some obvious abusive notations

$$\begin{aligned}
\Gamma^{(N,q)}_n &= \frac{1}{N^q} \sum_{\alpha \in \langle N\rangle^{\langle q\rangle}} \delta_{((\xi^{\alpha(i)}_0)_{1\leq i\leq q}, \ldots, (\xi^{\alpha(i)}_n)_{1\leq i\leq q})} \\
\Gamma^{\langle N,q\rangle}_n &= \frac{1}{(N)_q} \sum_{\alpha \in \langle q,N\rangle} \delta_{((\xi^{\alpha(i)}_0)_{1\leq i\leq q}, \ldots, (\xi^{\alpha(i)}_n)_{1\leq i\leq q})}
\end{aligned}$$

**Lemma 4.1** *For any pair of integers $1 \leq q \leq N$ and any test function $F \in \mathcal{B}_b(E^{(q)}_{[0,n]})$ with $\|F\| \leq 1$ we have*

$$|\mathbb{E}^N_{\eta_0}(\Gamma^{\langle N,q\rangle}_n(F)) - \mathbb{E}^N_{\eta_0}(F((\xi^i_0)_{1\leq i\leq q}, \ldots, (\xi^i_n)_{1\leq i\leq q}))| \leq (q-1)^2/N$$

**Proof:**
By lemma 3.1 we observe that

$$\|(\Gamma^N_n)^{\odot q} - (\Gamma^N_n)^{\otimes q}\|_{tv} \leq (q-1)^2/N \tag{22}$$

By the exchangeability property of the particle model we also have that for any $\alpha \in \langle q, N\rangle$ and $F \in \mathcal{B}_b(E^{(q)}_{[0,n]})$

$$\mathbb{E}^N_{\eta_0}(F((\xi^{\alpha(i)}_0)_{1\leq i\leq q}, \ldots, (\xi^{\alpha(i)}_n)_{1\leq i\leq q})) = \mathbb{E}^N_{\eta_0}(F((\xi^i_0)_{1\leq i\leq q}, \ldots, (\xi^i_n)_{1\leq i\leq q}))$$

This implies that

$$\mathbb{E}^N_{\eta_0}(\Gamma^{\langle N,q\rangle}_n(F)) = \mathbb{E}^N_{\eta_0}(F((\xi^i_0)_{1\leq i\leq q}, \ldots, (\xi^i_n)_{1\leq i\leq q}))$$

However (22) also ensures that

$$\|\Gamma_n^{(N,q)} - \Gamma_n^{\langle N,q\rangle}\|_{tv} \leq (q-1)^2/N \tag{23}$$

from which the end of the proof of the lemma is easily completed. ■

We are now in position to prove the theorem.

**Proof of theorem 1.2:** In a similar fashion as in the proof of theorem 1.1 we use the decomposition

$$\Gamma_n^{(N,q)} - \Gamma_n^{(q)} = \sum_{p=0}^{n} \left[\Omega_{p,n}^{(q)}(\Gamma_p^{(N,q)}) - \Omega_{p,n}^{(q)}(\Omega_{p-1,p}^{(q)}(\Gamma_{p-1}^{(N,q)}))\right] \tag{24}$$

As usually we take the convention for $p=0$, $\Omega_{-1,0}^{(q)}(\Gamma_{-1}^{(N,q)}) = \eta_0^{\otimes q}$. To describe more precisely each term in the above summand we first observe that for the $q$-tensor product measure $(\eta_p^N)^{\otimes q} \in \mathcal{P}(E_p^q)$ is the $p$-th time marginal of $\Gamma_n^{(N,q)}$. On the other hand, by definition of the semi-group $\Omega_{p,n}^{(q)}$ we have

$$\Omega_{p,n}^{(q)}(\Gamma_p^{(N,q)}) = \Gamma_p^{(N,q)} \otimes B_{p,n}^{(q)}((\eta_p^N)^{\otimes q(n-p)})U_{p,n}^{(q)}$$

and

$$\Omega_{p-1,p}^{(q)}(\Gamma_{p-1}^{(N,q)}) \quad = \quad \Gamma_{p-1}^{(N,q)} \otimes \Phi_{p-1,p}^{(q)}((\eta_{p-1}^N)^{\otimes q}) = \Gamma_{p-1}^{(N,q)} \otimes \Phi_p(\eta_{p-1}^N)^{\otimes q}$$

This implies that for any $1 \leq p \leq n$ we have

$$\begin{aligned}
\Omega_{p-1,n}^{(q)}(\Gamma_{p-1}^{(N,q)}) &= \Omega_{p,n}^{(q)}(\Omega_{p-1,p}^{(q)}(\Gamma_{p-1}^{(N,q)})) \\
&= \Gamma_{p-1}^{(N,q)} \otimes \Phi_p(\eta_{p-1}^N)^{\otimes q} \otimes [B_{p,n}^{(q)}(\Phi_p(\eta_{p-1}^N)^{\otimes q(n-p)})U_{p,n}^{(q)}]
\end{aligned}$$

