THE ROOTS OF POLYNOMIALS WITH PRIME COEFFICIENTS

by

Anirban DasGupta
Purdue University

Technical Report #06-06

Department of Statistics
Purdue University

August 2003

The Roots of Polynomials with Prime Coefficients

Anirban DasGupta Purdue University

August 18, 2003

ABSTRACT

We consider the polynomials defined as $Q_n(x) = \sum_{k=1}^{n+1} p_k x^{k-1}$, where p_k is the kth prime number. We call these 'Prime Polynomials'. The roots of Q_n are investigated, and certain properties commented on. The plots of the roots, considered as points in the plane, show visually appealing phenomena.

1 Polynomials with Prime Coefficients

There are only a few technical statements in this note. It is mostly about computations and pictures. But the computations suggest interesting theorems, perhaps even deep, and the pictures are certainly visually appealing. It seemed someone would be interested in the computations as well as the pictures. We are not sure if the computations and the resulting pictures are already available in the literature. We cannot rule that out at the time of writing this note.

This note is about the roots of the sequence of polynomials $Q_n(x) = \sum_{k=1}^{n+1} p_k x^{k-1}$, where p_k is the kth prime number. We denote the roots of $Q_n(x)$ by $r_{1,n}, r_{2,n}, \ldots, r_{n,n}$ and let $r_{j,n} = x_{j,n} + iy_{j,n}$, with i denoting as usual $\sqrt{-1}$.

1.1 A Few Technical Statements

Our main interest is in the values and properties of these roots $\{r_{j,n}\}_{j=1}^{n}$. We make a few technical statements.

1) If a rational root $\frac{p}{q}$ exists for some member $Q_n(x)$ of our sequence of polynomials, then from general theory about rational roots of polynomials, p must divide p_1 and q must divide p_{n+1}. This can happen only when $\frac{p}{q} = -\frac{2}{p_{n+1}}$. An instance of this is $-\frac{2}{3}$, which is the unique root of $Q_1(x) = 2 + 3x$. It follows, from this, that with the exception of $Q_1(x)$, all real roots of $Q_n(x), n \leq 100$, are irrational. The computations presented here suggest that there are in fact almost no real roots at all, rational or irrational.

2) Second, for any $Q_n(x)$, the sum of the real parts of its roots, i.e.,

$\sum_{j=1}^{n} x_{j,n}$ is equal to the ratio $-\frac{p_n}{p_{n+1}}$. This is because, the complex roots occur in pairs, and so $\sum_{j=1}^{n} y_{j,n}$ is always zero. Thus, the sum of the real parts is the same as $\sum_{j=1}^{n} r_{j,n}$, which from general theory about roots of polynomials, equals $-\frac{p_n}{p_{n+1}}$. Denoting the $(n+1)$th prime gap $p_{n+1} - p_n$ by d_{n+1}, this equals $-1 + \frac{d_{n+1}}{p_{n+1}}$. From the Prime number theorem, one has that $\frac{d_n}{p_n} \to 0$. Thus, for large n, the sum of the real parts of the roots is close to -1, and of course the sum of the imaginary parts is always zero.

3) Next, if the roots are treated as points $(x_{j,n}, y_{j,n})$ in the plane, then barring the very first one $Q_1(x)$, there seems to be, always, a fairly large circle centered at the origin that is completely free of any roots. We can make the following technically correct statement. For any given n, a circle with radius smaller than $min\{\frac{p_1}{p_2}, \frac{p_2}{p_3}, \ldots, \frac{p_n}{p_{n+1}}\}$ and centered at the origin is free of any roots of $Q_n(x)$ (this follows from Pólya and Szegö(1998, pp 107)). For the first 500 polynomials, this implies that the circle of any radius $c < .6$ cannot have any roots. The computations suggest that a circle of radius about .8 is free of roots for large n.

4) Another question of interest is the frequency of primes produced by the polynomials $Q_n(x)$ at integer arguments. There are infinitely many for the very first one $Q_1(x) = 2 + 3x$, as a consequence of the 1837 result of Dirichlet; see pp 148 in Ribenboim(1991). For $n > 1$, the only statement we are able to make is that the asymptotic density of primes produced by Q_n at integer arguments is zero. That is, fix $N > 1$; then, $\lim_{N\to\infty}$(Number of prime values of $Q_n(x) \leq N)/N = 0$. This follows from a more general result stated in pp 183 in Ribenboim(1991). The asymptotic zero density is true for the case of $n = 1$ also. The exact number of prime values among the first 10,000 values of Q_n are later reported for $n \leq 50$.

