Nonparametric Estimation of Time-changed Lévy Models Under High-frequency Data

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Abstract: Let $\{Z_t\}_{t\geq 0}$ be a Lévy process with Lévy measure ν and let $\tau(t) = \int_0^t r(u)du$ where $\{r(t)\}_{t\geq 0}$ is a positive ergodic diffusion independent from Z. Based upon discrete observations of the time-changed Lévy process $X_t := Z_{\tau_t}$ during a time interval $[0,T]$, we study the asymptotic properties of some estimators of the parameters $\beta(\varphi) := \int \varphi(x)\nu(dx)$, which in turn are well-known to be the building blocks of several nonparametric methods such as sieve-based estimation and kernel estimation. Under uniformly boundedness of the second moments of r and conditions on φ necessary for the standard short-term ergodic property $\lim_{t\to 0} \mathbb{E}\,\varphi(Z_t)/t = \beta(\varphi)$ to hold, consistency and asymptotic normality of the proposed estimators are ensured when the time horizon T increases in such as way that the sampling frequency is high enough relative to T.

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1. Introduction

Historically, Brownian motion has been the model of choice to describe the evolution of a random measurement which value is the result of a large-number of small shots occurring through time with high-frequency. This is indeed the situation with stock prices which value is the result of a high number of agents posting bid and ask prices almost at all times. However, processes exhibiting infinitely many jumps in any finite time horizon $[0,T]$ are arguably better approximations to such high-activity stochastic processes. A Lévy process is a natural extension of Brownian motion which preserves the tractable statistical properties of its increments, while relaxes the continuity of paths. The previous

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considerations motivated an explosion of financial-price models driven by Lévy processes with infinite jump activity. The simplest of these models postulates that the price of a commodity (say a stock) at time t is determined by

$$S_t := S_0 e^{X_t}, \tag{1.1}$$

where $X := \{X_t\}_{t\geq 0}$ is a Lévy process. Even this simple extension of the classical Black-Scholes model, in which X is simply a Brownian motion with drift, is able to accommodate several features commonly observed in the returns of financial assets such as heavy tails, high-kurtosis, and asymmetry. Among the better known models are the *variance Gamma model* of [6], the *CGMY model* of [4], and the *generalized hyperbolic motion* of [1, 10] (see also [2, 9]).

Even though the geometric Lévy paradigm (1.1) incorporates several desirable stylized features, the model has several shortcomings, especially to account for the so-called *volatility clustering* and *leverage phenomena* exhibited by real financial data. Roughly speaking the former effect refers to the fact that there are periods of high variability in the market, followed by periods of low variability. People usually say that "high-volatility" events tend to cluster in time. Leverage refers to the empirical observation that returns seem to be negatively correlated with volatility. These two effects cannot be captured by the model (1.1). To explain why this is the case and motivate the use of random clocks, let us study the *realized variation* up to time t of the log returns in the periods $[t_0, t_1], \dots, [t_{n-1}, t_n]$:

$$V_\pi(t) := \sum_{i:t_i\leq t} \log^2\left(S_{t_i}/S_{t_{i-1}}\right), \tag{1.2}$$

where $\pi : t_0 = 0 < t_1 < \cdots < t_n := T < \infty$. When the mesh $\bar{\pi} := \max_i\{t_i - t_{i-1}\}$ of the partition is small, we can think of the increment $V_\pi(t) - V_\pi(s)$ as a measure of the volatility of the stock during the period $[s, t]$. Under the model (1.1),

$$V_\pi(t) = \sum_{i:t_i\leq t} (X_{t_i} - X_{t_{i-1}})^2,$$

which is well-know to converge, in probability, to the quadratic variation of the process,

$$[X]_t := \sigma^2 t + \sum_{s\leq t} \Delta X_s,$$

as $\bar{\pi} \to 0$. In that case, the realized variations in consecutive time periods of equal time length Δ, say $V_\pi(\Delta)$, $V_\pi(2\Delta) - V_\pi(\Delta)$, etc., will look like white noise (i.e. independent identically distributed random variables) and will not exhibit the volatility clustering phenomenon.

In recent years subordinated Lévy processes have been proposed to incorporate the intermittency and leverage phenomena (c.f. [5, 7]). Concretely, these models postulate that the asset price at time t is given by (1.1) with

$$X_t := Z_{\tau(t)}, \tag{1.3}$$

where Z is a Lévy process and $\{\tau(t)\}_{t\geq 0}$ stands for a nondecreasing absolutely continuous process. This approach leads to a *geometric time-changed Lévy model*:

$$S_t = S_0 e^{Z_{\tau(t)}},$$

where the process τ plays the role of a "business" clock which may reflect non-synchronous trading effects or a "cumulative measure of economic activity". To incorporate volatility clustering, random clocks $\{\tau(t)\}_{t\geq 0}$ of the form

$$\tau(t) := \int_0^t r(u)du, \tag{1.4}$$

with $\{r(t)\}_{t\geq 0}$ being a positive *mean-revering process*, are plausible choices. This crucial observation was first noticed by Carr et.al. [5], who specialized further their model to consider particular parametric models for Z (such as normal inverse Gaussian or variance gamma processes) and explicit positive ergodic diffusions for r such as the Cox-Ingersoll-Ross (CIR) process. Roughly speaking, the rate process r controls the volatility of the process; for instance, in time periods where r is high, the "business time" τ runs faster resulting in more frequent jump times. More formally, under the model (1.1) with X as in (1.3) and assuming for that τ is independent of the Lévy process Z, the realized variation (1.2) of the log returns converges to

$$\sigma^2\tau(t) + \sum_{s\leq\tau(t)} \Delta Z_s,$$

where σ is the variance of the Brownian component of Z. The observable volatility during a time period $[t,u]$ will be given by

$$\sigma^2\{\tau(u)-\tau(t)\} + \sum_{\tau(t)<s\leq\tau(u)} \Delta Z_s. \tag{1.5}$$

Thus, under (1.4) with a mean-reverting process $\{r(t)\}_{t\geq 0}$, there will be periods $[t,u]$ of high volatility (which correspond to periods where the process r takes on a high level) and periods $[t,u]$ of low volatility (which corresponds to periods where r takes on a low level).

Time-changed Lévy processes are one step further in the trend of increasingly complex models that are aimed at incorporating the so-called stylized features of asset prices. Considerably less effort has been devoted to analyze the potential departures from the presumed model. One recent approach to deal with the later issue is the adoption of general nonparametric models for the functional parameters of the underlying process, hence reducing the estimation bias resulting from assuming an inadequate parametric model. In the case of a Lévy model Z, this parameter could be the so-called *Lévy density* $s(\cdot)$, which dictates the jump dynamics of the process and is the main object of interest in the present paper. The value of s at a point x_0 determines how frequent jumps of size close to x_0 are to occur per unit time. Concretely, the function s is such that

$$\int_A s(x)dx = \frac{1}{t}\,\mathbb{E}\left[\sum_{s\leq t}\chi_A\left(\Delta Z_s\right)\right],$$

for any Borel set A and $t > 0$. Here, $\Delta Z_t \equiv Z_t - Z_{t-}$ denotes the magnitude of the jump of Z at time t, and $\chi_A(x) = 1$ if $x \in A$, and 0 otherwise. Thus,

$$\nu(A) := \int_A s(x)dx,$$

called the Lévy measure of the process, is the average number of jumps (per unit time) whose magnitudes fall in the set A. For instance, if $\nu((0,\infty)) = 0$, then Z will exhibit only jumps of negative size. In the context of financial applications, an empirical assessment of the possible sudden price shifts of the underlying assets is critical as these shifts play a key role in developing appropriate risk-management and investment strategies.

The challenge of devising nonparametric methods for the Lévy density s of Z lies in the fact that the jumps are latent (unobservable) variables since in practice only discrete observations of the process are available. It is natural to devise statistical methodologies based on high-frequency observations since this type of data will contain more relevant information about the jumps of the process and hence, about the Lévy density s. Such a high-frequency based statistical approach has played a central role in the recent literature on nonparametric estimation for Lévy processes (see e.g. Figueroa-López [11, 14], Woerner [24, 25], and Mancini [20, 21]). For instance, under discrete observations of a pure Lévy process X at times $\pi : 0 = t_0 < \cdots < t_n = T$, Woerner [24], and also independently Figueroa-López [11], propose the estimators

$$\hat{\beta}^{\pi}(\varphi) := \frac{1}{t_n}\sum_{k=1}^{n}\varphi\left(X_{t_k} - X_{t_{k-1}}\right), \tag{1.6}$$

as consistent estimators for the integral parameter

$$\beta(\varphi) := \int \varphi(x)s(x)dx, \tag{1.7}$$

where φ is a given "test function". We can think of the statistic (1.6) as the realized φ-variation of the process X per unit time based on the sampling observations $X_{t_0}, \ldots, X_{t_n}$. In [11], the proposed estimators were used to devise nonparametric estimators $\hat{s}$ for s via Grenander's *method of sieves*. The problem of model selection was analyzed further in [14, 15], where it was proved that sieve estimators $\widetilde{s}_T$ can match the rate of convergence of the minimax risk of estimators $\hat{s}$. Concretely, it turns out that

$$\limsup_{T\to\infty} \frac{\mathbb{E}\,\|s-\widetilde{s}_T\|^2}{\inf_{\hat{s}}\sup_{s\in\Theta}\mathbb{E}\,\|s-\hat{s}\|^2}<\infty,$$

where $[0,T]$ is the time horizon over which we observe the process X, Θ is certain class of smooth functions, and the infimum in the denominator is over all estimators $\hat{s}$ which are based on whole trajectory $\{X_t\}_{t\leq T}$. The optimal rate of the estimator $\widetilde{s}_T$ is attained by appropriately choosing the dimension of the sieve and the sampling frequency in function of T and the smoothness of the class of functions Θ. In [12], the sieve estimators of [14] were also used to built confidence intervals (CI) and confidence bands for the Lévy density s.

In this paper, we consider the problem of drawing statistical inferences for the model (1.3) when we have at hand high-frequency sampling observations of X. A recent treatment of the problem of predicting (estimating) the business clock process $\tau(t) := \int_0^t r(u)du$ is given in Woerner [25]. We concentrate here in estimating the Lévy density s of the Lévy process Z. A natural question is the following: how does the random time τ affect the statistical properties of the estimator (1.6)? We prove that when the rate process r in (1.4) is a positive ergodic diffusion independent of the Lévy process Z, (1.6) is still a consistent estimator for (1.7) up to a constant, provided that the time horizon T and sampling frequency converge to infinite at suitable rates. Roughly speaking, suppose that the following conditions hold true:

(i) φ is a continuous, locally bounded function such that

$$\sup_{t>0}\frac{1}{t}\,\mathbb{E}\,|\varphi(Z_t)|<\infty, \tag{1.8}$$

and $\varphi(x)\to 0$ as $x\to 0$ at an "appropriate rate" (see Condition 2.1 below);

(ii) $\{r(t)\}_{t\geq 0}$ is an *ergodic positive solution* of the SDE:

$$dr(t)=b(r(t))dt+\sigma(r(t))\,dW_t, \tag{1.9}$$

such that r is *independent* of Z and also

$$\sup_{t\geq 0}\mathbb{E}\;r^2(t)<\infty; \tag{1.10}$$

(iii) The time horizon T and the sampling times π_T are such that $T\to\infty$ and $T\cdot\bar{\delta}_T^2 \stackrel{T\to\infty}{\longrightarrow} 0$, where $\bar{\delta}_T$ is the largest time span between consecutive observation.

Then, it follows that

$$\lim_{T\to\infty} \hat{\beta}^{\pi_T}(\varphi) \stackrel{\mathbb{P}}{=} \bar{\zeta} \cdot \beta(\varphi), \quad \lim_{T\to\infty} \mathbb{E}\,\hat{\beta}^{\pi_T}(\varphi) = \bar{\zeta} \cdot \beta(\varphi), \tag{1.11}$$

where $\bar{\zeta} := \bar{\zeta}(r)$ is the expectation of the invariant distribution of r. Furthermore, under the stronger assumption that

$$\sup_{t\geq 0} \mathbb{E}\, |r(t)|^{2+\varepsilon} < \infty, \tag{1.12}$$

for some $\varepsilon > 0$, the condition $T \cdot \bar{\delta}_T^2 \stackrel{T\to\infty}{\longrightarrow} 0$ is not needed for (1.11).

By an *ergodic diffusion*, we mean a strong continuous Markov process $\{r(t)\}_{t\geq 0}$ that takes values on an interval $I := (a, b) \subset \mathbb{R}$ and that is regular and recurrent (see e.g. [18]). Such a process admits a unique *invariant probability measure* ζ, which in turn satisfies the *ergodic property*:

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t g(r(u))du = \int_I g(x)\zeta(dx), \quad a.s. \tag{1.13}$$

for any $g \in \mathbb{L}^1(\zeta)$ (c.f. [18, Theorem 20.14]). A model that meets the condition (ii) above and that is a typical choice in applications (cf. [5]) is the Cox-Ingersoll-Ross (CIR) process

$$dr(t) = \alpha(m - r(t))dt + v\sqrt{r(t)}\, dW_t,$$

with positive α, v and m such that $\alpha m/v^2 > 1/2$. It will turn out that the consistency (1.11) is a consequence of the ergodic property (1.13) and the consistency of the estimator (1.6) when the underlying process X is a pure Lévy process.