Let $\Omega_{p,n}^{\langle q,N\rangle}$ be the random measures defined by

$$\Omega_{p,n}^{\langle q,N\rangle} = \Gamma_p^{\langle N,q\rangle} \otimes B_{p,n}^{(q)}((\eta_p^N)^{\otimes q(n-p)})U_{p,n}^{(q)}$$

Using lemma 3.1 we find that that for any $p \leq n$

$$\|\Omega_{p,n}^{\langle q,N\rangle} - \Omega_{p,n}^{(q)}(\Gamma_p^{(N,q)})\|_{tv} \leq (q-1)^2/N \tag{25}$$

As a parenthesis, using lemma 2.2 we already notice that

$$\begin{aligned}
\Omega_{p-1,n}^{\langle q,N\rangle} &= \Gamma_{p-1}^{\langle N,q\rangle} \otimes B_{p-1,n}^{(q)}((\eta_{p-1}^N)^{\otimes q(n-p+1)})U_{p-1,n}^{(q)} \\
&= \Gamma_{p-1}^{\langle q,N\rangle} \otimes \Phi_p(\eta_{p-1}^N)^{\otimes q} \otimes B_{p,n}^{(q)}(\Phi_p(\eta_{p-1}^N)^{\otimes q(n-p)})U_{p,n}^{(q)}
\end{aligned} \tag{26}$$

Now by (24) the estimates (25) imply that

$$\|\Gamma_n^{(N,q)} - \Gamma_n^{(q)} - \sum_{p=0}^{n} \left(\Omega_{p,n}^{\langle q,N\rangle} - \Omega_{p-1,n}^{\langle q,N\rangle}\right)\|_{tv} \leq 2(n+1)(q-1)^2/N \tag{27}$$

with the convention, for $p=0$, $\Omega_{-1,n}^{\langle q,N\rangle} = \Gamma_n^{(q)} = \eta_0^{\otimes q} \otimes B_{0,n}^{(q)}(\eta_0^{\otimes qn})U_{0,n}^{(q)}$. By symmetry arguments it is now convenient to observe that for any $p \leq n$, and any test function $\varphi \in \mathcal{B}_b(E^{(q)}_{[p,n]})$ the following sequence of random variables does not depend on the choice of $\alpha \in \langle q, N\rangle$

$$\mathbb{E}^N_{\eta_0}(\int [B^{(q)}_{p,n}((\eta^N_p)^{\otimes q(n-p)})U^{(q)}_{p,n}](dy)\ \varphi((\xi^{\alpha(i)}_p)_{1\leq i\leq q}, y)|F^N_{p-1})$$

$$= \mathbb{E}^N_{\eta_0}(\int [B^{(q)}_{p,n}((\eta^N_p)^{\otimes q(n-p)})U^{(q)}_{p,n}](dy)\ \varphi((\xi^{i}_p)_{1\leq i\leq q}, y)|F^N_{p-1})$$

$$= \mathbb{E}^N_{\eta_0}(\int ((\eta^N_n)^{\odot q} \otimes [B^{(q)}_{p,n}((\eta^N_p)^{\otimes q(n-p)})U^{(q)}_{p,n}])(d(x,y))\ \varphi(x,y)|F^N_{p-1})$$

where the integral is taken over the product space $E^{(q)}_{(p,n]}$. Using this property we prove that for any $f \in \mathcal{B}_b(E^{(q)}_{[0,n]})$

$$\mathbb{E}^N_{\eta_0}(\Omega^{\langle q,N\rangle}_{p,n}(f) \mid F^N_{p-1})$$

$$= \mathbb{E}^N_{\eta_0}(\ [\Gamma^{\langle N,q\rangle}_p \otimes B^{(q)}_{p,n}((\eta^N_p)^{\otimes q(n-p)})U^{(q)}_{p,n}](f) \mid F^N_{p-1})$$

$$= \mathbb{E}^N_{\eta_0}(\ [\Gamma^{\langle N,q\rangle}_{p-1} \otimes (\eta^N_p)^{\odot q} \otimes B^{(q)}_{p,n}((\eta^N_p)^{\otimes q(n-p)})U^{(q)}_{p,n}](f) \mid F^N_{p-1})$$

On the other hand using the fact that

$$\|(\eta^N_p)^{\odot q} - (\eta^N_p)^{\otimes q}\|_{tv} \leq (q-1)^2/N$$

we find the $\mathbb{P}^N_{\eta_0}$-almost sure estimate

$$|\mathbb{E}^N_{\eta_0}([\Omega^{\langle q,N\rangle}_{p,n} - \tilde{\Omega}^{\langle q,N\rangle}_{p,n}](f) \mid F^N_{p-1})| \leq \|f\|\ (q-1)^2/N \tag{28}$$

with

$$\widetilde{\Omega}^{\langle q,N\rangle}_{p,n} =_{\text{def.}} \Gamma^{\langle N,q\rangle}_{p-1} \otimes (\eta^N_p)^{\otimes q} \otimes B^{(q)}_{p,n}((\eta^N_p)^{\otimes q(n-p)})U^{(q)}_{p,n}$$

In the above display and for $p = 0$ we use have used the convention

$$\widetilde{\Omega}^{\langle q,N\rangle}_{0,n} = (\eta^N_0)^{\otimes q} \otimes B^{(q)}_{0,n}((\eta^N_0)^{\otimes qn})U^{(q)}_{0,n}$$