1.2 Distribution of the Roots : Spiders and Comets

First, we present six plots of the combined set of roots, treated as points in the plane, of the first 10,25,35,50,60 and 75 polynomials of the $\{Q_n(x)\}$ sequence. There are only five real roots, obviously all negative, of the first 10 polynomials. There is a huge empty space. When we look at the roots of the first 25 polynomials, there are about a dozen real roots. In fact, the exact number is 13. The empty space is as pronounced as the case for the first 10 polynomials. But some structure has begun to emerge. In particular, the two arcs approaching the edge of the positive real axis have closed up considerably compared to the previous plot. Also, the pattern of the points in the second quadrant is very different from the pattern in the first quadrant. We also see ten roots that are almost purely imaginary; this number was four in the first plot. Something really interesting happens in the next plot for the roots of the first 35 polynomials. All on a sudden, *two spiders* have appeared, symmetrically about the real axis, in the second and the third quadrant. The number of real roots has increased too; now the exact number is 18. The two arcs approaching the positive real axis have gotten much darker. The large empty space still remains. Most interestingly, when we look at the next plot for the roots of the first 50 polynomials, two *new spiders* have appeared in the first and the third quadrant. There is considerable darkening around the edges, almost everywhere. The whole picture now looks very structured to the eye, and appealing. And when we look at the roots of the first 75 polynomials, we see a new pair of spiders, now in the second and the third quadrant, just slightly hiding, at vertical heights of about $\pm.75$. The two arcs approaching the positive real axis now almost look like comets with tails, and the middle empty space remains. In fact, the empty space remains more or less unchanged over the six plots. There may be a theorem to this regard stronger than what we stated in 3) above for large n. The roots of just the 100th and the 250th degree polynomial are shown in the last two plots. The roots are near the boundary of the unit circle, with a few stray points here and there . This is like the bimodality phenomenon in the distribution of roots of random algebraic polynomials with standard normal coefficients, with the concentration of the real roots near ± 1; see Fig. 1.1 in Bharucha-Reid and Sambandham(1986) . However, the distribution of the roots here is less spherical than for normal random polynomials; compare Fig. 7.11 in Bharucha-Reid and Sambandham(1986) to the six plots here.

The lengths of the roots(with the roots being considered as points in the plane)are investigated in the next two plots. The smoothed histograms show a sharp peak near 1 when only the first 25 polynomials are considered. But if the first 45 are looked at, the peak looks more like a *Gibbs phenomenon*, with the histogram being more or less a *uniform distribution* in the range $[.8, 1]$. There may be a theorem here too, but we are not able to comment on how difficult it would be to prove.

Something rather interesting seems to be happening as regards the number of real roots of the polynomials $Q_n(x)$. Upto $n = 100$, the even degree ones have no real roots, and the odd degree ones have *exactly one* real root. If this were to be true in general, it would be a remarkable theorem! Q_5, Q_{10}, Q_{25} and Q_{50} are plotted in the domain $[-1, 0]$ in the next four plots. They are not monotone, but do not cross the x-axis twice.

The most interesting aspects of the plots are the spiders, and the comets, and the concentration of the roots near the boundary of the unit circle, with a huge intermediate empty space. *Unless* theorems to these regards are already known, it seems to us it would be very interesting to establish some of these visual phenomena as theorems. We just do not know how hard it would be to do so.

1.3 Frequency of Prime Values

We computed the number of values among the first 10,000 values of the polynomials $Q_1, Q_2, \ldots, Q_{50}$ that are primes. A little reflection shows that all the values of the even degree ones are even. So we report here the number of prime values for only the odd degree ones. Generally speaking, the early polynomials appear to produce more primes; but there are some interesting exceptions. $Q_{11}, Q_{17}, Q_{31}, Q_{37}$, and Q_{49} produce primes at a visibly higher rate than their neighbors. The frequencies of prime values are reported below.

Table 1

Degree	Number of Primes Among First 10,000 Values
1	1633
3	459
5	277
7	188
9	154
11	249
13	142
15	39
17	84

Degree	Number of Primes Among First 10,000 Values
19	31
21	43
23	27
25	30
27	39
29	24
31	56
33	15
35	22
37	75
39	14
41	38
43	26
45	43
47	38
49	62