Let us remark that the independence assumption between Z and r is a drawback from a financial point of view. One could think of ad hoc treatments to incorporate certain degree of dependence such as common driving factors for r and Z, but we won't explore this direction in this work. Note also that without loss of generality we can assume that $\bar{\zeta}(r) := 1$ since for an arbitrary Lévy process Z and a diffusion r, we can write the time-changed Lévy process (1.3-1.4) as follows

$$Z_{\tau(t)} = \hat{Z}_{\hat{\tau}(t)},$$

with $\hat{Z}_t := Z_{\bar{\zeta}(r)t}$, $\hat{\tau}(t) := \int_0^t \hat{r}(u)du$, and $\hat{r}(t) := r(t)/\bar{\zeta}(r)$. The process $\hat{Z}$ is again a Lévy process satisfying (1.8) with Lévy density $\bar{\zeta}(r)s$. Similarly, $\hat{r}$ is a positive ergodic solution of the SDE:

$$d\hat{r}(t) = \hat{b}(\hat{r}(t))dt + \hat{\sigma}(\hat{r}(t))\, dW_t,$$

with $\hat{b}(x) := \bar{\zeta}(r)^{-1}b(\bar{\zeta}(r)x)$ and $\hat{\sigma}(x) := \bar{\zeta}(r)^{-1}\sigma(\bar{\zeta}(r)x)$. Clearly, $\hat{r}$ satisfies (1.10) and $\bar{\zeta}(\hat{r}) = 1$.

In the last part of the paper, we obtain a central limit theorem for the estimators $\hat{\beta}^{\pi_T}(\varphi)$ in (1.6) with scaling constant $T^{1/2}$ and centering constants

$$\breve{\beta}^{\pi_T}(\varphi) := \frac{1}{t_n} \sum_{k=1}^{n} \mathbb{E}\left[\varphi\left(X_{t_k} - X_{t_{k-1}}\right) \middle| \mathcal{F}^X_{t_{k-1}}\right],$$

where $\mathcal{F}^X_t = \sigma(X_u : u \leq t)$. Concretely, using central limit theorems for martingale differences (see e.g. [3]), we show that

$$T^{1/2}\left(\hat{\beta}^{\pi_T}(\varphi) - \breve{\beta}^{\pi_T}(\varphi)\right) \stackrel{\mathfrak{D}}{\to} \sigma(\varphi)\mathcal{N}(0,1),$$

as $T \to \infty$ and $\bar{\delta}_T \to 0$, with $\sigma^2(\varphi) := \bar{\zeta}\beta(\varphi^2)$. Under certain conditions, the statistics $\breve{\beta}^{\pi_T}(\varphi)$ satisfy themselves the CLT:

$$T^{1/2}\left(\breve{\beta}^{\pi_T}(\varphi) - \bar{\zeta} \cdot \beta(\varphi)\right) \stackrel{\mathfrak{D}}{\to} \beta(\varphi)\Gamma^{1/2} \cdot \mathcal{N}(0,1),$$

whenever $T\bar{\delta}_T \stackrel{n\to\infty}{\longrightarrow} 0$, for certain positive constant Γ depending on the process r. Such a result suggests a CLT of the form

$$T^{1/2}\left(\hat{\beta}^{\pi_T}(\varphi) - \bar{\zeta} \cdot \beta(\varphi)\right) \stackrel{\mathfrak{D}}{\to} (\sigma^2(\varphi) + \beta^2(\varphi)\Gamma)^{1/2} \cdot \mathcal{N}(0,1);$$

however, the later limit is still under investigation and we expect to address this issue in a future work.

The paper is structured as follows. In Section 2, we show the consistency for time-changed Lévy models with a general random clock τ. We propose the limit

$$\lim_{n\to\infty} \mathbb{E}\left\{\frac{1}{\tau(t_n^n)} \sum_{k=1}^{n} (\Delta_k^n \tau)\, \mathbf{1}_{\{\Delta_k^n \tau \geq t_0\}}\right\} = 0, \tag{1.14}$$

for arbitrary $t_0 > 0$ and where $\Delta_k^n \tau := \tau(t_k^n) - \tau(t_{k-1}^n)$, as a key assumption on the random clock τ for consistency to hold. As an application, the case of a pure Lévy model is considered, extending a former result by Woerner [24] to non-regular sampling schemes and simpler functions φ. In Section 3, we proceed to investigate conditions under which (1.14) holds when τ is driven by a positive ergodic diffusion $\{r(t)\}_{t\geq 0}$ via (1.4). The case of general sampling schemes is discussed in Section 4. In particular, it is proved that under (1.12), the rate condition $T\bar{\delta}_T^2 \to 0$ is not needed for consistency. Finally, the asymptotic normality of the estimators is addressed in Section 5.

2. Estimation of integrals of the Lévy measure

We consider a time-changed Lévy model of the form (1.3), where $\{Z_t\}_{t\geq 0}$ is a Lévy process with generating triplet (b, σ^2, ν), and $\{\tau(t)\}_{t\geq 0}$ is as in (1.4) for a general non-negative process r that is *independent* of Z. Suppose we sample the process X over a finite time horizon $[0, T_n]$ at discrete times $0 = t_0^n < \cdots < t_n^n = T_n$. In this part we provide conditions for the convergence in probability of the realized φ-variations

$$\hat{\beta}_n(\varphi) := \frac{1}{t_n^n} \sum_{k=1}^{n} \varphi\left(X_{t_k^n} - X_{t_{k-1}^n}\right), \tag{2.1}$$

as the time-horizon T_n tends to infinity and the largest time span between observations, $\bar{\delta}^n := \max_k\{t_{k+1}^n - t_k^n\}$, tends to 0. In the case of a pure Lévy process (namely, $\tau(t) = t$ and $X = Z$) and equally-spaced time-points, Woerner [24] (see Theorems 4.2 and 5.1) considers this problem under some additional regularity conditions that can be greatly simplified as it will be shown here (see Theorem 2.5 below).

In order to study the behavior of (2.1), we first survey the asymptotics of the following statistics

$$\tilde{\beta}_n(\varphi) := \frac{1}{\tau(t_n^n)} \sum_{k=1}^{n} \varphi\left(Z_{\tau(t_k^n)} - Z_{\tau(t_{k-1}^n)}\right), \tag{2.2}$$

when the time horizon $T_n = t_n^n \to \infty$ and $\bar{\delta}^n \to 0$. Through this part, we shall write $\tau_k^n := \tau(t_k^n)$. Note that, due to the independence of Z and τ, the convergence in probability of $\tilde{\beta}_n(\varphi)$ will follow from the pure Lévy case ($Z = X$) if $\tau_n^n \stackrel{n\to\infty}{\longrightarrow} \infty$ a.s., and

$$\max_k \left(\tau_k^n - \tau_{k-1}^n\right) \stackrel{n\to\infty}{\longrightarrow} 0, \; a.s. \tag{2.3}$$

However, the condition (2.3) is rather unsatisfactory as it translates into asking that almost all paths $t \to \tau(t)$ are uniformly continuous in all $\mathbb{R}_+$ since the time horizon t_n^n is increasing.

We first review a crucial preliminary result. It is well-known that

$$\lim_{t\to 0} \frac{1}{t}\; \mathbb{E}\,\varphi(X_t) = \int \varphi(x)\nu(dx), \tag{2.4}$$

for any bounded $\nu-$continuous function φ vanishing in a neighborhood of the origin (cf. Sato [22, Corollary 8.9]). Consider the following class of locally bounded (but potentially unbounded) functions:

$$\mathcal{S}(\nu) := \{g : \mathbb{R} \to \mathbb{R}_+ : \int_{|x|>1} g(x)\nu(dx) < \infty, \quad g(x) = p(x)q(x), \tag{2.5}$$

$$\text{where } p \text{ is subadditive, and } k \text{ is submultiplicative}\}.$$

Building on results in [24] and [17], [13] proves that the limit

$$\breve{\beta}(\varphi) := \lim_{t\to 0} \frac{1}{t}\; \mathbb{E}\,\varphi(X_t), \tag{2.6}$$

exists provided that the following conditions hold:

Conditions 2.1.

1. *φ is ν-continuous and locally bounded,*
2. *There exists a function g in $S(\nu)$ such that*

$$\limsup_{|x|\to\infty} \frac{|\varphi(x)|}{g(x)} < \infty, \tag{2.7}$$

3. *$\varphi(x) \to 0$ as $x \to 0$ under any of the following conditions:*
   (a) $\varphi(x) = o(|x|^2)$;
   (b) $\varphi(x) = O(|x|^r)$, *for some* $r \in (1,2)$ *such that* $\int (|x|^r \wedge 1)\,\nu(dx) < \infty$, *and* $\sigma = 0$;
   (c) $\varphi(x) = o(|x|)$, $\int (|x| \wedge 1)\,\nu(dx) < \infty$, *and* $\sigma = 0$;
   (d) $\varphi(x) = O(|x|^r)$, *for some* $r \in (0,1)$ *such that* $\int (|x|^r \wedge 1)\,\nu(dx) < \infty$, $\sigma = 0$, *and* $\bar{b} := b - \int_{|x|\leq 1} x\nu(dx) = 0$;
   (e) $\varphi(x) \sim x^2$;
   (f) $\varphi(x) \sim |x|$, *and* $\sigma = 0$.

Moreover, (2.6) is given as follows depending on which condition in 3 is satisfied:

$$\breve{\beta}(\varphi) := \begin{cases} \beta(\varphi) & \text{if any (a)-(d) is true} \\ \sigma^2 + \beta(\varphi) & \text{if (e) is true} \\ |\bar{b}| + \beta(\varphi) & \text{if (f) is true,} \end{cases} \tag{2.8}$$

where as before $\beta(\varphi) := \int \varphi(x)\nu(x)$. Note that conditions 2.1 implies that $\beta(|\varphi|) < \infty$.

We are ready to study the asymptotic behavior of (2.2). The following result gives conditions for asymptotic unbiasedness.

Proposition 2.1. *Assume the following:*

(i) *φ is a continuous function satisfying Conditions 2.1 and also*

$$M_\varphi := \sup_{t>0} \frac{1}{t}\,\mathbb{E}\,|\varphi(Z_t)| < \infty; \tag{2.9}$$

(ii) *$\{\tau(t)\}_{t\geq 0}$ is a non-decreasing càdlàg process, independent of Z, such that $\tau(t) > 0$ a.s. for any $t > 0$, and*

$$\lim_{n\to\infty} \mathbb{E}\left\{\frac{1}{\tau(t_n^n)} \sum_{k=1}^{n} (\Delta_k^n \tau)\,\mathbf{1}_{\{\Delta_k^n\tau \geq t_0\}}\right\} = 0, \tag{2.10}$$

for any $t_0 > 0$, where $\Delta_k^n\tau := \tau(t_k^n) - \tau(t_{k-1}^n)$.

Then, the statistics $\tilde{\beta}_n(\varphi)$ in (2.2) are asymptotically unbiased estimators for the parameter $\breve{\beta}(\varphi)$ in (2.8) as $n \to \infty$.

Proof. Conditioning on $\{\tau_k^n\}_{k\leq n}$ and using the the independence between τ and X (see Appendix C for more details), we get

$$\mathbb{E}\ \tilde{\beta}_n(\varphi) = \mathbb{E}\,\frac{1}{\tau_n^n}\sum_{k=1}^{n} H_\varphi\left(\Delta_k^n\tau\right), \tag{2.11}$$

where $H_\varphi(t) := \mathbb{E}\,\varphi(Z_t)$. Then,

$$\left|\mathbb{E}\,\tilde{\beta}_n(\varphi) - \breve{\beta}(\varphi)\right| \leq \mathbb{E}\,\frac{1}{\tau_n^n}\sum_{k=1}^{n}\left|\frac{H_\varphi\left(\Delta_k^n\tau\right)}{\Delta_k^n\tau} - \breve{\beta}(\varphi)\right|\Delta_k^n\tau,$$

under the convention that $0/0 = 0$. For $\varepsilon > 0$, let $t_0 := t_0(\varepsilon) > 0$ such that if $t < t_0$, then $|H_\varphi(t)/t - \breve{\beta}(\varphi)| < \varepsilon$. Then, breaking up the summation above into the k's such that $\Delta_k^n\tau < t_0$ and its complement and using that $\sum_k \Delta_k^n\tau = \tau_n^n$, we obtain

$$\left|\mathbb{E}\,\tilde{\beta}_n(\varphi) - \breve{\beta}(\varphi)\right| \leq \varepsilon + \left(M_\varphi + |\breve{\beta}(\varphi)|\right)\,\mathbb{E}\,\frac{1}{\tau_n^n}\sum_{k=1}^{n}\Delta_k^n\tau\mathbf{1}_{\{\Delta_k^n\tau\geq t_0\}} \stackrel{n\to\infty}{\longrightarrow} \varepsilon.$$

This proves the result since ε is arbitrary. □

We proceed to show that the conditions above are also sufficient for the consistency of the estimator (2.2). We need first to introduce a truncated version of (2.2) via the following Lemma.

Lemma 2.2. *Let*

$$\tilde{\beta}_n^t(\varphi) := \frac{1}{\tau_n^n}\sum_{k=1}^{n}\varphi\left(Z_{\tau_k^n} - Z_{\tau_{k-1}^n}\right)\mathbf{1}_{\{|\varphi(Z_{\tau_k^n}-Z_{\tau_{k-1}^n})|\leq\tau_n^n\}}, \tag{2.12}$$

and assume that (i)-(ii) of Proposition 2.1 holds as well as

(iii) $\tau_n^n \to \infty$, a.s.