In these notations we have by (26) the formula

$$\begin{aligned}\widetilde{\Omega}^{\langle q,N\rangle}_{p,n} - \Omega^{\langle q,N\rangle}_{p-1,n} \quad = \quad & \Gamma^{\langle q,N\rangle}_{p-1} \otimes [(\eta^N_p)^{\otimes q} \otimes B^{(q)}_{p,n}((\eta^N_p)^{\otimes q(n-p)}) \\ & -\Phi_p(\eta^N_{p-1})^{\otimes q} \otimes B^{(q)}_{p,n}(\Phi_p(\eta^N_{p-1})^{\otimes q(n-p)})]U^{(q)}_{p,n}\end{aligned}$$

Let $\widetilde{B}^{(q)}_{p,n}$ be the extended Boltzmann-Gibbs transformation on $\mathcal{P}(E^{q(n-p+1)}_p)$ defined for any pair $(\nu, \varphi) \in (\mathcal{P}(E^{q(n-p+1)}_p) \times \mathcal{B}_b(E^{q(n-p+1)}_p))$ by

$$\widetilde{B}^{(q)}_{p,n}(\nu)(\varphi) = \nu(\tilde{T}^{(q)}_{p,n}(1)\ \varphi)/\nu(\tilde{T}^{(q)}_{p,n}(1))$$

with the non homogeneous potential functions $\tilde{T}^{(q)}_{p,n}(1)$ on $E^{q(n-p+1)}_p)$ given by

$$\tilde{T}^{(q)}_{p,n}(1) \quad = \quad G^{(q)}_{p,p} \otimes G^{(q)}_{p,p+1} \otimes \ldots \otimes G^{(q)}_{p,n}$$

In these notations, recalling that $G_{p,p}=1$, we find that for any $\mu\in\mathcal{P}(E_p^q)$ and $\nu\in\mathcal{P}(E_p^{q(n-p)})$ we have

$$\mu\otimes B^{(q)}_{p,n}(\nu)=\tilde{B}^{(q)}_{p,n}(\mu\otimes\nu)$$

This readily yields that

$$\begin{aligned}&\widetilde{\Omega}^{\langle q,N\rangle}_{p,n}-\Omega^{\langle q,N\rangle}_{p-1,n}\\&=\Gamma^{\langle q,N\rangle}_{p-1}\otimes[\widetilde{B}^{(q)}_{p,n}((\eta^N_p)^{\otimes q(n-p+1)})-\widetilde{B}^{(q)}_{p,n}(\Phi_p(\eta^N_{p-1})^{\otimes q(n-p+1)})]U^{(q)}_{p,n}\end{aligned}$$

Using theorem 3.1 and arguing as in (20) we obtain the following $\mathbb{P}^N_{\eta_0}$-almost sure estimate

$$\begin{aligned}&|\mathbb{E}^N_{\eta_0}([\widetilde{\Omega}^{\langle q,N\rangle}_{p,n}-\Omega^{\langle q,N\rangle}_{p-1,n}](f)|F^N_{p-1})|\\&\leq c\;\frac{(q(n-p+1))^2}{N}\;[1+e_{p,n}\left(\frac{2(q(n-p+1))^2}{N}\right)]\end{aligned}$$

as soon as $(n+1)q\leq N$ and $\|f\|\leq 1$. This readily implies the rather crude and almost sure upper bound

$$|\mathbb{E}^N_{\eta_0}([\widetilde{\Omega}^{\langle q,N\rangle}_{p,n}-\Omega^{\langle q,N\rangle}_{p-1,n}](f)|F^N_{p-1})|\leq c\;\frac{(q(n+1))^2}{N}\;[1+e_n(\frac{2(q(n+1))^2}{N})] \tag{29}$$

with the mapping $e_n(u)$ defined as in (7) by replacing the constants $r_{p,n}$ by $r_n=\sup_{p\leq n}r_{p,n}$. Combining (27), (28) and (29) we conclude that for any $f\in\mathcal{B}_b(E^{(q)}_{[0,n]})$ with $\|f\|\leq 1$

$$\left|\mathbb{E}^N_{\eta_0}([\Gamma^{(N,q)}_n-\Gamma^{(q)}_n](f))\right|\leq c\;(n+1)\;\tfrac{(q(n+1))^2}{N}\;[1+e_n(\tfrac{2(q(n+1))^2}{N})]$$

By definition of $\mathbb{P}^{(N,q)}_{\eta_0,n}$ and $\Gamma^{(N,q)}_n$ the total variation estimate (8) is now a simple application of lemma 4.1. This completes the proof of theorem 1.2.■