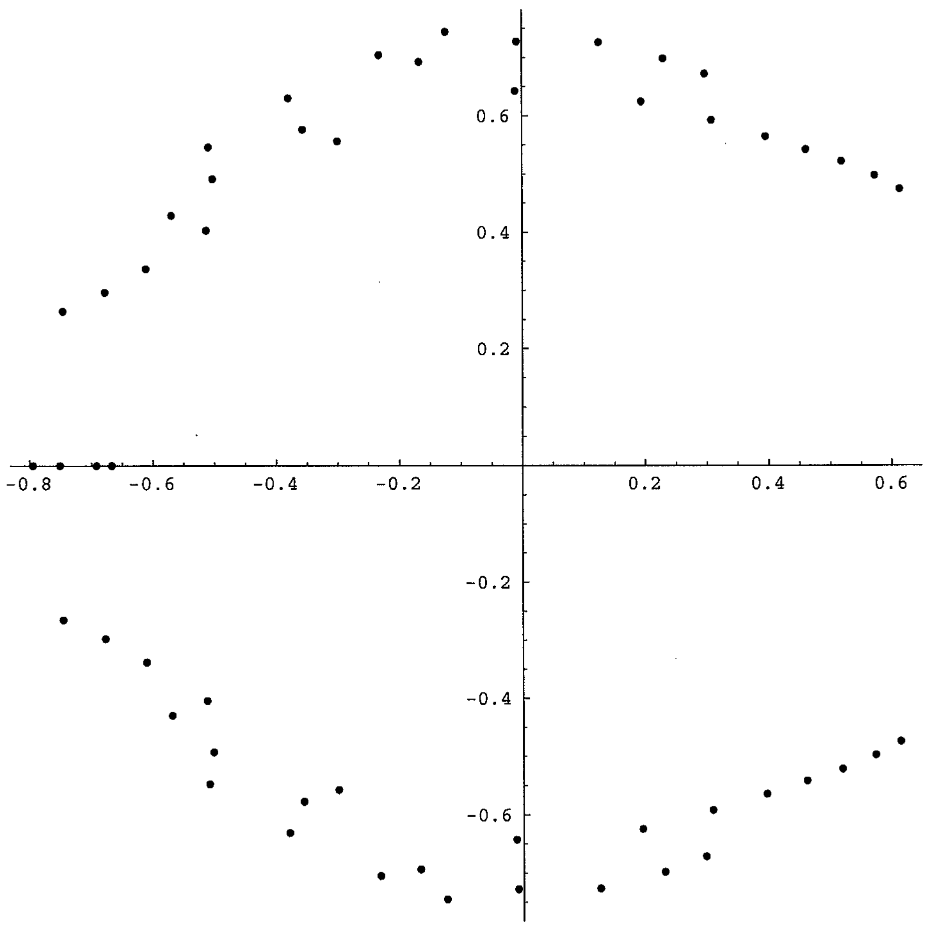
References

Bharucha-Reid,A.T. and Sambandham,M.(1986). *Random Polynomials, Academic Press, New York*

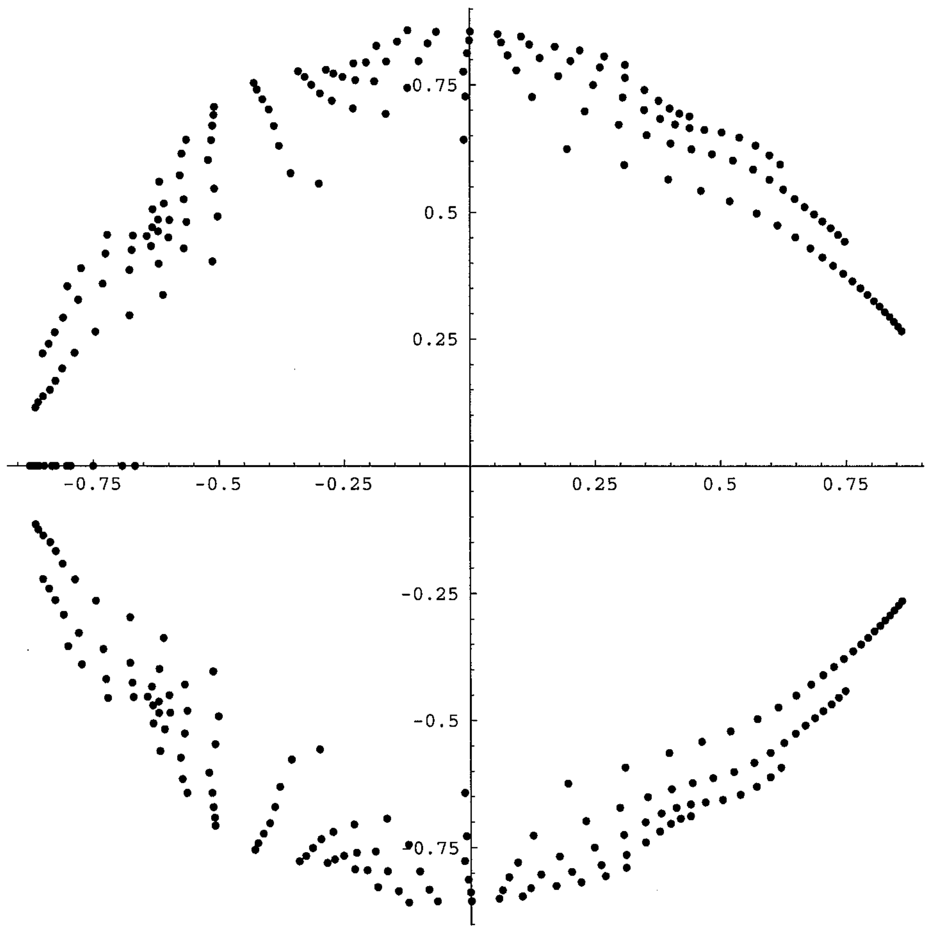
Pólya,George and Szegö,Gabor(1998). *Problems and Theorems in Analysis, I, Springer, Berlin*