Then,

$$\lim_{n\to\infty}\ \mathbb{E}\ \left\{\tilde{\beta}_n^t(\varphi) - \tilde{\beta}_n(\varphi)\right\} = 0. \tag{2.13}$$

Proof. For a given $T_0 > 0$, let $t_0 := t_0(T_0)$ be such that if $0 < t < t_0$, then

$$\mathbb{E}\left\{|\varphi(Z_t)|\mathbf{1}_{\{|\varphi(Z_t)|>T_0\}}\right\} \leq 2t\left(\int|\varphi(x)|\mathbf{1}_{\{|\varphi(x)|>T_0\}}\nu(dx)\vee T_0^{-1}\right). \tag{2.14}$$

Such a $t_0 > 0$ exists since $|\varphi(\cdot)|\mathbf{1}_{\{|\varphi(\cdot)|>T_0\}}$ satisfies the conditions 2.1 with the behavior 3(a) and thus,

$$\lim_{t\to 0}\frac{1}{t}\,\mathbb{E}\left\{|\varphi(Z_t)|\mathbf{1}_{\{|\varphi(Z_t)|>T_0\}}\right\} = \int|\varphi(x)|\mathbf{1}_{\{|\varphi(x)|>T_0\}}\nu(dx).$$

Note that

$$\begin{aligned}\mathbb{E}\,|\tilde{\beta}_n^t(\varphi)-\tilde{\beta}_n(\varphi)| &\leq \mathbb{E}\,\frac{1}{\tau_n^n}\sum_{k=1}^n |\varphi(Z_{\tau_k^n}-Z_{\tau_{k-1}^n})|\mathbf{1}_{\{|\varphi(Z_{\tau_k^n}-Z_{\tau_{k-1}^n})|>\tau_n^n\}} \qquad (2.15)\\ &\leq \mathbb{E}\,\frac{1}{\tau_n^n}\sum_{k=1}^n |\varphi(Z_{\tau_k^n}-Z_{\tau_{k-1}^n})|\mathbf{1}_{\{\tau_n^n\leq T_0\}}\\ &\quad+\mathbb{E}\,\frac{1}{\tau_n^n}\sum_{k=1}^n |\varphi(Z_{\tau_k^n}-Z_{\tau_{k-1}^n})|\mathbf{1}_{\{|\varphi(Z_{\tau_k^n}-Z_{\tau_{k-1}^n})|>T_0\}}\end{aligned}$$

Conditioning on $\{\tau_k^n\}_{k\leq n}$ in the last two expectations and using the stationary increments of Z, it is evident that $\mathbb{E}\,|\tilde{\beta}_n^t(\varphi)-\tilde{\beta}_n(\varphi)|$ can be bounded by

$$\mathbb{E}\,\frac{1}{\tau_n^n}\sum_{k=1}^n H_{|\varphi|}(\Delta_k^n\tau)\mathbf{1}_{\{\tau_n^n\leq T_0\}}+\mathbb{E}\,\frac{1}{\tau_n^n}\sum_{k=1}^n H_{|\varphi|\mathbf{1}_{\{|\varphi|>T_0\}}}(\Delta_k^n\tau), \qquad (2.16)$$

where as before $H_\varphi(t):=\mathbb{E}\,\varphi(Z_t)$. Using that $M_\varphi:=\sup_{t>0}\frac{1}{t}\,\mathbb{E}\,|\varphi(Z_t)|<\infty$ and that $\sum_{k=1}^n\tau_k^n=\tau_n^n$, the first term in (2.16) is bounded by $M_\varphi\mathbb{P}(\tau_n^n\leq T_0)$. Similarly, using the same identities and (2.14), the second term can be bounded as follows

$$\begin{aligned}&\mathbb{E}\,\frac{1}{\tau_n^n}\sum_{k=1}^n H_{|\varphi|\mathbf{1}_{\{|\varphi|>T_0\}}}(\Delta_k^n\tau)\left\{\mathbf{1}_{\{\Delta_k^n\tau<t_0\}}+\mathbf{1}_{\{\Delta_k^n\tau\geq t_0\}}\right\}\\ &\leq 2\Big(\int|\varphi(x)|\mathbf{1}_{\{|\varphi(x)|>T_0\}}\nu(dx)\vee T_0^{-1}\Big)+M_\varphi\,\mathbb{E}\left\{\frac{1}{\tau(t_n^n)}\sum_{k=1}^n(\Delta_k^n\tau)\,\mathbf{1}_{\{\Delta_k^n\tau\geq t_0\}}\right\}\end{aligned}$$

Putting the previous estimates together and using (2.10) and (iii),

$$\limsup_{n\to\infty}\left|\mathbb{E}\,\left\{\tilde{\beta}_n^t(\varphi)-\tilde{\beta}_n(\varphi)\right\}\right|\leq 2\left(\int|\varphi(x)|\mathbf{1}_{\{|\varphi(x)|>T_0\}}\nu(dx)\vee T_0^{-1}\right),$$

which can be made arbitrarily small taking T_0 large enough. □

Theorem 2.3. *Under the conditions of Proposition 2.1 and (iii) in Lemma 2.2, the statistics* $\tilde{\beta}_n(\varphi)$ *in (2.2) are consistent estimators for* $\breve{\beta}(\varphi)$ *in (2.8).*

Proof. We apply arguments similar to the weak law of large numbers for row-wise independent arrays as described in e.g. [8, Theorem 5.2.3][1]. Let $\varepsilon>0$. We first note that in light of Lemma 2.2 and Proposition 2.1, there exists $n_0>0$ such that

$$\left|\mathbb{E}\,\tilde{\beta}_n^t(\varphi)-\breve{\beta}(\varphi)\right|<\varepsilon/2, \qquad (2.17)$$

[1]The results in Chung are proved for i.i.d. random variables, but they can be readily extended for row-wise independent arrays

for any $n \geq n_0$. Next, assuming (2.17) and noticing that $\tilde{\beta}_n(\varphi) = \tilde{\beta}_n^t(\varphi)$ on the event $E = \{|\varphi(Z_{\tau_k^n} - Z_{\tau_{k-1}^n})| \leq \tau_n^n, \text{for all } k\}$,

$$\begin{aligned}\mathbb{P}(|\tilde{\beta}_n(\varphi) - \breve{\beta}(\varphi)| > \varepsilon) &\leq \mathbb{P}(|\tilde{\beta}_n(\varphi) - \breve{\beta}(\varphi)| > \varepsilon, E^c) + \mathbb{P}(|\tilde{\beta}_n^t(\varphi) - \breve{\beta}(\varphi)| > \varepsilon, E) \\ &\leq \mathbb{P}(E^c) + \mathbb{P}(|\tilde{\beta}_n^t(\varphi) - \breve{\beta}(\varphi)| > \varepsilon) \\ &\leq \mathbb{P}(E^c) + \mathbb{P}(|\tilde{\beta}_n^t(\varphi) - \mathbb{E}\,\tilde{\beta}_n^t(\varphi)| > \frac{\varepsilon}{2})\end{aligned}$$

Using Chebyshev, we obtain that for $n \geq n_0$,

$$\mathbb{P}\left\{\left|\tilde{\beta}_n(\varphi) - \breve{\beta}(\varphi)\right| > \varepsilon\right\} \leq B_n + \frac{4}{\varepsilon^2} C_n, \tag{2.18}$$

where

$$B_n := \sum_{k=1}^{n} \mathbb{P}\left[\left|\varphi(Z_{\tau_k^n} - Z_{\tau_{k-1}^n})\right| > \tau_n^n\right], \quad C_n := \operatorname{Var}\left(\tilde{\beta}_n^t(\varphi)\right).$$

We show that B_n and C_n vanish. First, using Markov inequality, under the convention that $0/0 = 0$,

$$B_n \leq \mathbb{E}\left\{\frac{1}{\tau_n^n} \sum_{k=1}^{n} |\varphi|(Z_{\tau_k^n} - Z_{\tau_{k-1}^n}) \mathbf{1}_{\{|\varphi|(Z_{\tau_k^n} - Z_{\tau_{k-1}^n}) > \tau_n^n\}}\right\},$$

which is the expression in the right-hand side of (2.15) and hence, it can be proved to converge to 0 along the same lines as in the proof of Lemma 2.2.

Next, using the law of total variance, conditioning on $\tau_\cdot^n = (\tau_1^n, \ldots, \tau_n^n)$,

$$\operatorname{Var}\left(\tilde{\beta}_n^t(\varphi)\right) = \operatorname{Var}\left(\mathbb{E}\left\{\tilde{\beta}_n^t(\varphi)\middle|\tau_\cdot^n\right\}\right) + \mathbb{E}\left\{\operatorname{Var}\left(\tilde{\beta}_n^t(\varphi)\middle|\tau_\cdot^n\right)\right\}. \tag{2.19}$$

Let us denote D_n and E_n the two terms on the right-hand side of (2.19). Conditioning on $\tau_\cdot^n$, the terms in (2.12) are independent, and we can write:

$$D_n = \operatorname{Var}\left(\frac{1}{\tau_n^n} \sum_{k=1}^{n} \mathbb{E}\left[\varphi(Z_t)\mathbf{1}_{\{|\varphi(Z_t)| \leq s\}}\right]\Big|_{t=\Delta_k^n\tau, s=\tau_n^n}\right) \tag{2.20}$$

$$E_n \leq \mathbb{E}\left(\frac{1}{(\tau_n^n)^2} \sum_{k=1}^{n} \mathbb{E}\left[\varphi^2(Z_t)\mathbf{1}_{\{|\varphi(Z_t)| \leq s\}}\right]\Big|_{t=\Delta_k^n\tau, s=\tau_n^n}\right) \tag{2.21}$$

Clearly,

$$D_n \leq 2\operatorname{Var}\left(\frac{1}{\tau_n^n} \sum_{k=1}^{n} \mathbb{E}\left[\varphi(Z_t)\mathbf{1}_{\{|\varphi(Z_t)| > s\}}\right]\Big|_{t=\Delta_k^n\tau, s=\tau_n^n}\right) \tag{2.22}$$

$$+ 2\operatorname{Var}\left(\frac{1}{\tau_n^n} \sum_{k=1}^{n} H_\varphi\left(\Delta_k^n\tau\right) - \breve{\beta}(\varphi)\right), \tag{2.23}$$

where as before $H_\varphi(t) := \mathbb{E}\{\varphi(Z_t)\}$. Using the inequality $\mathrm{Var}(X) \leq EX^2$, the estimate (2.9), and that $\tau_n^n = \sum_{k=1}^n \tau_k^n$, the term on the right-hand side of (2.22) can be bounded by

$$M_\varphi \mathbb{E}\left(\frac{1}{\tau_n^n}\sum_{k=1}^n \mathbb{E}\left[|\varphi|(Z_t)\mathbf{1}_{\{|\varphi(Z_t)|>s\}}\right]\Big|_{t=\Delta_k^n\tau, s=\tau_n^n}\right)$$

which again converges to 0 by the same arguments as the reasoning following (2.15) in Lemma 2.2. Using again $\mathrm{Var}(X) \leq EX^2$, (2.9), and $\tau_n^n = \sum_{k=1}^n \tau_k^n$, the term in line (2.23) can be bounded as follows for any fixed $\varepsilon > 0$:

$$\begin{aligned}&(M_\varphi + |\breve{\beta}(\varphi)|)\,\mathbb{E}\,\frac{1}{\tau_n^n}\sum_{k=1}^n\left|\frac{1}{\Delta_k^n\tau}H_\varphi(\Delta_k^n\tau) - \breve{\beta}(\varphi)\right|\Delta_k^n\tau \\ &\leq (M_\varphi + |\breve{\beta}(\varphi)|)^2\,\mathbb{E}\,\frac{1}{\tau_n^n}\sum_{k=1}^n \Delta_k^n\tau\mathbf{1}_{\{\Delta_k^n\tau\geq t_0\}} + (M_\varphi + |\breve{\beta}(\varphi)|)\varepsilon,\end{aligned}$$

where $t_0 := t_0(\varepsilon)$ is chosen such that $|H_\varphi(t)/t - \breve{\beta}(\varphi)| \leq \varepsilon$. In view of (2.10) and since $\varepsilon > 0$ is arbitrary, we conclude that term (2.23) converges to 0 and so does D_n.

We now prove that the term on the right-hand side of (2.21) converges to 0. We shall use the inequality

$$\mathbb{E}\,|Z|^2\mathbf{1}_{\{|Z|\leq s\}} \leq 2s\int_{(0,1)} \mathbb{E}\left\{|Z|\mathbf{1}_{\{|Z|>us\}}\right\}du, \tag{2.24}$$

which can be easily deduced as follows:

$$\begin{aligned}\mathbb{E}\,|Z|^2\mathbf{1}_{\{|Z|\leq s\}} &= 2\int_0^s v\mathbb{P}[v < |Z| \leq s]dv \leq 2\int_0^s v\mathbb{P}[|Z| > v]dv \\ &\leq 2s^2\int_0^1 u\mathbb{P}[|Z| > us]du \leq 2s\int_0^1 \mathbb{E}\left\{|Z|\mathbf{1}_{\{|Z|>us\}}\right\}du.\end{aligned}$$

Applying (2.24) to each term in (2.21), E_n can be bounded by

$$\mathbb{E}\left(\frac{2}{\tau_n^n}\sum_{k=1}^n\int_0^1 \mathbb{E}\left[|\varphi(Z_t)|\,\mathbf{1}_{\{|\varphi(Z_t)|>us\}}\right]\Big|_{t=\Delta_k^n\tau, s=\tau_n^n}du\right) \leq 2\int_0^1 s_n(u)du, \tag{2.25}$$

where we defined

$$s_n(u) := \mathbb{E}\left(\frac{1}{\tau_n^n}\sum_{k=1}^n \mathbb{E}\left[|\varphi(Z_t)|\,\mathbf{1}_{\{|\varphi(Z_t)|>us\}}\right]\Big|_{t=\Delta_k^n\tau, s=\tau_n^n}\right).$$