# Appendix

## 4.1 Proof of lemma 3.2

We first use a symmetrization technique. We consider a collection of independent copies $X'=(X'^i)_{i\geq 1}$ of the random variables $X=(X^i)_{i\geq 1}$. We also assume that $(X,X')$ are independent. As usually we slight abuse notations and we denote by $m(X')=\frac{1}{N}\sum_{i=1}^N\delta_{X'^i}$ the $N$-empirical distribution associated to $X'$. We observe that

$$m(X)(h)=\mathbb{E}(m(X)(h)-m(X')(h)|\;X)$$

This clearly implies that for any $p\geq 1$ we have that

$$\mathbb{E}(|m(X)(h)|^p)\leq\mathbb{E}(|m(X)(h)-m(X')(h)|^p)$$

We first examine the case $p = 2n$ with $n \geq 0$. In this situation we have

$$N^{2n}\mathbb{E}(|m(X)(h) - m(X')(h)|^{2n})$$

$$= \sum_{k=1}^{2n} \sum_{p_1+\ldots+p_k=2n} \frac{(2n)!}{p_1!\ldots p_k!} \sum_{\alpha\in\langle k,N\rangle} \prod_{i=1}^{k} \mathbb{E}((h(X^{\alpha(i)}) - h(X'^{\alpha(i)})^{p_i})$$

where $\sum_{p_1+\ldots+p_k=2n}$ indicates summation over all ordered sets of strictly positive integers $p_i \geq 1$ such that $p_1 + \ldots + p_k = 2n$. Since we have

$$\mathbb{E}([h(X^j) - h(X'^j)]^p) = -\mathbb{E}([h(X^j) - h(X'^j)]^p) = 0$$

for any $1 \leq j \leq N$ and any odd integer $p$ we check easily that

$$N^{2n}\mathbb{E}(|m(X)(h) - m(X')(h)|^{2n})$$

$$= \sum_{k=1}^{n} \sum_{p_1+\ldots+p_k=n} \frac{(2n)!}{(2p_1)!\ldots(2p_k)!} \sum_{\alpha\in\langle k,N\rangle} \mathbb{E}(\prod_{i=1}^{k}[h(X^{\alpha(i)}) - h(X'^{\alpha(i)})]^{2p_i})$$

$$\leq (2n)_n \left(\sup_{1\leq k\leq n} \sup_{p_1+\ldots+p_k=n} \prod_{i=1}^{k}(2p_i)_{p_i}^{-1}\right) \mathbb{E}((\textstyle\sum_{i=1}^{N}[h(X^i) - h(X'^i)]^2)^n)$$

Using the fact that for any $p \geq 1$ we have

$$\begin{aligned}(2p)_p &= (2p)!/p! = 2p\ (2p-1)\ldots(2p-(p-1)) \\ &= \prod_{k=1}^{p}(p+k) \geq 2^p\end{aligned}$$

we conclude that

$$N^n\mathbb{E}(|m(X)(h)-m(X')(h)|^{2n}) \leq (2n)_n\ 2^{-n}\ \mathbb{E}((\frac{1}{N}\sum_{i=1}^{N}[h(X^i)-h(X'^i)]^2)^n)$$

and therefore

$$N^n\mathbb{E}(|m(X)(h)|^{2n}) \quad \leq \quad (2n)_n\ 2^{-n}\ \mathbb{E}((\frac{1}{N}\sum_{i=1}^{N}\ [h(X^i) - h(X'^i)]^2)^n)$$

We readily conclude that

$$N^n\mathbb{E}(|m(X)(h)|^{2n}) \quad \leq \quad (2n)_n\ 2^{-n}\ \mathrm{osc}(h)^{2n}$$

as soon as $\mathrm{osc}(h) < \infty$ In the same way if we have $\mu(h^{2n}) < \infty$ then

$$\begin{aligned}N^n\mathbb{E}(|m(X)(h)|^{2n}) &\leq (2n)_n\ \mathbb{E}((m(X)(h^2) + m(X')(h^2))^n) \\ &\leq (2n)_n\ 2^n\ \mathbb{E}(m(X)(h^2)^n) \leq (2n)_n\ 2^n\ \mu(h^{2n})\end{aligned}$$

This ends the proof of lemma 3.2. For odd integers $p = 2n + 1$ we use Cauchy-Schwartz' inequality to check that

$$\mathbb{E}(|m(X)(h)|^{2n+1})^2 \leq \mathbb{E}(|m(X)(h)|^{2n})\ \ \mathbb{E}(|m(X)(h)|^{2(n+1)})$$

From previous estimates we find that

$$N^{2n+1}\ \mathbb{E}(|m(X)(h)|^{2n+1})^2 \leq (2n)_n\ (2(n+1))_{n+1}\ 2^{-(2n+1)}\ \mathrm{osc}(h)^{2(2n+1)}$$

as soon as $\mathrm{osc}(h) < \infty$. Since

$$\begin{aligned}(2(n+1))_{n+1} &= \frac{(2(n+1))!}{(n+1)!} = 2\ \frac{2n+1!}{n!} = 2\ (2n+1)_{n+1}\\ (2n)_n &= \frac{2n!}{n!} = \frac{1}{2n+1}\ \frac{2n+1!}{n!} = \frac{(2n+1)_{n+1}}{(2n+1)}\end{aligned}$$

we get

$$N^{n+1/2}\ \mathbb{E}(|m(X)(h)|^{2n+1}) \leq \frac{(2n+1)_{n+1}}{\sqrt{n+1/2}}\ 2^{-(n+1/2)}\ \mathrm{osc}(h)^{(2n+1)}$$