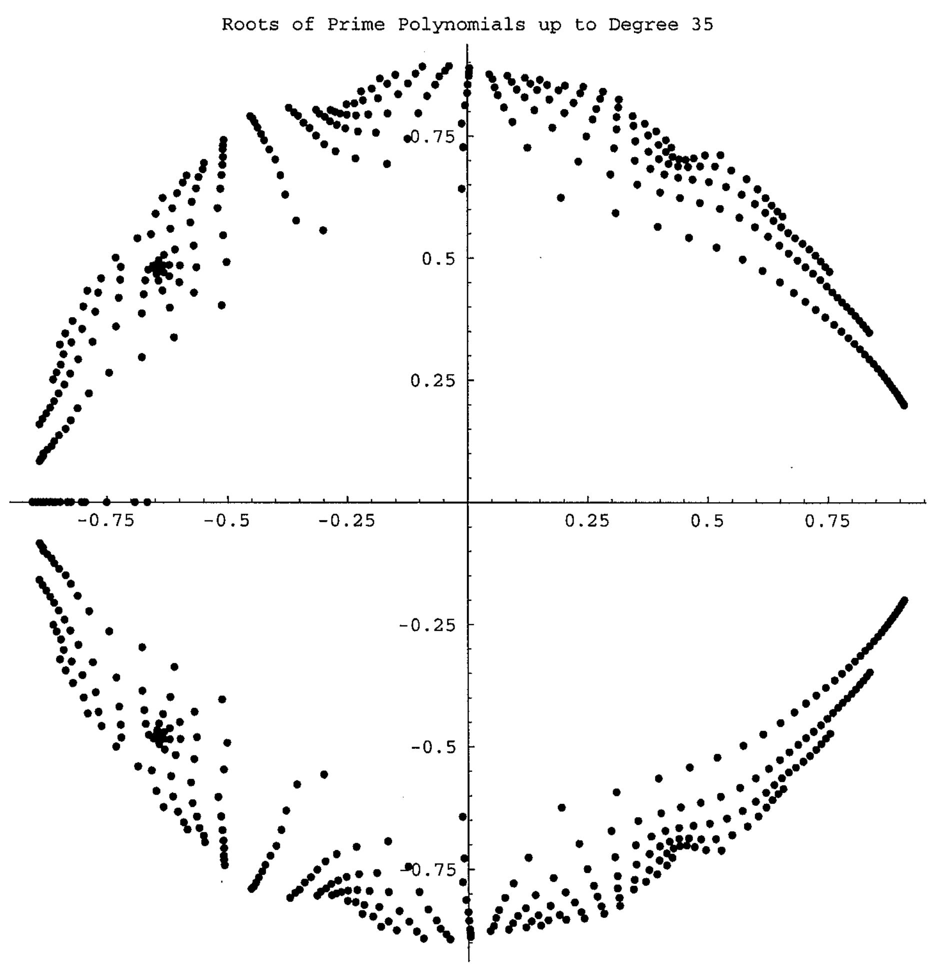
Ribenboim,P.(1991). *The Little Book of Big Primes, Springer-Verlag, New York*