Note that $s_n(u) \leq M_\varphi$, for all $u \in [0,1]$. Fix $u_0 \in (0,1)$ and $0 < T_0 < \infty$. There exists $t_0 := t_0(u_0, T_0)$ such that

$$0 \leq \mathbb{E}\left[|\varphi(Z_t)|\,\mathbf{1}_{\{|\varphi(Z_t)|>u_0T_0\}}\right] \leq 2t\int |\varphi(x)|\,\mathbf{1}_{\{|\varphi(x)|>u_0T_0\}}\nu(dx), \tag{2.26}$$

for all $0 < t < t_0$. Using that

$$\mathbf{1}_{\{|\varphi(Z_t)|>us\}} \leq \mathbf{1}_{\{s\leq T_0\}} + \mathbf{1}_{\{|\varphi(Z_t)|>u_0T_0\}},$$

for any $u \geq u_0$, we can have:

$$s_n(u) \leq 2M_\varphi \mathbb{P}(\tau_n^n \leq T_0) + \mathbb{E}\left(\frac{2}{\tau_n^n}\sum_{k=1}^n \mathbb{E}\left[|\varphi(Z_t)|\,\mathbf{1}_{\{|\varphi(Z_t)|>u_0T_0\}}\right]\Big|_{t=\Delta_k^n\tau}\right),$$

for $u \geq u_0$. Next, by breaking up the summation above into those k for which $\Delta_k^n \geq t_0$ and those for which $\Delta_k^n < t_0$ and using that $\mathbb{E}\,[|\varphi(Z_t)|] \leq M_\varphi t$ and (2.26), for any $u_0 \leq u \leq 1$,

$$s_n(u) \leq 2M_\varphi \mathbb{P}(\tau_n^n \leq T_0) + 2M_\varphi\,\mathbb{E}\,\frac{1}{\tau_n^n}\sum_{k=1}^n \Delta_k^n\tau \mathbf{1}_{\{\Delta_k^n\tau\geq t_0\}} + 4\int |\varphi|\,\mathbf{1}_{\{|\varphi|>u_0T_0\}}d\nu.$$

Breaking the integral in (2.25),

$$\begin{aligned} E_n &\leq 2u_0M_\varphi + 2M_\varphi\,\mathbb{E}\,\frac{1}{\tau_n^n}\sum_{k=1}^n \Delta_k^n\tau\mathbf{1}_{\{\Delta_k^n\tau\geq t_0\}} \\ &\quad + 2M_\varphi\mathbb{P}(\tau_n^n \leq T_0) + 4\int |\varphi(x)|\,\mathbf{1}_{\{|\varphi(x)|>u_0T_0\}}\nu(dx). \end{aligned}$$

In view of conditions (ii) and (iii) of Proposition 2.1 and Lemma 2.2,

$$\limsup_{n\to\infty} E_n \leq 2u_0M_\varphi + 4\int |\varphi(x)|\,\mathbf{1}_{\{|\varphi(x)|>u_0T_0\}}\nu(dx),$$

which can be made arbitrarily small taking u_0 small enough and T_0 large enough. This proves that the second term on the right-hand side of (2.19), and thereof (2.18), vanishes as $n \to \infty$. □

Remark 2.4.

1. *Clearly, (2.9) will be satisfied if the Conditions 2.1 are true and φ is bounded. Moreover, (2.9) holds as well, if φ has linear growth, Conditions 2.1 are satisfied, and Z has finite first moment. Indeed, in light of (2.6), there exists $t_0 > 0$ such that $\sup_{0<t<t_0} \mathbb{E}\,|\varphi(Z_t)|/t < \infty$. Suppose that $|\varphi(x)| \leq c|x|$, whenever $|x| > x_0$ for some $x_0 > 0$. Then,*

$$\sup_{t\geq t_0}\frac{1}{t}\,\mathbb{E}\,|\varphi(Z_t)| \leq c\sup_{t\geq t_0}\frac{1}{t}\,\mathbb{E}\,|Z_t| + \frac{1}{t_0}\sup_{|x|\leq x_0}|\varphi(x)|.$$

Hence, it remains to show that the first term on the right-hand side of the above inequality is bounded. This follows from the inequality:

$$\mathbb{E}\,|Z_t| \leq \lfloor t\rfloor\,\mathbb{E}\,|Z_1| + \mathbb{E}\,Z_1^*,$$

where $Z_1^* := \sup_{t\leq 1}|Z_t|$. $\mathbb{E}\, Z_1^* < \infty$ in light of Theorem 25.18 in [22], which is actually stated for submultiplicative moment functions g, but which can be readily modified to cover subadditive g as well.

2. *Suppose now that φ has quadratic growth, Conditions 2.1 are satisfied, and Z has finite second moment. In that case, there exist constants $c_1, c_2 > 0$ such that*

$$\mathbb{E}\,|\varphi(Z_t)| \leq c_1 t + c_2 t^2,$$

for all $t > 0$. Proposition 2.1 and Lemma 2.2 hold true if one impose (iii) of Lemma 2.2 and the following condition instead of (2.10):

$$\lim_{n\to\infty} \mathbb{E}\left\{ \frac{1}{\tau(t_n^n)} \sum_{k=1}^{n} (\Delta_k^n \tau)^2 \mathbf{1}_{\{\Delta_k^n \tau \geq t_0\}} \right\} = 0,$$

for all $t_0 > 0$. For Theorem 2.3 to hold true, it suffices (ii) and that, for any $t_0 > 0$,

$$\lim_{n\to\infty} \mathbb{E}\left\{ \frac{1}{\tau(t_n^n)} \sum_{k=1}^{n} (\Delta_k^n \tau)^3 \mathbf{1}_{\{\Delta_k^n \tau \geq t_0\}} \right\} = 0. \tag{2.27}$$

More precisely, when trying to show that the different terms of (2.18) vanish, the following limit naturally appears:

$$\lim_{n\to\infty} \mathbb{E}\left\{ \frac{1}{\tau(t_n^n)} \sum_{k=1}^{n} (\Delta_k^n \tau)^2 \mathbf{1}_{\{\Delta_k^n \tau \geq t_0\}} \right\}^2,$$

which can be linked to (2.27) in view of the Jensen's inequality:

$$\left(\frac{1}{\tau_n^n} \sum_{k=1}^{n} c_k \Delta_k^n \right)^2 \leq \frac{1}{\tau_n^n} \sum_{k=1}^{n} c_k^2 \Delta_k^n, \quad a.s.$$

We finish this part with a closer look into the pure Lévy model where $\tau(t) \equiv t$ and $X = Z$. In that case, $\hat{\beta}_n(\varphi) = \tilde{\beta}_n(\varphi)$ and the condition (ii) of Proposition 2.1 is satisfied whenever $\bar{\delta}^n := \max_k\{t_{k+1}^n - t_k^n\} \to 0$. Thus,

$$\lim_{n\to\infty} \hat{\beta}_n(\varphi) \stackrel{\mathbb{P}}{=} \check{\beta}(\varphi), \tag{2.28}$$

provided that $T_n \to \infty$, $\bar{\delta}^n \to 0$, and φ satisfies condition (i) in Proposition 2.1. It turns out that (2.9) is not needed. The following result generalizes Theorem 5.1 in Woerner [24] for non-regular sampling schemes and simpler functions φ. For the sake of readability and to continue presenting the main subject of the paper, we postpone the proof of the below result to the Appendix A. The proof we presented shows the explicit connection between $\varepsilon > 0$ and the thresholds time horizon T and mesh δ.

Theorem 2.5. *Suppose X is a Lévy process with Lévy triplet (b,σ^2,ν) and let φ be a function satisfying conditions 2.1. Then, the estimator $\hat{\beta}^{\pi}(\varphi)$ of (1.6) is such that for any $0<\varepsilon<1$, there exist $T<\infty$ and $\delta>0$ for which*

$$\mathbb{P}\left\{\left|\hat{\beta}^{\pi}(\varphi)-\breve{\beta}(\varphi)\right|>\varepsilon\right\}<\varepsilon, \tag{2.29}$$

whenever $t_n>T$ and $\max_k(t_k-t_{k-1})<\delta$.

3. Random clocks driven by ergodic diffusions

In this part we consider random clocks $\{\tau(t)\}_{t\geq 0}$ of the form (1.4) with $r(t):=g(\tilde{r}(t))$, where g is non-negative function and $\{\tilde{r}_t\}_{t\geq 0}$ is an ergodic diffusion process; that is, $\{\tilde{r}(t)\}_{t\geq 0}$ is a regular recurrent strong Markov process with continuous paths taking values on an interval $I=(a,b)\subset\mathbb{R}$ (see e.g. [18]). As it was explained in the introduction, mean-reverting processes $\tilde{r}$ and monotone continuous functions g are especially attractive since in that case the resulting time-changed Lévy process $X(t):=Z_{\tau(t)}$ will exhibit the volatility clustering effect.

Proposition 3.1. *Assume the model (1.3)-(1.4) under the following setting:*

(A) *Z is a Lévy process with Lévy triplet (σ^2,b,ν);*
(B) *The instantaneous rate process r is independent of Z and is given by $r(t):=g(\tilde{r}(t))$ for a measurable non-negative function g and an ergodic diffusion $\{\tilde{r}_t\}_{t\geq 0}$ with*

$$m_2(g):=\sup_{t\geq 0}\mathbb{E}\,g^2(\tilde{r}(t))<\infty, \tag{3.1}$$

and invariant measure ζ satisfying that $\bar{\zeta}(g):=\int g(x)\,\zeta(dx)\in(0,\infty)$.

Then, the statistics

$$\hat{\beta}_n(\varphi):=\frac{1}{T_n}\sum_{k=1}^{n}\varphi\left(X_{t_k^n}-X_{t_{k-1}^n}\right), \tag{3.2}$$

are consistent and asymptotically unbiased estimators for $\bar{\zeta}(g)\breve{\beta}(\varphi)$ when $T_n\to\infty$ and $\bar{\delta}^n:=\max_k(t_k^n-t_{k-1}^n)\to 0$, provided that

(i) *φ is a continuous function satisfying Conditions 2.1 and (2.9);*
(ii) *(2.10) holds true;*

Proof. Note that

$$\hat{\beta}_n(\varphi)=\bar{\beta}_n(\varphi)\,\frac{\int_0^{T_n}g(\tilde{r}(u))du}{T_n}.$$

By the ergodic theorem (1.13) the last factor converges a.s. to $\bar{\zeta}(g)$ and hence, (iii) of Lemma 2.2 is satisfied. Consistency is now clear in light of Theorem 2.3. For unbiasedness, first note that $\bar{r}(t) := \int_0^t r(u)du/t$ is uniformly integrable since

$$\sup_{t>0} \mathbb{E}\left(\frac{1}{t}\int_0^t r(u)du\right)^2 \leq \sup_{t>0} \frac{1}{t}\int_0^t \mathbb{E}\, g^2(r(u))du \leq m_2(g) < \infty.$$

Also, by the ergodic theorem (1.13), $\lim_{t\to\infty} \bar{r}(t) = \bar{\zeta}(g)$ a.s., and thus,

$$\lim_{t\to\infty} \mathbb{E}\left|\bar{r}(t) - \bar{\zeta}(g)\right| = 0. \tag{3.3}$$

Next, we write

$$\mathbb{E}\,\hat{\beta}_n(\varphi) = \bar{\zeta}(g)\,\mathbb{E}\,\tilde{\beta}_n(\varphi) + \mathbb{E}\left[\left(\frac{\tau_n^n}{t_n^n} - \bar{\zeta}(g)\right)\frac{1}{\tau_n^n}\sum_{k=1}^n H_\varphi\left(\Delta_k^n \tau\right)\right].$$

The first term on the right hand side converges to $\bar{\zeta}(g)\,\breve{\beta}(\varphi)$, while the absolute value of the second term is bounded by

$$M_\varphi\,\mathbb{E}\left|\frac{\tau_n^n}{t_n^n} - \bar{\zeta}(g)\right| = M_\varphi\,\mathbb{E}\left|\bar{r}(t_n^n) - \bar{\zeta}(g)\right| \xrightarrow{n\to} 0.$$

□

In the case that g is bounded, the conditions (3.1), (2.10), and $\bar{\zeta}(g) < \infty$ hold automatically, and thus, the following two limits are true:

$$\lim_{n\to\infty} \hat{\beta}_n(\varphi) \stackrel{\mathbb{P}}{=} \bar{\zeta}\cdot\breve{\beta}(\varphi), \qquad \lim_{n\to\infty} \mathbb{E}\,\hat{\beta}_n(\varphi) = \bar{\zeta}\cdot\beta(\varphi), \tag{3.4}$$

Given that $r(t) = g(\tilde{r})$ plays the role of volatility, one could argue that there is no reason to assume that the volatility will take arbitrarily large values, and thus, the boundedness assumption for g is not completely implausible. Nevertheless, since an upper bound for g cannot be determined in principle, it is natural to consider the case $g(x) = x\mathbf{1}_{x\geq 0}$. Note that in this case, for (3.1) to hold, it suffices that

$$m_2 := \sup_{t\geq 0} \mathbb{E}\,\tilde{r}^2(t) < \infty. \tag{3.5}$$

The following lemma gives a useful sufficient condition for (2.10) to hold.

Lemma 3.2. *Under the setting (A)-(B) of Proposition 3.1 with $g(x) := x\mathbf{1}_{x\geq 0}$, condition (2.10) is satisfied if*

$$\lim_{m\to\infty}\limsup_{n\to\infty} \mathbb{P}\left[\exists\, k \text{ s.t. } \sup_{t\in I_k^n}\left|\tilde{r}(t) - \tilde{r}(t_{k-1}^n)\right| \geq \frac{1}{2}\left(\frac{t_0}{\delta_k^n} - m\right)\right] = 0,$$

for any $t_0, m > 0$, where $I_k^n := [t_{k-1}^n, t_k^n]$ and $\delta_k^n := t_{k-1}^n - t_k^n$.