In the same way for any $h$ such that $\mu(h^{2(n+1)}) < \infty$ we have

$$N^{2n+1}\ \mathbb{E}(|m(X)(h)|^{2n+1})^2 \quad \leq \quad \frac{(2n+1)^2_{n+1}}{n+1/2}\ 2^{2n+1}\ \mu(h^{2n})\mu(h^{2(n+1)})$$

Since

$$\mu(h^{2n})\mu(h^{2(n+1)}) \leq \mu(h^{2(n+1)})^{2-\frac{1}{n+1}}$$

we conclude that

$$N^{n+1/2}\mathbb{E}(|m(X)(h)|^{2n+1}) \leq \frac{(2n+1)_{n+1}}{\sqrt{n+1/2}}\ 2^{n+1/2}\ \mu(h^{2(n+1)})^{1-\frac{1}{2(n+1)}}$$

## 4.2 Proof of lemma 3.3

We first prove (16) for $n=1$. Using the decomposition

$$\prod_{i=1}^{q}(1+a_i) = 1 + \sum_{1\leq p\leq q}\ \sum_{1\leq i_1<\cdots<i_p\leq q}\ \prod_{j=1}^{p} a_{i_j}$$

which is valid for any $q \geq 0$ and any collection of real numbers $(a_i)_{i\geq 1}$ we find that

$$\mathbb{E}(\prod_{i=1}^{q} m(X)(g_i)) - 1 = \sum_{2\leq p\leq q}\ \sum_{1\leq i_1<\cdots<i_p\leq q}\ \mathbb{E}(\ \prod_{j=1}^{p}[m(X)(g_j) - 1]\ )$$

Using Holders' inequality we find that

$$|\mathbb{E}(\prod_{i=1}^{q} m(X)(g_i)) - 1| = \sum_{2\leq p\leq q} C_q^p\ \ \mathbb{E}(|m(X)(g) - 1|^p)$$

with

$$\mathbb{E}(|m(X)(g) - 1|^p) = \sup_{i\geq 1}\ \mathbb{E}(|m(X)(g_j) - 1|^p)$$

Suppose $q = 2q'$ is an even integer. In this case using lemma 3.2 we find that

$$|\mathbb{E}(\prod_{i=1}^{2q'} m(X)(g_i)) - 1|$$

$$\leq \sum_{p=1}^{q'} C_{2q'}^{2p} \ (2p)_p \ \left(\frac{\text{osc}^2(g)}{2N}\right)^p + \sum_{p=1}^{q'-1} C_{2q'}^{2p+1} \ \frac{(2p+1)_{p+1}}{\sqrt{p+1/2}} \left(\frac{\text{osc}^2(g)}{2N}\right)^{p+1/2}$$

In the above display we have used the notation $\text{osc}(g) = \sup_{i\geq 1} \text{osc}(g_i)$. Since we have the estimates

$$\begin{aligned} C_{2q'}^{2p} \ (2p)_p &= \frac{1}{p!} \frac{(2q')!}{(2q'-2p)!} = \frac{(2q')_{2p}}{p!} \leq \frac{(2q')^{2p}}{p!} = \frac{q^{2p}}{p!} \\ C_{2q'}^{2p+1} \ (2p+1)_{p+1} &= \frac{1}{p!} \frac{(2q')!}{(2q'-(2p+1))!} = \frac{(2q')_{2p+1}}{p!} \leq \frac{q^{2p+1}}{p!} \end{aligned}$$

this also yields that

$$\begin{aligned} |\mathbb{E}(\prod_{i=1}^{2q'} m(X)(g_i)) - 1| &\leq \sum_{p=1}^{q'} \frac{1}{p!} \left(\frac{q^2}{2N} \text{osc}^2(g)\right)^p + \sum_{p=1}^{q'-1} \frac{1}{p!} \left(\frac{q^2}{2N} \text{osc}^2(g)\right)^{p+1/2} \\ &\leq (1 + \text{osc}(g) \ q/\sqrt{2N}) \sum_{p=1}^{q/2} \frac{1}{p!} \left(\frac{q^2}{2N} \text{osc}^2(g)\right)^p \end{aligned}$$

Recalling that for any $n \geq 0$ and $\epsilon \geq 0$ we have

$$\sum_{p=1}^{n} \frac{\epsilon^p}{p!} \ \leq \epsilon \sum_{p=0}^{n-1} \frac{\epsilon^p}{p!} \ \leq \epsilon \ e^{\epsilon}$$

we arrive at

$$|\mathbb{E}(\prod_{i=1}^{q} m(X)(g_i)) - 1| \ \leq \ \frac{q^2}{2N} \ e_g(\frac{q^2}{2N})$$

with $e_g(u) = \text{osc}^2(g)(1 + \text{osc}(g) \ \sqrt{u}) \exp(\text{osc}^2(g) \ u)$. The proof for odd integers $q = 2q' + 1$ is derived in a completely analogous fashion. This ends the proof of (16) when $n = 1$. Next we prove (16) for even integers $n = 2n'$, $n' \in \mathbb{N}$. We use the decomposition