Roots of Prime Polynomials up to Degree 10

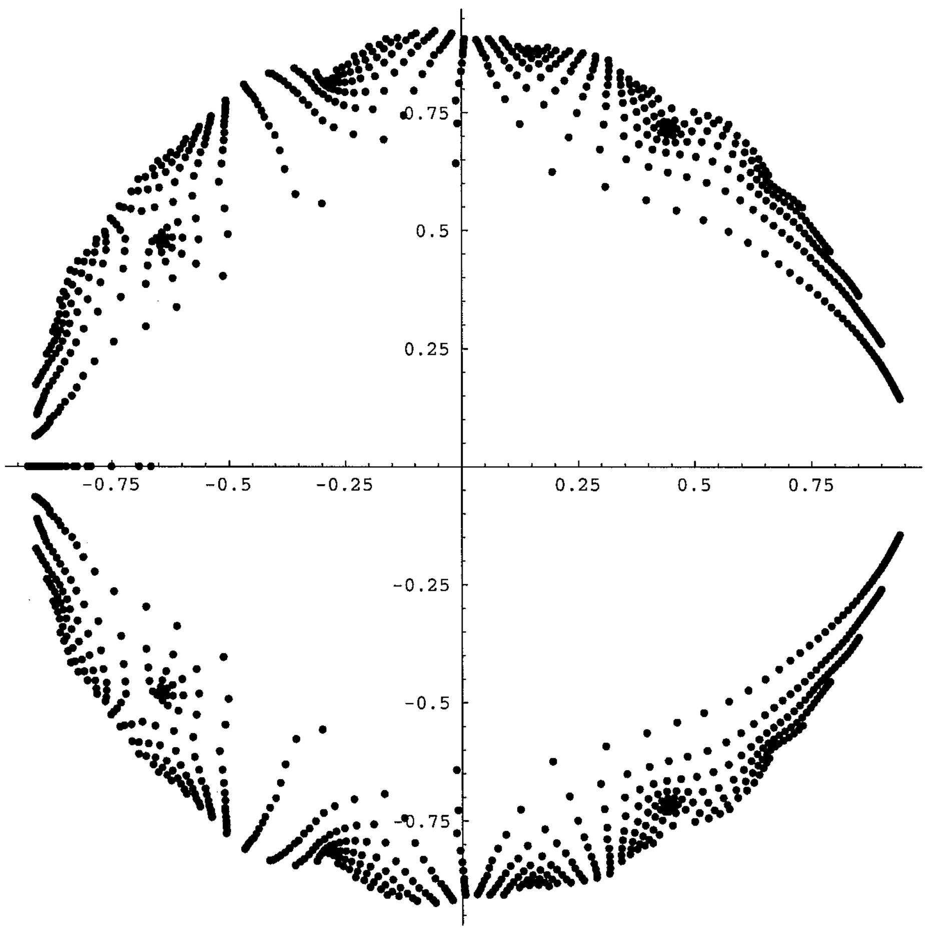


Roots of Prime Polynomials up to Degree 25

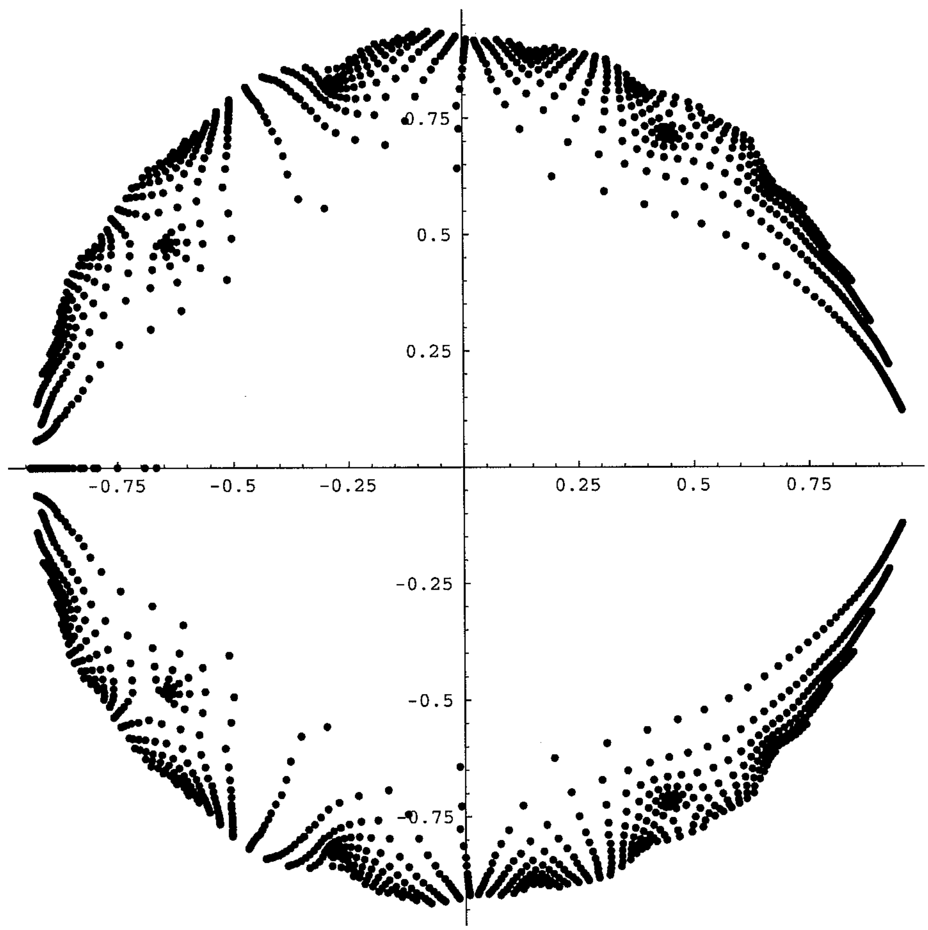



Roots of Prime Polynomials up to Degree 35
0.75
0.5
0.25
-0.75
-0.5
-0.25
0.25
0.5
0.75
-0.25
-0.5
-0.75

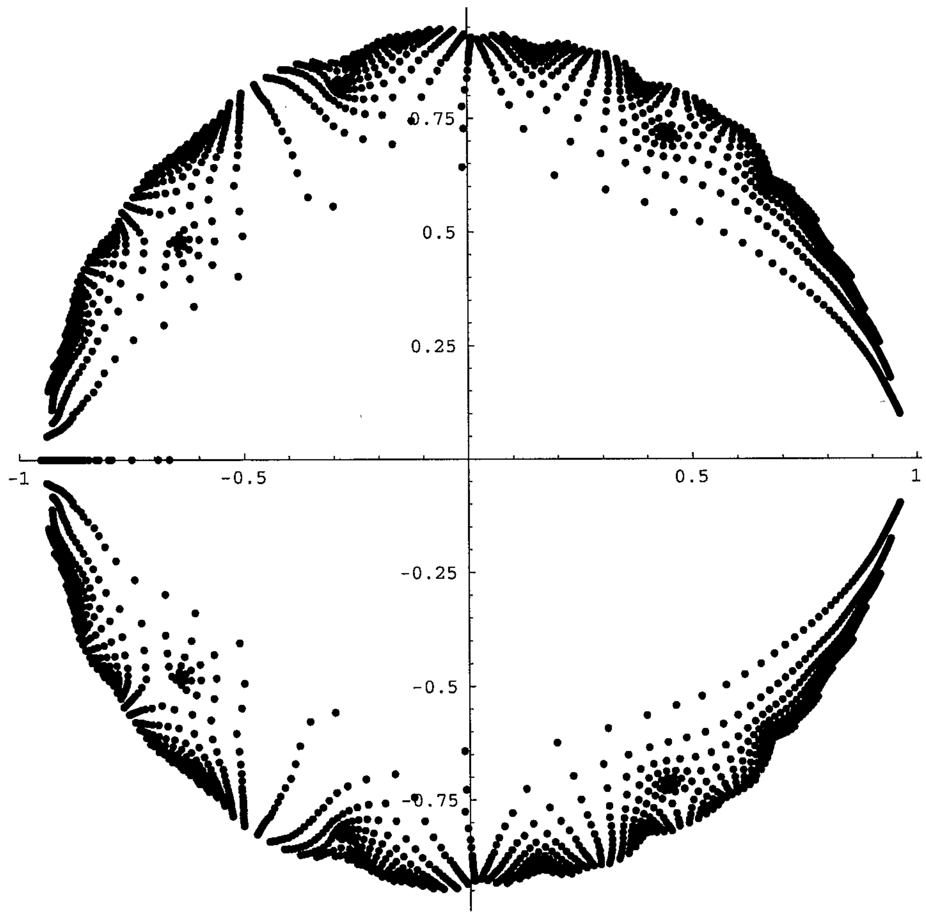
Roots of Prime Polynomials up to Degree 50