Proof. Fix $m > 0$. For $n \geq 1$, let $B_n^m := \{k \in \{1, \ldots, n\} : \sup_{t \in [t_{k-1}^n, t_k^n]} r(t) < m\}$. Clearly,

$$\limsup_{n\to\infty} \mathbb{E}\, \frac{1}{\tau_n^n} \sum_{k \in B_n^m} (\Delta_k^n \tau)\, \mathbf{1}_{\{\Delta_k^n \tau \geq t_0\}} \leq \limsup_{n\to\infty} \mathbb{E}\, \frac{1}{\tau_n^n} \sum_{k \in B_n^m} (\Delta_k^n \tau)\, \mathbf{1}_{\{\bar{\delta}^n \geq t_0/m\}} = 0,$$

where the last limit follows from the fact that $\bar{\delta}^n \to 0$. Next, let $C_n^m := \{k \in \{1, \ldots, n\} : \inf_{t \in [t_{k-1}^n, t_k^n]} \tilde{r}(t) \geq m\}$. Then,

$$\begin{aligned} \limsup_{n\to\infty} \mathbb{E}\, \frac{1}{\tau_n^n} \sum_{k \in C_n^m} (\Delta_k^n \tau)\, \mathbf{1}_{\{\Delta_k^n \tau \geq t_0\}} &\leq \limsup_{n\to\infty} \mathbb{E}\, \frac{\int_0^{T_n} \tilde{r}(u) \mathbf{1}_{\{\tilde{r}(u) \geq m\}} du}{\int_0^{T_n} \tilde{r}(u) \mathbf{1}_{\tilde{r}(u) \geq 0} du}, \\ &= \frac{\int_m^\infty x\, \zeta(dx)}{\int_0^\infty x\, \zeta(dx)}, \end{aligned}$$

by the dominated convergence theorem and the ergodic theorem (1.13). Next, let $D_n^m := \{k \in \{1, \ldots, n\} : \exists\, u, v \in [t_{k-1}^n, t_k^n] \text{ with } \tilde{r}(u) < m < \tilde{r}(v)\}$. Now, if $k \in D_n^m$ is such that $\Delta_k^n \tau > t_0$, then $\sup_{t \in [t_{k-1}^n, t_k^n]} \tilde{r}(t) \geq \frac{t_0}{\delta_k^n}$ and thus, the following inequalities must be true:

$$\sup_{t \in [t_{k-1}^n, t_k^n]} \left| \tilde{r}(t) - \tilde{r}(t_{k-1}^n) \right| \geq \frac{1}{2} \left(\frac{t_0}{\delta_k^n} - m \right).$$

If Ω_n^m denotes the event that there exists a k satisfying the above inequality, then

$$\limsup_{n\to\infty} \mathbb{E}\, \frac{1}{\tau_n^n} \sum_{k \in D_n^m} (\Delta_k^n \tau)\, \mathbf{1}_{\{\Delta_k^n \tau \geq t_0\}} \leq \limsup_{n\to\infty} \mathbb{P}[\Omega_n^m].$$

Putting together the previous estimates, for each $m > 0$,

$$\limsup_{n\to\infty} \mathbb{E}\, \frac{1}{\tau_n^n} \sum_{k=1}^n (\Delta_k^n \tau)\, \mathbf{1}_{\{\Delta_k^n \tau \geq t_0\}} \leq \frac{\int_m^\infty x\, \zeta(dx)}{\int_0^\infty x\, \zeta(dx)} + \limsup_{n\to\infty} \mathbb{P}[\Omega_n^m].$$

We finally make $m \to \infty$. □

The most well-known examples of diffusions are solutions to Stochastic Differential Equations (SDE) of the form

$$d\tilde{r}(t) = b(\tilde{r}(t))dt + \sigma(\tilde{r}(t))\, dW_t. \tag{3.6}$$

Two important instances of mean-reverting diffusions of this kind are the Ornstein-Uhlenbeck process and Cox-Ingersoll-Ross processes (see Examples 3.4 and 3.5 below). Conditions for the solution of (3.6) to be ergodic can be found in e.g. Van Zanten [26]. We will make use of moment estimates for (3.6) in order to conclude the sufficient conditions of Lemmas 3.2. Under a linear growth condition of the form

$$|b(x)| + |\sigma(x)| \leq K\,(1 + |x|), \tag{3.7}$$

for all x and certain constant $K < \infty$, it turns out that

$$\mathbb{E}\left[\sup_{0\leq s\leq h}|\tilde{r}(s_0+s)-\tilde{r}(s_0)|^{2m}\right]\leq k_m h^m\left(1+\mathbb{E}\,|\tilde{r}(s_0)|^{2m}\right)e^{k_m h}, \tag{3.8}$$

for any $s_0 \geq 0$, $0 < h \leq 1$, and $m \geq 1$, where k_m is a constant depending only on m and K. We present the proof of the above estimate in the Appendix B for the sake of completeness. We are now ready to establish the consistency of the estimators (3.2).

Proposition 3.3. *Under the setting (A)-(B) of Proposition 3.1 with* $g(x) := x\mathbf{1}_{x\geq 0}$, *suppose also that*

(B') $\tilde{r}$ *satisfies the equation (3.6) with the linear growth condition (3.7).*

Then, the statistics $\hat{\beta}_n(\varphi)$ *in (3.2) are both consistent and asymptotically unbiased estimators for the parameter* $\bar{\zeta}\breve{\beta}(\varphi)$ *with* $\bar{\zeta} := \int_0^\infty x\zeta(dx)$, *provided that (i) in Proposition 3.1 holds and also that* $T_n \to \infty$ *and* $T_n(\bar{\delta}^n)^2 \stackrel{n\to\infty}{\longrightarrow} 0$.

Proof. From Propositions 3.1 and Lemma 3.2, it suffices to prove that for all $t_0, m > 0$,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\mathbb{P}\left[\sup_{t\in I_k^n}\left|\tilde{r}(t)-\tilde{r}(t_{k-1}^n)\right|\geq\frac{1}{2}\left(\frac{t_0}{\Delta_k^n}-m\right)\right]=0.$$

Let n large enough that $\bar{\delta}^n < t_0/(2m)$ and write $c_k^n = (t_0 - m\delta_k^n)/2$ and $\kappa = 4/t_0^2$. Using the bound in (3.8), one can find a constant K such that

$$\begin{aligned}\sum_{k=1}^{n}\mathbb{P}\left[\sup_{t\in I_k^n}\left|\tilde{r}(t)-\tilde{r}(t_{k-1}^n)\right|\geq\frac{c_k^n}{\delta_k^n}\right]&\leq\kappa\sum_{k=1}^{n}(\delta_k^n)^2\ \mathbb{E}\left[\sup_{t\in I_k^n}\left|\tilde{r}(t)-\tilde{r}(t_{k-1}^n)\right|^2\right]\\&\leq(\kappa)(k_2)\sum_{k=1}^{n}(\delta_k^n)^3\left(1+\mathbb{E}\ \tilde{r}^2(t_{k-1}^n)\right)e^{k_2\delta_k^n}\\&\leq K\sum_{k=1}^{n}(\delta_k^n)^3\leq K(\bar{\delta}^n)^2T_n\stackrel{n\to\infty}{\longrightarrow}0.\end{aligned}$$

□

Example 3.4. *For positive* α, v *and* m, *consider the mean-reverting Cox-Ingersoll-Ross (CIR) process*

$$dr(t)=\alpha(m-r(t))dt+v\sqrt{r(t)}\,dW_t, \tag{3.9}$$

where $\{W_t\}_{t\geq 0}$ *is a standard Brownian motion independent of the Lévy process* X *and* $\alpha m/v^2 > 1/2$. *The equation above has a weak non-negative solution with* *unique positive stationary distribution* $\Gamma(\frac{2m\alpha}{v^2}, \frac{v^2}{2\alpha})$. *Also, the conditional mean and variance given* $r(0)$ *are determined by*

$$\mathbb{E}\,[r(t)|r(0)] = r(0)e^{-\alpha t} + m\left(1 - e^{-\alpha t}\right),$$
$$\mathrm{Var}\,(r(t)|r(0)) = r(0)\frac{v^2}{\alpha}\left(e^{-\alpha t} - e^{-2\alpha t}\right) + m\frac{v^2}{2\alpha}\left(1 - e^{-\alpha t}\right).$$

Clearly, this equation satisfies the linear growth condition (3.7) and all the conditions of Proposition 3.3. Then, $\tilde{\beta}_n(\varphi)$ *is asymptotically unbiased estimator of* $\breve{\beta}(\varphi)$ *and* $\hat{\beta}_n(\varphi)$ *in (3.2) are asymptotically consistent and unbiased estimators of* $m\breve{\beta}$.

Example 3.5. *Consider the mean-reverting Ornstein-Uhlenbeck process determined by the Stochastic Differential Equation (SDE)*

$$d\tilde{r}(t) = \alpha(m - \tilde{r}(t))dt + v\sqrt{2\alpha}\,dW_t, \tag{3.10}$$

where $\{W_t\}_{t\geq 0}$ *is a standard Brownian motion independent of the Lévy process* X. *The solution to (3.10) is*

$$\tilde{r}(t) = m + (\tilde{r}(0) - m)e^{-\alpha t} + v\sqrt{2\alpha}\int_0^t e^{-\alpha(t-s)}dW_s,$$

and thus, given $\tilde{r}(0)$, $\tilde{r}(t) - \tilde{r}(0)e^{-\alpha t} \sim \mathcal{N}\left(m\left(1 - e^{-\alpha t}\right), v^2\left(1 - e^{-2\alpha t}\right)\right)$. *Note that* $\lim_{t\to\infty}\mathbb{E}\,\tilde{r}^2(t) = v^2 + m^2$, *and the invariant distribution of* $\tilde{r}$ *is* $\mathcal{N}(m, v^2)$. *Let* $b(x) := \alpha(m-x)$ *be the drift and* $\sigma := v\sqrt{2\alpha}$ *the diffusion of (3.10). Clearly, this equation satisfies the linear growth condition (3.7) and all the conditions of Proposition 3.3. Then,* $\tilde{\beta}_n(\varphi)$ *is asymptotically unbiased estimator of* $\breve{\beta}(\varphi)$ *and* $\hat{\beta}_n(\varphi)$ *in (3.2) are asymptotically consistent and unbiased estimators of* $\mu\breve{\beta}$, *where* $\mu := \mathbb{E}\,(vZ + m)_+$.

4. Consistency of the estimators for general sampling schemes

A rather relevant question is whether the condition $T_n(\bar{\delta}^n)^2 \stackrel{n\to\infty}{\longrightarrow} 0$ of Proposition 3.3 is actually necessary. This condition came from the path of proof we chose in working with the modified estimator (2.2). It is of interest to know whether or not one could directly apply the same reasonings to (3.2). We will discuss this point here. In short, we find that the following condition plays a similar role to (2.10) in this direction of proof:

$$\lim_{n\to\infty}\mathbb{E}\left\{\frac{1}{t_n^n}\sum_{k=1}^{n}\left(\Delta_k^n\tau\right)1_{\{\Delta_k^n\tau\geq t_0\}}\right\}^2 = 0. \tag{4.1}$$

We will show that under the condition

$$m_{2+\varepsilon}(g) := \sup_{t\geq 0}\mathbb{E}\,|g(\tilde{r}(t))|^{2+\varepsilon} < \infty, \tag{4.2}$$

for some $\varepsilon > 0$, the rate $T_n(\bar{\delta}^n)^2 \stackrel{n\to\infty}{\longrightarrow} 0$ is not needed.

Theorem 4.1. *Consider the model (1.3)-(1.4) assuming the setting (A)-(B) of Proposition 3.1 and also assuming (4.2) for some $\varepsilon > 0$. Then, the estimators (3.2) are consistent and asymptotically unbiased for $\bar{\zeta}(g)\breve{\beta}(\varphi)$ when $T_n \to \infty$ and $\bar{\delta}^n \to 0$, provided that (i) in Proposition 3.1 is satisfied.*

Proof. Let us assume for now that (4.1) is true. We shall see at the end of the proof that (4.2) implies (4.1) whenever $T_n \to \infty$ and $\bar{\delta}^n \to 0$. The proof is quite similar to that of Theorem 2.3 working with $\hat{\beta}_n$ instead of $\tilde{\beta}_n(\varphi)$ and with

$$\hat{\beta}_n^t(\varphi) := \frac{1}{t_n^n}\sum_{k=1}^n \varphi\left(Z_{\tau(t_k^n)} - Z_{\tau(t_{k-1}^n)}\right) \mathbf{1}_{\left\{\left|\varphi(Z_{\tau(t_k^n)} - Z_{\tau(t_{k-1}^n)})\right| \le t_n^n\right\}} \tag{4.3}$$

instead of $\tilde{\beta}_n^t$. I will outline the general steps. First, we check that $\hat{\beta}_n(\varphi)$ is asymptotically unbiased. This will follow because, conditioning on $\{\tau_k^n\}_{k\le n}$,

$$\begin{aligned} |\,\mathbb{E}\,\hat{\beta}_n(\varphi) - \bar{\zeta}\breve{\beta}(\varphi)| \le \frac{1}{t_n^n}\,\mathbb{E}\sum_{k=1}^n \Delta_k^n\tau\left|\frac{1}{\Delta_k^n\tau}H_\varphi(\Delta_k^n\tau) - \breve{\beta}(\varphi)\right| \\ + \breve{\beta}(\varphi)\left|\frac{1}{t_n^n}\,\mathbb{E}\int_0^{t_n^n} g(\tilde{r}(u))\,du - \bar{\zeta}(g)\right|. \end{aligned}$$