$$\begin{aligned} \mathbb{E}( \ [m(X)^{\otimes q}(g^{(q)}) - 1]^{2n'} \ ) &= \sum_{p=0}^{2n'} C_{2n'}^{p} \ (-1)^p \ \mathbb{E}([m(X)^{\otimes q}(g^{(q)})]^p) \\ &= I_1 + I_2 + I_3 \end{aligned}$$

with

$$\begin{aligned} I_1 &= \sum_{p=0}^{n'} C_{2n'}^{2p} \ [\mathbb{E}([m(X)^{\otimes q}(g^{(q)})]^{2p} - 1] \\ I_2 &= -\sum_{p=0}^{n'-1} C_{2n'}^{2p+1} \ [\mathbb{E}([m(X)^{\otimes q}(g^{(q)})]^{2p+1} - 1] \\ I_3 &= \sum_{p=0}^{n'} C_{2n'}^{2p} - \sum_{p=0}^{n'-1} C_{2n'}^{2p+1} = 0 \end{aligned}$$

Next we observe that for any $n \geq 1$ we have

$$\mathbb{E}([m(X)^{\otimes q}(g^{(q)})]^n) = \mathbb{E}([m(X)^{\otimes (q,n)}(g^{(q,n)})]$$

with

$$m(X)^{\otimes (q,n)} = \underbrace{m(X)^{\otimes q} \otimes \ldots \otimes m(X)^{\otimes q}}_{n \text{ times}} \quad \text{and} \quad g^{(q,n)} = \underbrace{g^{(q)} \otimes \ldots \otimes g^{(q)}}_{n \text{ times}}$$

From previous considerations we find that

$$\begin{aligned} |I_1| &= \sum_{p=1}^{n'} C_{2n'}^{2p} \, |\mathbb{E}([m(X)^{\otimes (q,2p)}(g^{(q,2p)})] - 1| \\ &\leq \sum_{p=1}^{n'} C_{2n'}^{2p} \frac{(2pq)^2}{2N} \; e_g\left(\frac{(2qp)^2}{2N}\right) \leq \frac{(nq)^2}{2N} \; e_g\left(\frac{(nq)^2}{2N}\right) \sum_{p=1}^{n'} C_{2n'}^{2p} \end{aligned}$$

Using similar arguments we find that

$$|I_2| \leq \frac{(nq)^2}{2N} \; e_g\left(\frac{(nq)^2}{2N}\right) \sum_{p=0}^{n'-1} C_{2n'}^{2p+1}$$

from which we conclude that

$$|\mathbb{E}( \; [m(X)^{\otimes q}(g^{(q)}) - 1]^n \; )| \leq 2^{n-1} \; \frac{(nq)^2}{N} \; e_g\left(\frac{(nq)^2}{2N}\right)$$

The proof of this estimate for odd integers $n = 2n' + 1$ follows the same arguments. This completes the proof of the lemma. ■

## 4.3 Combinatorial techniques for empirical measures

Throughout this section $(E, \mathcal{E})$ denotes an arbitrary measurable space. In the further development of this section we fix the integer $N \geq 1$ and for any $x = (x^1, \ldots, x^N) \in E^N$ we slight abuse notations and we set

$$m(x) = \frac{1}{N} \sum_{i=1}^{N} \delta_{x^i} \in \mathcal{P}(E)$$

the empirical measure associated to the $N$-uple $x$. For any $1 \leq q \leq N$ we introduce the empirical measures on $E^q$ defined by

$$\begin{aligned} m(x)^{\otimes q} &= \frac{1}{N^q} \sum_{\alpha \in \langle N \rangle^{\langle q \rangle}} \delta_{(x^{\alpha(1)}, \ldots, x^{\alpha(q)})} \\ m(x)^{\odot q} &= \frac{1}{(N)_q} \sum_{\alpha \in \langle q, N \rangle} \delta_{(x^{\alpha(1)}, \ldots, x^{\alpha(q)})} \end{aligned}$$

Note that each mapping $\alpha \in \langle N \rangle^{\langle q \rangle}$ induces an unique equivalence relation $\sim_\alpha$ on $\langle q \rangle$ defined for any $i, j \in \langle q \rangle$ by

$$i \sim_\alpha j \Longleftrightarrow \alpha(i) = \alpha(j)$$

The corresponding set of equivalence classes $\langle q \rangle_\alpha$ can alternatively be regarded as a partition $\pi_\alpha$ of the set $\langle q \rangle$. More precisely if $b(\pi_\alpha)$ stands for the cardinality of the set $\alpha(\langle q \rangle)$ then we have

$$\pi_\alpha = \{\pi_\alpha(1), \ldots, \pi_\alpha(b(\pi_\alpha))\} \quad \text{with} \quad \pi_\alpha(i) \neq \pi_\alpha(j) \quad \text{for any} \quad i \neq j$$

and

$$\langle q \rangle = \cup_{i=1}^{b(\pi_\alpha)} \pi_\alpha(i) \quad \text{with} \quad \pi_\alpha(i) = \{j \in \langle q \rangle \ : \ \alpha(j) = \alpha(i)\}$$