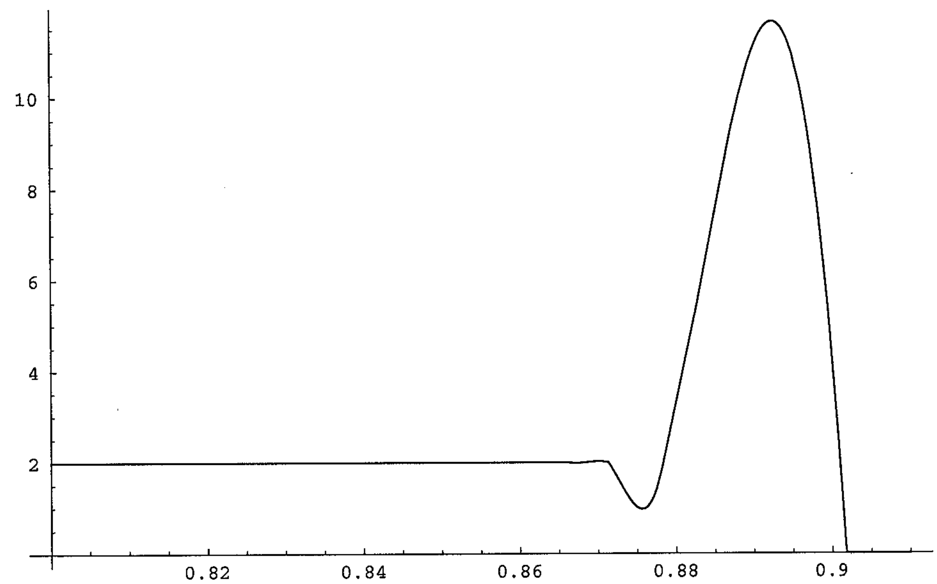
Roots of Prime Polynomials up to Degree 60

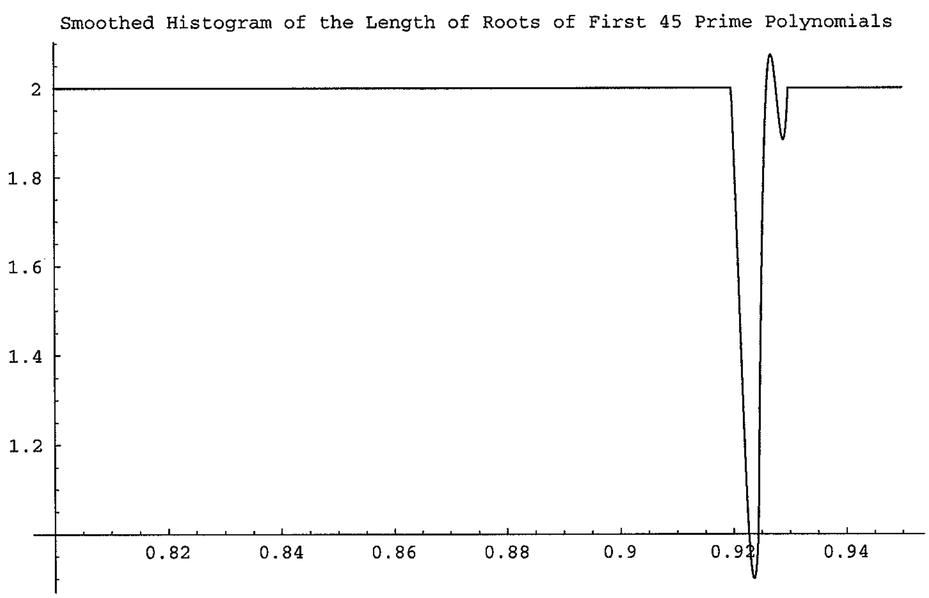


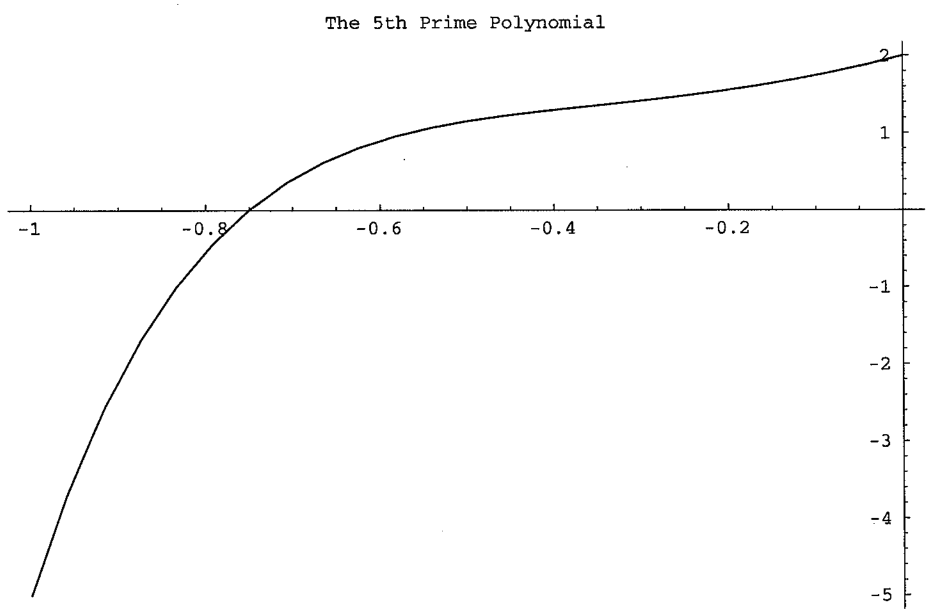
Roots of Prime Polynomials up to Degree 75