The second term on the rhs vanishes as $n \to \infty$ because of the ergodicity of $\tilde{r}$ and (3.1), similar to the verification of (3.3). The first term on the rhs can be bounded by

$$\varepsilon\frac{1}{t_n^n}\,\mathbb{E}\int_0^{t_n^n} g(\tilde{r}(u))du + \frac{c}{t_n^n}\,\mathbb{E}\sum_{k=1}^n (\Delta_k^n\tau)\,\mathbf{1}_{\{\Delta_k^n\tau\ge t_0\}},$$

for any $\varepsilon > 0$ and some $t_0 = t_0(\varepsilon)$ such that $|H_\varphi(t)/t - \breve{\beta}(\varphi)| < \varepsilon$ for any $0 < t < t_0$. The limit of the second term above converges to 0 in light of (4.1), while the first term converges to $\varepsilon\bar{\zeta}(g)$, which is arbitrarily small. The second step is to show that $\mathbb{E}\,\{\hat{\beta}_n\{\varphi) - \hat{\beta}_n^t(\varphi)\} = 0$, which can be done almost identically to the proof of (2.13). The next step will be to bound $\mathbb{P}\left\{\left|\hat{\beta}_n(\varphi) - \bar{\zeta}\breve{\beta}(\varphi)\right| > \varepsilon\right\}$ as in (2.18) with τ_n^n and $\tilde{\beta}_n^t$ replaced by t_n^n and $\hat{\beta}_n^t(\varphi)$, respectively, in the definition of B_n and C_n. The limit of B_n can be treated as before. For C_n, we use a decomposition similar to (2.19) with D_n and E_n being defined and bounded as in (2.20-2.21) with τ_n^n replaced by t_n^n. The convergence of E_n to 0 can be proved in a similar manner to the proof of Theorem 2.3. The term

$$D_n := \mathrm{Var}\left(\frac{1}{t_n^n}\sum_{k=1}^n \mathbb{E}\,\left[\varphi(Z_t)\mathbf{1}_{\{|\varphi(Z_t)|\le t_n^n\}}\right]\Big|_{t=\Delta_k^n\tau}\right),$$

requires some care. As before,

$$D_n \leq 2\,\mathbb{E}\left(\frac{1}{t_n^n}\sum_{k=1}^{n}\mathbb{E}\left[|\varphi|(Z_t)\mathbf{1}_{\{|\varphi(Z_t)|>t_n^n\}}\right]\Big|_{t=\Delta_k^n\tau}\right)^2 \tag{4.4}$$

$$+ 2\mathrm{Var}\left(\frac{1}{t_n^n}\sum_{k=1}^{n}H_\varphi\left(\Delta_k^n\tau\right)-\bar{\zeta}(g)\breve{\beta}(\varphi)\right), \tag{4.5}$$

Fix $T_0>0$ and let $t_0>0$ (depending on T_0) such that

$$\mathbb{E}\left[|\varphi|(Z_t)\mathbf{1}_{\{|\varphi(Z_t)|>T_0\}}\right]\leq t\left(2\int|\varphi|\mathbf{1}_{|\varphi|\geq T_0}d\nu\vee T_0^{-1}\right),$$

for any $0<t<t_0$. Then, when $t_n^n>T_0$, the term in (4.4) can be bounded by

$$c\,\mathbb{E}\left\{\frac{1}{t_n^n}\int_0^{t_n^n}r(u)du\right\}^2\left\{\int|\varphi|\mathbf{1}_{|\varphi|\geq T_0}d\nu\vee T_0^{-1}\right\}^2+c'M_\varphi\,\mathbb{E}\left\{\frac{1}{t_n^n}\sum_{k=1}^{n}\Delta_k^n\tau\mathbf{1}_{\{\Delta_k^n\tau\geq t_0\}}\right\}^2,$$

for some constants $c,c'>0$. The above bound converges to

$$c\bar{\zeta}^2(g)\left\{\int|\varphi|\mathbf{1}_{|\varphi|\geq T_0}d\nu\vee T_0^{-1}\right\}^2,$$

in view of (4.1), the ergodicity of $\tilde{r}$, and (4.2). Making $T_0\to\infty$, we conclude that the term in (4.4) vanishes. The term in (4.5), that we denote F_n, can be bounded in the following manner:

$$F_n\leq 2\,\mathbb{E}\left\{\frac{1}{t_n^n}\sum_{k=1}^{n}\Delta_k^n\tau\left(\frac{1}{\Delta_k^n\tau}H_\varphi\left(\Delta_k^n\tau\right)-\breve{\beta}(\varphi)\right)\right\}^2$$
$$+2\breve{\beta}(\varphi)^2\,\mathbb{E}\left\{\frac{1}{t_n^n}\int_0^{t_n^n}r(u)du-\bar{\zeta}(g)\right\}^2$$

The second term on the right-hand side above converges to 0 because of the ergodicity of $\tilde{r}$ and (4.2). For a fixed ε, the first term can be decomposed into two sums, when $\Delta_k^n\tau<t_0$ and when $\Delta_k^n\tau\geq t_0$, where $t_0=t_0(\varepsilon)$ is such that $|H_\varphi(t)/t-\breve{\beta}(\varphi)|<\varepsilon$ whenever $0<t<t_0$. We then take the limits when $n\to\infty$ and use that $\varepsilon>0$ is arbitrary.

It only remains to check that (4.2) implies (4.1) whenever $\bar{\delta}^n\to 0$. Indeed, by Jensen's inequality,

$$\mathbb{E}\left\{\frac{1}{t_n^n}\sum_{k=1}^{n}\Delta_k^n\tau\mathbf{1}_{\{\Delta_k^n\tau\geq t_0\}}\right\}^2\leq\frac{1}{t_0^\varepsilon}\,\mathbb{E}\left\{\frac{1}{t_n^n}\sum_{k=1}^{n}(\Delta_k^n\tau)^{1+\varepsilon/2}\right\}^2 \tag{4.6}$$
$$\leq\frac{1}{t_0^\varepsilon}\,\mathbb{E}\left\{\frac{1}{t_n^n}\sum_{k=1}^{n}(t_k^n-t_{k-1}^n)^\varepsilon\int_{t_{k-1}^n}^{t_k^n}|r(u)|^{2+\varepsilon}du\right\}$$
$$\leq\frac{m_{2+\varepsilon}(g)}{t_0^\varepsilon}(\bar{\delta}^n)^\varepsilon,$$

which converges to 0. □

5. Central limit theorems

In this part we investigate conditions for the asymptotic normality of the estimators (1.6). In the case of a true Lévy process, Figueroa-López [14] proves this result assuming that φ is bounded, ν-continuous, and such that $\varphi(x) = o(|x|)$ as $x \to 0$. The random clock case is more challenging as in this case $\hat{\beta}_n$ is not the sum of independent random variables. We use the central limit theorems for martingale differences (see e.g. Billingsley [3, Theorem 18.1]). Specifically, given a filtration $\{\mathcal{F}_k^n\}_{k\geq 0}$ for each $n \geq 0$, if $\xi_{k,n}$ is $\mathcal{F}_k^n$-measurable and $\mathbb{E}\left[\xi_{k,n}|\mathcal{F}_{k-1}^n\right] = 0$, then

$$S_n := \sum_{k=1}^{\infty} \xi_{k,n} \stackrel{\mathfrak{D}}{\to} \sigma \mathcal{N}(0,1), \tag{5.1}$$

for a constant $\sigma \geq 0$ provided that the two conditions below are satisfied as $n \to \infty$, for any $\varepsilon > 0$:

$$\sum_{k=1}^{\infty} \mathbb{E}\left[\xi_{k,n}^2|\mathcal{F}_{k-1}^n\right] \xrightarrow{\mathbb{P}} \sigma^2, \quad \sum_{k=1}^{\infty} \mathbb{E}\left[\xi_{k,n}^2 \mathbf{1}_{|\xi_{k,n}|\geq\varepsilon}\right] \to 0. \tag{5.2}$$

In this part, we take

$$\mathcal{F}_k^n := \sigma\left(X_u : u \leq t_k^n\right) \vee \sigma\left(\tau(u) : u \leq t_k^n\right), \tag{5.3}$$

for given sampling points $0 = t_0^n < \cdots < t_n^n := T_n$. We consider the following martingale difference sequence:

$$\xi_{k,n} := T_n^{-1/2}\left(\varphi(X_{t_k^n} - X_{t_{k-1}^n}) - \mathbb{E}\left[\varphi(X_{t_k^n} - X_{t_{k-1}^n})|\mathcal{F}_{k-1}^n\right]\right), \tag{5.4}$$

for $1 \leq k \leq n$, and $\xi_{k,n} = 0$, otherwise. Define $\mathcal{F}_t^Z := \sigma(Z_u : u \leq t)$, and $\mathcal{F}_t^\tau := \sigma(\tau_u : u \leq t)$. Note that if φ satisfies the condition (i) of Proposition 3.1 and $\mathbb{E}\,\tau(t) < \infty$, for all $t \geq 0$, then $\mathbb{E}\,|H_\varphi(\Delta_k^n \tau)| < \infty$ and

$$\mathbb{E}\left[\varphi(X_{t_k^n} - X_{t_{k-1}^n})|\mathcal{F}_{k-1}^n\right] = \mathbb{E}\left[H_\varphi(\Delta_k^n\tau)|\mathcal{F}_{t_{k-1}}^\tau\right], \tag{5.5}$$

where we recall that $\Delta_k^n\tau := \tau(t_k^n) - \tau(t_{k-1}^n)$ and $H_\varphi(t) := \mathbb{E}\,\varphi(Z_t)$ (see the end of Appendix C for more details on (5.5)). We can then write (5.4) as

$$\xi_{k,n} := T_n^{-1/2}\left(\varphi(X_{t_k^n} - X_{t_{k-1}^n}) - \mathbb{E}\left[H_\varphi(\Delta_k^n\tau)|\mathcal{F}_{t_{k-1}}^\tau\right]\right). \tag{5.6}$$

Also, we have that

$$S_n := \sum_{k=1}^{\infty} \xi_{k,n} = T_n^{1/2}\left(\hat{\beta}_n(\varphi) - \check{\beta}_n(\varphi)\right),$$

where

$$\breve{\beta}_n(\varphi) := \frac{1}{T_n}\sum_{k=1}^{n} \mathbb{E}\,[H_\varphi(\Delta_k^n\tau)|\mathcal{F}_{t_{k-1}}^{\tau}].$$

Our first task is to find conditions for

$$\lim_{n\to\infty} \breve{\beta}_n(\varphi) \stackrel{\mathbb{P}}{=} \bar{\zeta}(g)\breve{\beta}(\varphi). \tag{5.7}$$

Lemma 5.1. *Consider the model (1.3)-(1.4) assuming the setting (A)-(B) and condition (i) of Proposition 3.1. Also, assume that g is Lipschitz on $\mathbb{R}$ satisfying (4.2) for some $\varepsilon > 0$, and that $\tilde{r}$ satisfies (3.5) and condition (B') in Proposition 3.3. Then, (5.7) holds whenever $T_n \nearrow \infty$ and $\bar{\delta}^n \to 0$*

Proof. Since $\mathbb{E}\,\breve{\beta}_n(\varphi) = \mathbb{E}\,\beta_n(\varphi)$, it follows that $\lim_{n\to\infty}\mathbb{E}\,\breve{\beta}_n(\varphi) = \bar{\zeta}(g)\breve{\beta}(\varphi)$ in light of Theorem 4.1. Hence, it suffices to show that

$$\lim_{n\to\infty} \mathrm{Var}\left(\breve{\beta}_n(\varphi)\right) = 0. \tag{5.8}$$

For a given $\varepsilon > 0$, let $t_0 := t_0(\varepsilon)$ be such that $\left|H_\varphi(t)/t - \breve{\beta}(\varphi)\right| < \varepsilon$, whenever $0 < t < t_0$. Next, decomposing $H_\varphi(\Delta_k^n\tau)$ as follows

$$\left(\frac{1}{\Delta_k^n\tau}H_\varphi(\Delta_k^n\tau) - \breve{\beta}(\varphi)\right)\Delta_k^n\tau\left(\mathbf{1}_{\Delta_k^n\tau<t_0} + \mathbf{1}_{\Delta_k^n\tau\geq t_0}\right) + \breve{\beta}(\varphi)\Delta_k^n\tau,$$

with the convention that $0/0 = 0$, and using (2.9), it follows that

$$\mathrm{Var}\left(\breve{\beta}_n(\varphi)\right) \leq 4\varepsilon^2\,\mathbb{E}\,(\rho_{n,1}^2) + 4(M_\varphi + |\breve{\beta}(\varphi)|)\,\mathbb{E}\,(\rho_{n,2}^2) + 4\breve{\beta}(\varphi)^2\mathrm{Var}\,(\rho_{n,1})\,, \tag{5.9}$$

where

$$\rho_{n,1} := \frac{1}{T_n}\sum_{k=1}^{n}\mathbb{E}\,[\Delta_k^n\tau|\mathcal{F}_{t_{k-1}}^{\tau}], \quad \rho_{n,2} := \frac{1}{T_n}\sum_{k=1}^{n}\mathbb{E}\,[\Delta_k^n\tau\mathbf{1}_{\{\Delta_k^n\tau\geq t_0\}}|\mathcal{F}_{t_{k-1}}^{\tau}].$$

First, we note that (4.2) implies (3.1), and by Jensen's inequality,

$$\mathbb{E}\,\rho_{n,1}^2 \leq \frac{1}{T_n}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\mathbb{E}\,r^2(u)du \leq m_2(g).$$

Let $I_k^n := [t_{k-1}^n, t_k^n]$ and $\delta_k^n := t_k^n - t_{k-1}^n$. Following a procedure similar to (4.6), we can obtain that

$$\rho_{n,2}^2 \leq \frac{t_0^{-\varepsilon}}{T_n}\sum(\delta_k^n)^\varepsilon\,\mathbb{E}\left[\int_{t_{k-1}^n}^{t_k^n}|g(\tilde{r}(u))|^{2+\varepsilon}du\,\middle|\,\mathcal{F}_{t_{k-1}}^{\tau}\right],$$

Then, using (4.2), $\mathbb{E}\,\rho_{n,2}^2 \le t_0^{-\varepsilon}(\bar{\delta}^n)^{\varepsilon} m_{2+\varepsilon}(g) \to 0$. Let us analyze the last term in (5.9). First,

$$\operatorname{Var}(\rho_{n,1}) \le 2\operatorname{Var}\left(\rho_{n,1} - \frac{1}{T_n}\tau(T_n)\right) + 2\operatorname{Var}\left(\frac{1}{T_n}\tau(T_n)\right). \tag{5.10}$$