Inversely to each partition $\pi$ of the set $\langle q \rangle$ with $b(\pi)$ blocks we can associate in an unique way $(N)_q$ different mappings $\alpha \in \langle N \rangle^{\langle q \rangle}$. To be more precise let $\leq$ be the order relation on the subsets of $\langle q \rangle$ defined for any $A, B \subset \langle q \rangle$ by

$$A \leq B \Longleftrightarrow \inf\{i \ : \ i \in A\} \leq \inf\{i \ : \ i \in B\}$$

Notice that the $b(\pi_\alpha)$ blocks of partition $\pi$ of $\langle q \rangle$ can be written in the increasing order

$$\pi_1 \leq \pi_2 \leq \ldots \leq \pi_{b(\pi_\alpha)}$$

We associate to $\pi$ and to each one to one mapping $\beta \in \langle b(\pi), N \rangle$ the mapping $\alpha_\beta^\pi \in \langle N \rangle^{\langle q \rangle}$ defined by

$$\alpha_\beta^\pi = \sum_{i=1}^{b(\pi)} \beta(i) 1_{\pi_i}$$

From these one to one associations we find the decomposition

$$\langle N \rangle^{\langle q \rangle} = \cup_{p=1}^{q} \cup_{\pi : b(\pi) = p} \{\alpha_\beta^\pi \ : \ \beta \in \langle p, N \rangle\}$$

In these notations, for any $x \in E^N$ and any numerical function $f$ on $E^q$ we have that

$$\begin{aligned} m(x)^{\otimes q}(f) &= \frac{1}{N^q} \sum_{p=1}^{q} \sum_{\pi : b(\pi) = p} \sum_{\beta \in \langle p, N \rangle} f(x^{\alpha_\beta^\pi(1)}, \ldots, x^{\alpha_\beta^\pi(q)}) \\ &= \frac{1}{N^q} \sum_{p=1}^{q} \sum_{\pi : b(\pi) = p} \sum_{\beta \in \langle p, N \rangle} C_\pi^p(f)(x^{\beta(1)}, \ldots, x^{\beta(p)}) \end{aligned}$$

with the Markov kernel $C_\pi^p$ from $E^p$ into $E^q$ defined by

$$C_\pi^p(f)(x^1, \ldots, x^p) = f(\sum_{i=1}^{p} x^i 1_{\pi_i}(1), \ldots, \sum_{i=1}^{p} x^i 1_{\pi_i}(p))$$

It is now convenient to observe that for any $p \leq q$ we have

$$\frac{1}{(N)_p} \sum_{\beta \in \langle p, N \rangle} C_\pi^p(f)(x^{\beta(1)}, \ldots, x^{\beta(p)})$$

$$= \frac{1}{(N)_q} \sum_{\beta \in \langle q, N \rangle} C_{q,\pi}^p(f)(x^{\beta(1)}, \ldots, x^{\beta(q)})$$

with the extended Markov kernel $C^p_{q,\pi}$ from $E^q$ into $E^q$ defined by

$$C^p_{q,\pi}(f)(x^1,\ldots,x^q) = C^p_\pi(f)(x^1,\ldots,x^p)$$

From previous considerations we arrive at

$$m(x)^{\otimes q} = \frac{1}{N^q}\sum_{p=1}^{q} (N)_p\ S(p,q)\ m(x)^{\odot q} C^p_q$$

with the Markov transitions $C^p_q$, $p \leq q$, on $E^q$ defined by the formula

$$C^p_q = \frac{1}{(N)_p}\sum_{\beta\in\langle q,N\rangle} \frac{1}{S(p,q)}\sum_{\pi:b(\pi)=p} C^p_{q,\pi}$$

In the above displayed formulae $S(p,q)$ stands for the Stirling number of the second kind corresponding to the number of partitions of $q$ elements in $p$ blocks. Using the fact that $S(q,q)=1$ and $C^q_{q,\pi}=Id$ we prove easily the following result.

**Proposition 4.1** *For any $x \in E^N$ and $1 \leq q \leq N$ we have*

$$m(x)^{\otimes q} = m(x)^{\odot q} R^{(q)}_N \quad \text{with} \quad R^{(q)}_N = \frac{(N)_q}{N^q}\ Id + (1-\frac{(N)_q}{N^q})\ \tilde{R}^{(q)}_N$$

*and the Markov kernel $\tilde{R}^{(q)}_N$ on $E^q$ defined by*

$$\tilde{R}^{(q)}_N = \frac{1}{N^q-(N)_q}\sum_{p=1}^{q-1} (N)_p\ S(p,q)\ C^p_q$$

One easy consequence of this formula is that

$$\begin{aligned}
\|m(x)^{\otimes q} - m(x)^{\odot q}\|_{tv} &= \left(1-\frac{(N)_q}{N^q}\right)\|m(x)^{\odot q}(\tilde{R}^{(q)}_N - Id)\|_{tv}\\
&\leq \left(1-\frac{(N)_q}{N^q}\right)\\
&\leq 1-\left(1-\frac{q-1}{N}\right)^{q-1} \leq \frac{(q-1)^2}{N}
\end{aligned} \tag{30}$$