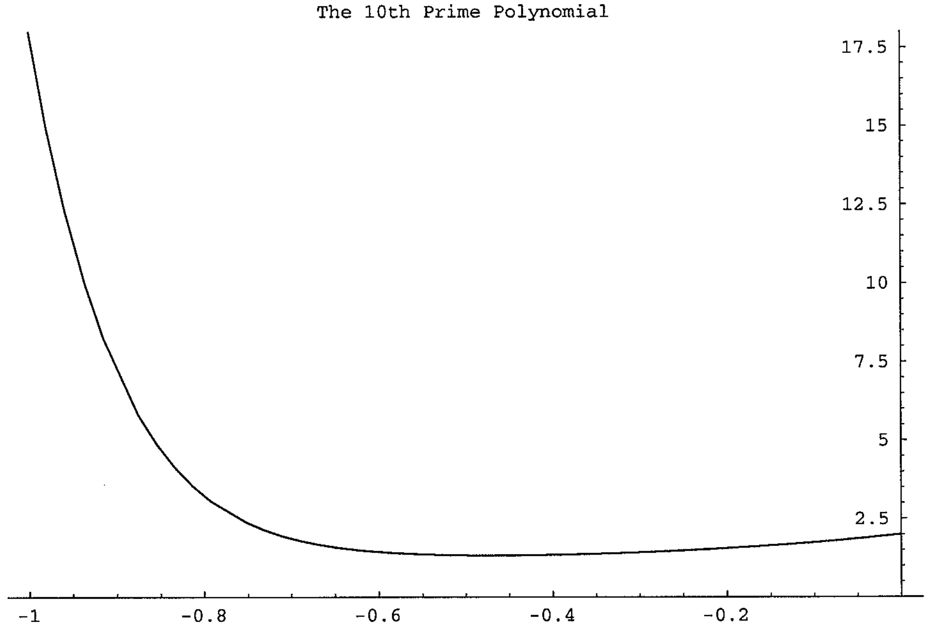
Smoothed Histogram of the Length of Roots of First 25 Prime Polynomials



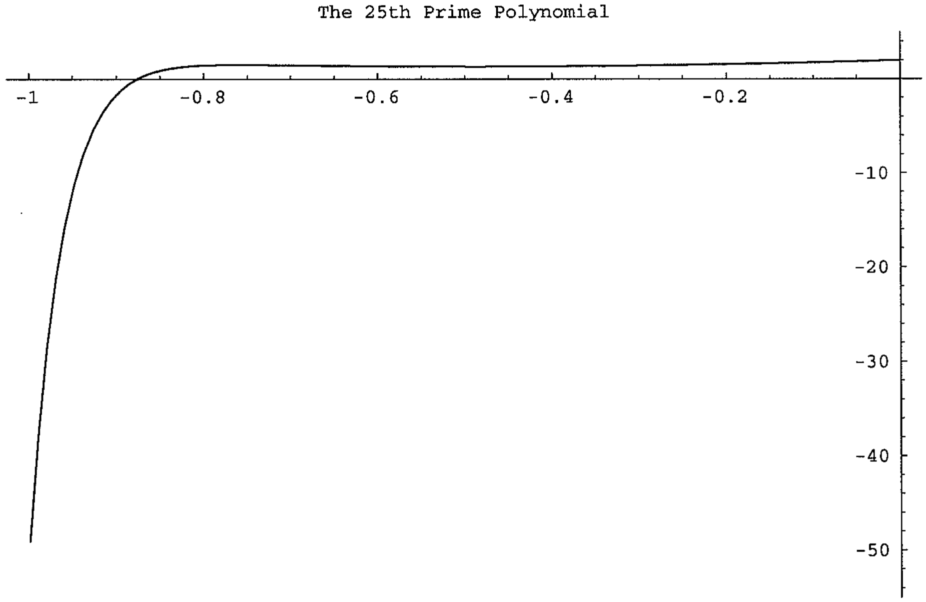

Smoothed Histogram of the Length of Roots of First 45 Prime Polynomials
2
1.8
1.6
1.4
1.2
0.82
0.84
0.86
0.88
0.9
0.92
0.94