Using Jensen's inequality and the fact that

$$\mathbb{E}\,\left(\mathbb{E}\left[r(u)|\mathcal{F}^{\tau}_{t^n_{k-1}}\right] - r(u)\right)^2 \le \mathbb{E}\,\left(r(t^n_{k-1}) - r(u)\right)^2,$$

the first term on the right-hand side of (5.10) can be bounded as follows:

$$\frac{1}{T_n}\sum_{k=1}^{n}\int_{t^n_{k-1}}^{t^n_k} \mathbb{E}\,\left(r(t^n_{k-1}) - r(u)\right)^2 du \le \frac{K}{T_n}\sum_{k=1}^{n}\delta^n_k\,\mathbb{E}\sup_{u\in I^n_k}\left|\tilde{r}(u) - \tilde{r}(t^n_{k-1})\right|^2,$$

where K is the Lipschitz constant of g. This converges to 0 in light of (3.8) and (3.5). The second term in the right hand side of (5.10) converges to 0 since

$$\operatorname{Var}\left(\frac{1}{T_n}\tau(T_n)\right) \le \mathbb{E}\,\left(\frac{1}{T_n}\int_0^{T_n} r(u)du - \bar{\zeta}(g)\right)^2 \stackrel{n\to\infty}{\longrightarrow} 0,$$

The above limit is a consequence of the ergodic theorem (1.13) and the fact that $(T_n^{-1}\int_0^{T_n} r(u)du)^2$ is uniformly integrable, which in turn is guaranteed by (4.2). We finally conclude that

$$\limsup_{n\to\infty} \operatorname{Var}\left(\check{\beta}_n(\varphi)\right) \le 4m_2(g)\varepsilon^2,$$

and since ε is arbitrary, (5.8) follows. □

Proposition 5.2. *Suppose that the conditions of Lemma 5.1 hold true and also that φ^2 satisfies Conditions 2.1 and (2.9). Then, when $T_n \nearrow \infty$ and $\bar{\delta}^n \to 0$,*

$$T_n^{1/2}\left(\hat{\beta}_n(\varphi) - \check{\beta}_n(\varphi)\right) \stackrel{\mathfrak{D}}{\to} \sigma(\varphi)\mathcal{N}(0,1), \tag{5.11}$$

with $\sigma^2(\varphi) := \bar{\zeta}(g)\check{\beta}(\varphi^2)$

Proof. We need to check that (5.2) holds for (5.3) and (5.6). First,

$$\sigma_n^2 := \sum_{k=1}^{\infty}\mathbb{E}\,[\xi_{k,n}^2|\mathcal{F}^n_{k-1}] = \check{\beta}_n(\varphi^2) - \frac{1}{T_n}\sum_{k=1}^{n}\left\{\mathbb{E}\,\left[H_\varphi(\Delta^n_k\tau)|\mathcal{F}^{\tau}_{t_{k-1}}\right]\right\}^2.$$

In light of Lemma 5.1, $\check{\beta}_n(\varphi^2) \stackrel{\mathbb{P}}{\longrightarrow} \sigma^2(\varphi)$. The second term on the right-hand side, that we denote A_n, converges in probability to 0 since

$$\begin{aligned}\mathbb{P}\left(|A_n| \ge \varepsilon\right) &\le \frac{1}{\varepsilon T_n}\mathbb{E}\sum_{k=1}^{n}\left\{\mathbb{E}\,\left[H_\varphi(\Delta^n_k\tau)|\mathcal{F}^{\tau}_{t_{k-1}}\right]\right\}^2 \\ &\le \frac{M_\varphi^2 m_2(g)}{\varepsilon}\frac{1}{T_n}\sum_{k=1}^{n}(t^n_k - t^n_{k-1})^2 \stackrel{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

Now consider

$$B_n := \frac{1}{T_n}\sum_{k=1}^{n}\mathbb{E}\left\{\varphi^2(X_{t_k}-X_{t_{k-1}})\mathbf{1}_{\{|\varphi(X_{t_k}-X_{t_{k-1}})|\geq T_n^{1/2}\varepsilon/2\}}\right\}.$$

Fix a $T_0>0$ and let $t_0>0$ be such that $|H_{\varphi^2\mathbf{1}_{|\varphi|\geq T_0}}(t)|\leq 2t\beta(\varphi^2\mathbf{1}_{|\varphi|\geq T_0})$, for all $0<t<t_0$. Then, conditioning on $\{\tau_k^n\}_{k=1}^n$, for n large enough,

$$B_n \leq 2\beta(\varphi^2\mathbf{1}_{|\varphi|\geq T_0})\frac{1}{T_n}\sum_{k=1}^{n}\mathbb{E}\,\Delta_k^n\tau + M_{\varphi^2}\frac{1}{T_n}\sum_{k=1}^{n}\mathbb{E}\left\{\Delta_k^n\tau\mathbf{1}_{\{\Delta_k^n\tau>t_0\}}\right\},$$

which limsup is bounded by $2\beta(\varphi^2\mathbf{1}_{|\varphi|\geq T_0})$ since $(1/T_n)\sum_{k=1}^n\mathbb{E}\,\Delta_k^n\tau\leq m_2^{1/2}$ and (4.2) implies (4.1) (see the last part in the proof of Theorem 4.1), which in turn implies that the second term on the right-hand side above vanishes. Since T_0 can be made arbitrarily large, $\limsup_{n\to\infty}B_n=0$. Next, conditioning on $\mathcal{F}_{k-1}^n$,

$$\begin{aligned}
C_n &:= \frac{1}{T_n}\sum_{k=1}^{n}\mathbb{E}\left\{\varphi^2(X_{t_k}-X_{t_{k-1}})\mathbf{1}_{\{|\mathbb{E}[H_\varphi(\Delta_k^n\tau)|\mathcal{F}_{t_{k-1}}^\tau]|\geq T_n^{1/2}\varepsilon/2\}}\right\}\\
&\leq M_{\varphi^2}\frac{1}{T_n}\sum_{k=1}^{n}\mathbb{E}\left\{\mathbb{E}[\Delta_k^n\tau|\mathcal{F}_{t_{k-1}}^\tau]\mathbf{1}_{\{|\mathbb{E}[\Delta_k^n\tau|\mathcal{F}_{t_{k-1}}^\tau]|\geq M_\varphi^{-1}T_n^{1/2}\varepsilon/2\}}\right\},\\
&\leq M_{\varphi^2}M_\varphi\varepsilon^{-1}\frac{1}{T_n^{3/2}}\sum_{k=1}^{n}\mathbb{E}\left\{\mathbb{E}[\Delta_k^n\tau|\mathcal{F}_{t_{k-1}}^\tau]^2\right\},
\end{aligned}$$

which can be shown to converges to 0 as D_n below converges to 0. Using Jensen's inequality,

$$D_n := \frac{1}{T_n}\sum_{k=1}^{n}\mathbb{E}\left\{\mathbb{E}[H_\varphi(\Delta_k^n\tau)|\mathcal{F}_{t_{k-1}}^\tau]^2\right\}\leq\frac{M_\varphi^2}{T_n}\sum_{k=1}^{n}(t_k^n-t_{k-1}^n)\int_{t_{k-1}^n}^{t_k^n}\mathbb{E}\,r^2(u)du,$$

which clearly converges to 0 in light of (4.2). Thus, we obtain the second limit in (5.2) because

$$\sum_{k=0}^{\infty}\mathbb{E}[\xi_{k,n}^2\mathbf{1}_{|\xi_{k,n}|\geq\varepsilon}]\leq 2B_n+2C_n+D_n.$$

In light of the Central Limit Theorem for martigale differences stated at the beginning of this section, we obtain (5.11). □

We proceed to show a central limit theorem for $\breve{\beta}_n(\varphi)$ of the form

$$T_n^{1/2}\left(\breve{\beta}_n(\varphi)-\bar{\zeta}(g)\breve{\beta}(\varphi)\right)\xrightarrow{\mathfrak{D}}\breve{\beta}(\varphi)\Gamma^{1/2}(g)\mathcal{N}(0,1), \qquad (5.12)$$

for certain positive constant $\Gamma(g)$. This result suggests a CLT of the form

$$T_n^{1/2}\left(\hat{\beta}_n(\varphi)-\bar{\zeta}(g)\breve{\beta}(\varphi)\right)\xrightarrow{\mathfrak{D}}(\sigma^2(\varphi)+\breve{\beta}(\varphi)^2\Gamma(g))^{1/2}\mathcal{N}(0,1). \tag{5.13}$$

However, we haven't been able to obtain such a result and we expect to address this issue in a future work. We shall make an assumption on the rate of convergence in (2.6).

Conditions 5.1. *There exists a $t_0>0$ such that*

$$\left|\frac{1}{t}\ \mathbb{E}\,\varphi(X_t)-\breve{\beta}(\varphi)\right|\leq k_0 t, \tag{5.14}$$

for any $0<t<t_0$ and for a constant k_0 independent of t.

Remark 5.3. *Condition 5.1 turns out to hold for a wide class of functions φ such as the following:*

1. *φ is supported on an interval $[c,d]\subset\mathbb{R}\backslash\{0\}$, where φ is continuous with continuous derivative (c.f. [12]).*
2. *$\varphi\in C^2$ vanishes in a neighborhood of the origin, and for each $i=0,1,2$, $|\varphi^{(i)}|$ is bounded by an element g_i in the class $\mathcal{S}(\nu)$ of (2.5) (c.f. [13]).*
3. *φ is bounded, ν-continuous, satisfying any of the conditions (a)-(f) in Conditions 2.1. This result can be shown using a second order polynomial expansion for smooth functions and mollifiers techniques similar to the expansions obtained in [16].*

Proposition 5.4. *Suppose that Condition 5.1 above holds as well as the conditions of Proposition 5.2 with $T_n\bar{\delta}^n\stackrel{n\to\infty}{\longrightarrow}0$. Then, (5.12) holds true.*

Proof. We recall that under the stated conditions, the diffusion $\{\tilde{r}(t)\}_{t\geq 0}$ obeys the central limit theorem

$$\sqrt{t}\left(\frac{1}{t}\int_0^t g(\tilde{r}(u))du-\int g(x)\zeta(dx)\right)\xrightarrow{\mathfrak{D}}\mathcal{N}(0,\Gamma(g)), \tag{5.15}$$

for a certain constant $\Gamma(g)\geq 0$ (see [23] for an explicit formula for $\Gamma(g)$ and references therein for a proof). Also,

$$T_n^{1/2}\left(\breve{\beta}_n(\varphi)-\bar{\zeta}(g)\breve{\beta}(\varphi)\right)=\breve{\beta}(\varphi)T_n^{-1/2}\int_0^{T_n}\left(g(\bar{r}(u))-\bar{\zeta}(g)\right)du+R_n,$$

where

$$R_n:=T_n^{-1/2}\sum_{k=1}^n\mathbb{E}\left[\left(\frac{1}{\Delta_k^n}H_\varphi(\Delta_k^n\tau)-\breve{\beta}(\varphi)\right)\Delta_k^n\tau\middle|\mathcal{F}^\tau_{t_{k-1}}\right] \tag{5.16}$$

$$+\breve{\beta}(\varphi)T_n^{1/2}\left\{\frac{1}{T_n}\sum_{k=1}^n\mathbb{E}\left[\Delta_k^n\tau|\mathcal{F}^\tau_{t_{k-1}}\right]-\frac{1}{T_n}\int_0^{T_n}r(u)du\right\}. \tag{5.17}$$

Thus, to show (5.12) it suffices that R_n converge to 0 in probability. Denote A_n the first term on the right-hand side of (5.16). Note that without loss of generality, one can assume that (5.14) holds for all $t > 0$. Then,

$$\mathbb{P}\left[|A_n| \geq \varepsilon\right] \leq \frac{k_0}{\varepsilon} T_n^{-1/2} \sum_{k=1}^{n} \mathbb{E}\,(\Delta_k^n \tau)^2$$
$$\leq \frac{k_0}{\varepsilon} T_n^{-1/2} \sum_{k=1}^{n} (t_k^n - t_{k-1}^n)\, \mathbb{E} \int_{t_{k-1}^n}^{t_k^n} r^2(u) du \leq \frac{m_2(g) k_0}{\varepsilon} T_n^{1/2} \bar{\delta}^n,$$

which converges to 0. For (5.17), we proceed as before when proving that the first term on the right-hand side of (5.10) converges to 0. Indeed, denoting B_n the left-hand side on the left-hand side of (5.17) and using Markov's inequality, Lipschitz condition of g, (3.5), and (3.8), we obtain that

$$\mathbb{P}\left[|B_n| \geq \varepsilon\right] \leq \frac{1}{\varepsilon^2} \sum_{k=1}^{n} \delta_k^n\, \mathbb{E} \sup_{u \in I_k^n} \left| r(u) - r(t_{k-1}^n) \right|^2 \leq \frac{K}{\varepsilon^2} T_n \bar{\delta}^n \overset{n \to \infty}{\longrightarrow} 0.$$

This concludes the proof. □

Note that $T_n^{1/2} \left(\hat{\beta}_n(\varphi) - \bar{\zeta}(g) \breve{\beta}(\varphi) \right)$ can be decomposed as follows:

$$T_n^{1/2} \left(\hat{\beta}_n(\varphi) - \breve{\beta}_n(\varphi) \right) + \breve{\beta}(\varphi) T_n^{-1/2} \int_0^{T_n} \left(g(\tilde{r}(u)) - \bar{\zeta}(g) \right) du + R_n, \qquad (5.18)$$

where R_n is defined as in (5.16). In the proof of Proposition 5.4, it was shown that R_n converges to 0 in probability, while each of the first two terms converge to a normal distribution in light of (5.11) and (5.15). To conclude (5.13), it will suffice that the first two terms converge jointly in distribution, an issue that we are actively pursuing as of this writing.

Appendix A: The pure Lévy case

In this part we present the proof of Theorem 2.5. Even though the proof below can be simplified, we choose this line of reasoning to show in an explicit manner how the constants T and δ depend on $\varepsilon > 0$. For instance, in the case that φ is bounded, one can deduce that T grows of the order of ε^{-3} when $\varepsilon \to 0$, while δ will have to be chosen such that (a)-(c) below are satisfied.