We end this article with a more probabilistic connection between $m(x)^{\otimes q}$ and $m(x)^{\odot q}$. We first observe that for any $q \geq 1$ and any $f$ on $E^{q+1}$

$$\begin{aligned}
&(m(x)\otimes m(x)^{\odot q})(f)\\
&= \frac{1}{N(N)_q}\sum_{i=1}^{N}\sum_{\alpha\in\langle q,N\rangle} f(x^i, x^{\alpha(1)}, x^{\alpha(2)},\ldots,x^{\alpha(q)})\\
&= \frac{1}{N(N)_q}\sum_{\alpha\in\langle q+1,N\rangle} f(x^{\alpha(1)}, x^{\alpha(2)},\ldots,x^{\alpha(q+1)})\\
&\qquad +\frac{1}{N(N)_q}\sum_{\alpha\in\langle q,N\rangle}\sum_{i=1}^{q} f(x^{\alpha(i)}, x^{\alpha(1)}, x^{\alpha(2)},\ldots,x^{\alpha(q)})\\
&= \left(1-\frac{q}{N}\right)\ m(x)^{\odot(q+1)}(f) + \frac{q}{N}\ m(x)^{\odot(q+1)}(\tilde{r}^{(q+1)}(f))
\end{aligned}$$

with the Markov transition $\tilde{r}_{q+1}$ on $E^{q+1}$ defined by

$$\tilde{r}^{(q+1)}(f)(x^0, x^1, \ldots, x^q) = \frac{1}{q}\sum_{i=1}^{q} f(x^i, x^1, \ldots, x^q)$$

This readily yields that

$$m(x)\otimes m(x)^{\odot q} = m(x)^{\odot (q+1)} r_N^{(q+1)} \quad \text{with} \quad r_N^{(q+1)} \left(1-\frac{q}{N}\right) Id + \frac{q}{N}\,\tilde{r}^{(q+1)}$$

The probabilistic interpretation of $r_N^{(q+1)}$ is quite elementary. Starting from a given configuration $(x^0, x^1, \ldots, x^q) \in E^{q+1}$ the Markov transition consists to keep this $(q+1)$-uple with a probability $\left(1-\frac{q}{N}\right)$ and otherwise we replace the first component $x^0$ by choosing randomly and uniformly one of the another components $x^1, \ldots, x^q$. To develop an inductive construction we associate to a given transition $r$ on some product space $E^q$ a transition $\mathrm{Ext}(r)$ on some product space $E^{q+1}$ by setting

$$\mathrm{Ext}(r)((x^0, x^1, \ldots, x^q), d(y^0, x^1, \ldots, y^q))$$

$$= \delta_{x^0}(dy^0)\; r((x^1, \ldots, x^q), d(x^1, \ldots, y^q))$$

In these somehow abusive notations we have for instance

$$\begin{aligned} m(x)^{\otimes 2} &= m(x)^{\odot 2} r_N^{(2)} \\ m(x)^{\otimes 3} &= m(x) \otimes m(x)^{\otimes 2} \\ &= m(x) \otimes (m(x)^{\odot 2} r_N^{(2)}) \\ &= (m(x) \otimes m(x)^{\odot 2})\mathrm{Ext}(r_N^{(2)}) = m(x)^{\odot 3} r_N^{(3)} \mathrm{Ext}(r_N^{(2)}) \end{aligned}$$

More generally if we define using backward induction

$$\mathcal{R}_N^{(q+1)} = r_N^{(q)} \mathrm{Ext}(\mathcal{R}_N^{(q)}) \quad \text{with} \quad \mathcal{R}_N^{(2)} = r_N^{(2)}$$

then we conclude that

$$m(x)^{\otimes q} = m(x)^{\odot q} \mathcal{R}_N^{(q)}$$

To describe more precisely the Markov transition $\mathcal{R}_N^{(q)}$ we introduce a sequence $\epsilon^{(q)} = (\epsilon_1^{(q)}, \ldots, \epsilon_q^{(q)})$ of $q$ independent and $\{0,1\}$-valued random variables with respective distributions

$$\mathbb{P}(\epsilon_i^{(q)} = 0) = 1 - \mathbb{P}(\epsilon_i^{(q)} = 1) = \frac{q-i}{N}$$

Notice that for $i = q$ we have $\epsilon_q^{(q)} = 1$. We also associate to a given configuration $(x^1, \ldots, x^q) \in E^q$ a collection of independent (and independent of $\epsilon^{(q)}$) random variables $(\tilde{x}^{(q,1)}, \ldots, \tilde{x}^{(q,q)})$ with respective distributions

$$\mathbb{P}(\tilde{x}^{(q,i)} \in dy) = \frac{1}{q-i} \sum_{j=i+1}^{q} \delta_{x^j}$$

with the convention $\tilde{x}^{(q,q)} = x^q$. From the inductive construction of $\mathcal{R}_N^{(q)}$ we observe that the $E^q$-valued random variable

$$\widehat{x}^{(q)} = (\widehat{x}^{(q,1)}, \ldots, \widehat{x}^{(q,q)}) \quad \text{with} \quad \widehat{x}^{(q,i)} = \epsilon_i^{(q)} x^i + (1-\epsilon_i^{(q)})\tilde{x}^{(q,i)}$$

is distributed according to $\mathcal{R}_N^{(q)}((x^1, \ldots, x^q), .)$. ■

# References


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