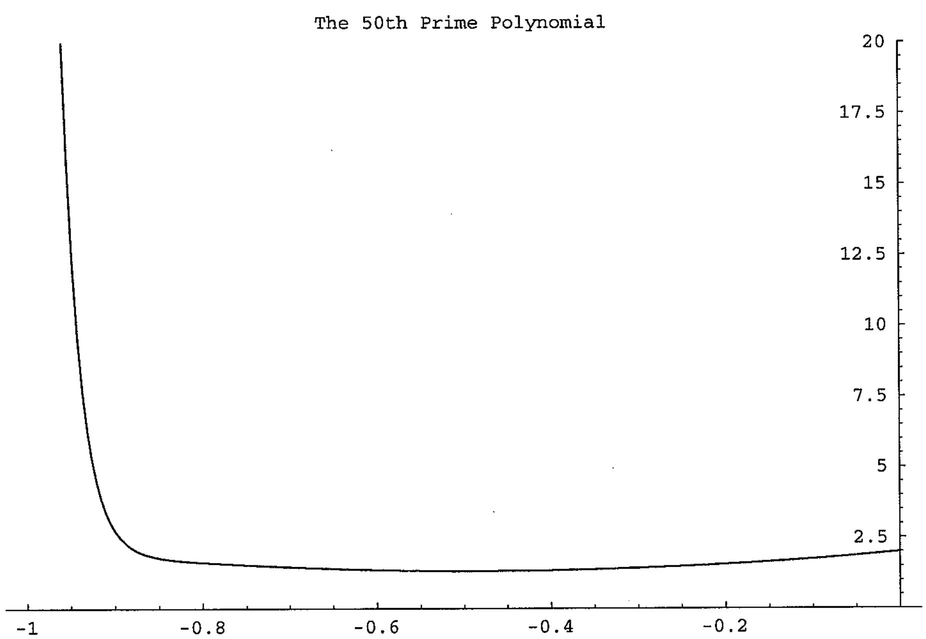

The 5th Prime Polynomial
2
1
-1
-2
-3
-4
-5
-1
-0.8
-0.6
-0.4
-0.2

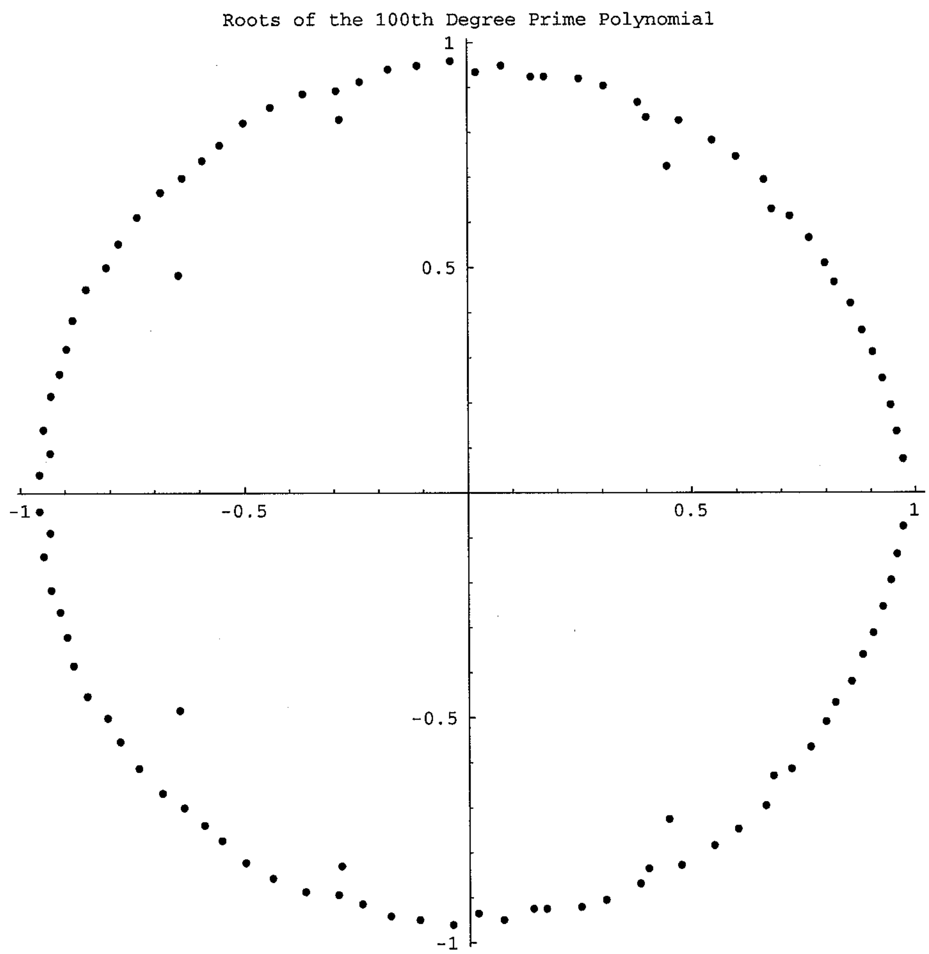
The 10th Prime Polynomial



The 25th Prime Polynomial




The 50th Prime Polynomial
20
17.5
15
12.5
10
7.5
5
2.5
-1
-0.8
-0.6
-0.4
-0.2


Roots of the 100th Degree Prime Polynomial
1
0.5
-1
-0.5
0.5
1
-0.5
-1

Roots of the 250th Degree Prime Polynomial