Proof. Through this part, c_h denotes an even continuous function such that $\mathbf{1}_{\{|x|>2h\}} \leq c_h(x) \leq \mathbf{1}_{\{|x|>h\}}$. Let h be large enough that $(\int |\varphi(x)| c_h(x) \nu(dx) \vee h^{-1}) < \varepsilon^3/2^6$, and let $0 < u_0 < 1$ be small enough that $u_0 \left(\breve{\beta}(|\varphi|) \vee 1 \right) < \varepsilon^3/2^6$.

Take T large enough that $u_0 T > \sup_{|x|\leq 2h} |\varphi(x)|$. Choose $\delta > 0$ such that for all $0 < t < \delta$,

$$(a)\ \mathbb{E}\left[|\varphi(X_t)|\, c_h(X_t)\right] \leq 2t \left(\int |\varphi(x)| c_h(x) \nu(dx) \vee h^{-1} \right),$$

$$(b)\ \mathbb{E}\left[|\varphi(X_t)|\right] \leq 2t \left(\breve{\beta}(|\varphi|) \vee 1 \right), \quad (c)\ \left| \frac{1}{t} \mathbb{E}\, \varphi(X_t) - \breve{\beta}(\varphi) \right| < \varepsilon/4.$$

The existence of such a δ is guaranteed because (2.6) holds for φ and also for $|\varphi|\, c_h$ and $|\varphi|$. Suppose that $\pi : 0 = t_0 < \cdots < t_n$ is any sequence such that $t_n > T$ and $\max_k (t_k - t_{k-1}) < \delta$. We apply similar arguments as in the proof of Theorem 2.3. Let $\delta_k := t_k - t_{k-1}$ and $A^\pi := \sum_{k=1}^{n} \mathbb{E} \left[\varphi(X_{\delta_k}) \mathbf{1}_{\{|\varphi(X_{\delta_k})| \leq t_n\}} \right]$. As in (2.18), it can be prove that if

$$\left| \frac{A^\pi}{t_n} - \breve{\beta}(\varphi) \right| < \varepsilon/2, \tag{A.1}$$

then

$$\mathbb{P}\left\{ \left| \hat{\beta}^\pi(\varphi) - \breve{\beta}(\varphi) \right| > \varepsilon \right\} \leq B^\pi + \frac{4}{\varepsilon^2} C^\pi, \tag{A.2}$$

where

$$B^\pi := \sum_{k=1}^{n} \mathbb{P}\left[|\varphi(X_{\delta_k})| > t_n \right], \quad C^\pi := \frac{1}{t_n^2} \sum_{k=1}^{n} \mathbb{E} \left[|\varphi(X_{\delta_k})|^2 \mathbf{1}_{\{|\varphi(X_{\delta_k})| \leq t_n\}} \right].$$

Due to the way h, T, and δ were chosen, it follows that

$$\begin{aligned} \mathbb{E} \left[|\varphi(X_{\delta_k})|\, \mathbf{1}_{\{|\varphi(X_{\delta_k})| > t_n\}} \right] &\leq \mathbb{E}\left[|\varphi(X_{\delta_k})|\, c_h(X_{\delta_k}) \right] \\ &\leq 2\delta_k \left(\int |\varphi(x)| c_h(x) \nu(dx) \vee h^{-1} \right), \end{aligned}$$

which is smaller than $\delta_k \varepsilon^2 / 2^5 < \delta_k \varepsilon / 4$, and thus, using Markov's inequality,

$$B^\pi \leq \frac{1}{t_n} \sum_{k=1}^{n} \mathbb{E} \left[|\varphi(X_{\delta_k})|\, \mathbf{1}_{\{|\varphi(X_{\delta_k})| > t_n\}} \right] < \varepsilon/4. \tag{A.3}$$

Next, applying (2.24),

$$\frac{1}{t_n^2} \mathbb{E} \left[|\varphi(X_{\delta_k})|^2 \mathbf{1}_{\{|\varphi(X_{\delta_k})| \leq t_n\}} \right] \leq \frac{2}{t_n} \int_{(0,1)} \mathbb{E} \left[|\varphi(X_{\delta_k})|\, \mathbf{1}_{\{|\varphi(X_{\delta_k})| > u t_n\}} \right] du,$$

and thus, $C^\pi \leq 2 \int_{(0,1)} s_n(u) du$, where

$$s_n(u) := \frac{1}{t_n} \sum_{k=1}^{n} \mathbb{E} \left[|\varphi(X_{\delta_k})| \mathbf{1}_{\{|\varphi(X_{\delta_k})| > u t_n\}} \right].$$

By (b) above, $s_n(u) \le 2\left(\breve{\beta}(|\varphi|) \vee 1\right)$, for any $u \in (0,1)$. Furthermore, $s_n(u) \le \varepsilon^3/2^5$, if $u \in (u_0, 1)$, in light of the following bounds:

$$\begin{aligned}\mathbb{E}\left[|\varphi(X_{\delta_k})|\mathbf{1}_{\{|\varphi(X_{\delta_k})|>ut_n\}}\right] &\le \mathbb{E}\left[|\varphi(X_{\delta_k})|\mathbf{1}_{\{|\varphi(X_{\delta_k})|>u_0t_n\}}\right]\\ &\le \mathbb{E}\left[|\varphi(X_{\delta_k})|c_h(X_{\delta_k})\right] \le 2\delta_k \int |\varphi(x)|c_h(x)\nu(dx) < \delta_k \frac{\varepsilon^3}{2^5}.\end{aligned}$$

Combining the two bounds for s_n,

$$C^\pi \le 2\int_0^{u_0} s_n(u)du + 2\int_{u_0}^1 s_n(u)du \le 4u_0\left(\breve{\beta}(|\varphi|) \vee 1\right) + \frac{\varepsilon^3}{2^4} < \frac{\varepsilon^3}{8}.$$

Using the previous inequality and (A.3), we conclude that (A.2) is bounded by ε. It only remains to prove (A.1). Using the second inequality in (A.3),

$$\left|\frac{A^\pi}{t_n} - \frac{1}{t_n}\sum_{k=1}^n \mathbb{E}\,\varphi(X_{\delta_k})\right| \le \frac{1}{t_n}\sum_{k=1}^n \mathbb{E}\left[|\varphi(X_{\delta_k})|\mathbf{1}_{\{|\varphi(X_{\delta_k})|>t_n\}}\right] < \varepsilon/4,$$

By (c) above, $\left|\frac{1}{t_n}\sum_{k=1}^n \mathbb{E}\,\varphi(X_{\delta_k}) - \breve{\beta}(\varphi)\right| \le \varepsilon/4$, implying that $\left|\frac{A^\pi}{t_n} - \breve{\beta}(\varphi)\right| < \varepsilon/2$. □

Appendix B: A moment estimate for diffusions

In this part we prove the moment estimate (3.8) for the solution $\tilde{r}$ of the SDE (3.6) under the linear growth condition (3.7). The ideas are classical (see e.g. the solution to Problem 5.3.15 in [19]). Below, k_m stands for a generic constant depending on m. First, note that

$$|\tilde{r}(s_0+s) - \tilde{r}(s_0)|^{2m} \le k_m \left\{\left|\int_{s_0}^{s_0+s} b(\tilde{r}(u))du\right|^{2m} + \left|\int_{s_0}^{s_0+s} \sigma(\tilde{r}(u))dW_u\right|^{2m}\right\}.$$

By Jensen's inequality and (3.7), $|\int_{s_0}^{s_0+s} b(\tilde{r}(u))du|^{2m}$ can be bounded as follows:

$$\begin{aligned}s^{2m-1}\int_{s_0}^{s_0+s} |b(\tilde{r}(u))|^{2m}du &\le k_m s^{2m-1}\int_{s_0}^{s_0+s}\left(1+|\tilde{r}(u)|^{2m}\right)du\\ &\le k_m s^{2m-1}\left(s + s|\tilde{r}(s_0)|^{2m} + \int_{s_0}^{s_0+s}|\tilde{r}(u)-\tilde{r}(s_0)|^{2m}du\right).\end{aligned}$$

Let $\tau_k := \inf\{s \ge 0 : |\tilde{r}(s+s_0)| \ge k\}$. By Davis-Burkhölder-Gundy inequality,

$$\begin{aligned}\mathbb{E}\sup_{s\le h\wedge\tau_k}\left|\int_{s_0}^{s_0+s}\sigma(\tilde{r}(u))dW_u\right|^{2m} &\le k_m\,\mathbb{E}\left|\int_{s_0}^{s_0+h\wedge\tau_k}\sigma^2(\tilde{r}(u))du\right|^m\\ &\le k_m h^{m-1}\,\mathbb{E}\int_{s_0}^{s_0+h\wedge\tau_k}\sigma^{2m}(\tilde{r}(u))du.\end{aligned}$$

As with $b(\cdot)$, we have the following bound for any $s \geq 0$,

$$\int_{s_0}^{s_0+s} \sigma^{2m}(\tilde{r}(u))du \leq k_m \left(s + s|\tilde{r}(s_0)|^{2m} + \int_{s_0}^{s_0+s} |\tilde{r}(u) - \tilde{r}(s_0)|^{2m} du \right).$$

Then, using that $0 < h \leq 1$ (and hence, $h^{2m} \leq h^m$),

$$\begin{aligned} \mathbb{E} \sup_{s \leq h \wedge \tau_k} |\tilde{r}(s_0+s) - \tilde{r}(s_0)|^{2m} &\leq k_m \mathbb{E} \sup_{s \leq h \wedge \tau_k} \left| \int_{s_0}^{s_0+s} b(\tilde{r}(u))du \right|^{2m} \\ &\quad + k_m \mathbb{E} \sup_{s \leq h \wedge \tau_k} \left| \int_{s_0}^{s_0+s} \sigma(\tilde{r}(u))dW_u \right|^{2m} \\ &\leq k_m h^m + k_m h^m \mathbb{E} |\tilde{r}(s_0)|^{2m} \\ &\quad + k_m h^{m-1} \int_0^h \mathbb{E} \sup_{s \leq u \wedge \tau_k} |\tilde{r}(s_0+s) - \tilde{r}(s_0)|^{2m} du. \end{aligned}$$

Denoting $\gamma_k(h) := \mathbb{E} \sup_{s \leq h \wedge \tau_k} |\tilde{r}(s_0+s) - \tilde{r}(s_0)|^{2m}$, we obtained the inequality

$$\gamma_k(h) \leq k_m h^m \left(1 + \mathbb{E} |\tilde{r}(s_0)|^{2m}\right) + k_m \int_0^h \gamma_k(u)du.$$

Finally, by Gronwall inequality (see [19]), $\gamma_k(h) \leq k_m h^m \left(1 + \mathbb{E} |\tilde{r}(s_0)|^{2m}\right) e^{k_m h}$. Inequality (3.8) will follow from making $k \to \infty$.

Appendix C: Conditional expectation given the random clock

In several occasions, we use conditional expectations of the time-changed Lévy model $X_t := Z_{\tau(t)}$ given the random clock τ and/or past evolution of X. In this part we intend to formalize this procedure under the assumption that Z and τ are independent.

(1) Let $0 \leq t_0 < \cdots < t_n < \infty$ and let $\tau_k := \tau(t_k)$. For given $0 \leq s_0 \leq \cdots \leq s_n < \infty$, we first show that the distribution of $Z_{\tau_1} - Z_{\tau_0}, \ldots, Z_{\tau_n} - Z_{\tau_{n-1}}$ given $\tau_0 = s_0, \ldots, \tau_n = s_n$ is the same as that of $Z_{s_1} - Z_{s_0}, \ldots, Z_{s_n} - Z_{s_{n-1}}$. Let

$$M_{s_0,\ldots,s_n}(u_1, \ldots, u_n) = \mathbb{E} \prod_{k=1}^n e^{iu_k(Z_{s_k} - Z_{s_{k-1}})},$$

let $A \in \sigma(\tau_0, \ldots, \tau_n)$, and let $\kappa_m(t) = \sum_{j=1}^{m^2} \frac{j}{m} \mathbf{1}_{[\frac{j-1}{m}, \frac{j}{m})}(t) + m\mathbf{1}_{[m,\infty)}(t)$. First, by the right-contunuity of Z and dominated convergence theorem,

$$\mathbb{E} \prod_{k=1}^n e^{iu_k(Z_{\tau_k} - Z_{\tau_{k-1}})} \chi_A = \lim_{m \to \infty} \mathbb{E} \prod_{k=1}^n e^{iu_k(Z_{\kappa_m(\tau_k)} - Z_{\kappa_m(\tau_{k-1})})} \chi_A.$$

Using the independence of Z and τ, the expectation after the limit in the previous equation can be expressed as follows:

$$\sum_{1\leq j_0\leq\cdots\leq j_n\leq m^2} \mathbb{E}\prod_{k=1}^{n} e^{iu_k(Z_{\frac{j_k}{m}}-Z_{\frac{j_{k-1}}{m}})}\prod_{k=0}^{n}\mathbf{1}_{[\frac{j_k-1}{m},\frac{j_k}{m})}(\tau_k)\chi_A$$
$$=\sum_{j_0\leq\cdots\leq j_n} M_{\frac{j_0}{m},\ldots,\frac{j_n}{m}}(u_1,\ldots,u_n)\,\mathbb{E}\prod_{k=0}^{n}\mathbf{1}_{[\frac{j_k-1}{m},\frac{j_k}{m})}(\tau_k)\chi_A$$
$$=\mathbb{E}\,M_{\kappa_m(\tau_0),\ldots,\kappa_m(\tau_n)}(u_1,\ldots,u_n)\chi_A.$$

Using dominated convergence and right-continuity of Z, the last expression converges to $\mathbb{E}\,M_{\tau_0,\ldots,\tau_n}(u_1,\ldots,u_n)\chi_A$, as $m\to\infty$. Thus, we proved that

$$\mathbb{E}\left(\prod_{k=1}^{n} e^{iu_k(Z_{\tau_k}-Z_{\tau_{k-1}})}\chi_A\right)=\mathbb{E}\left(M_{\tau_0,\ldots,\tau_n}(u_1,\ldots,u_n)\chi_A\right),$$

and hence,

$$\mathbb{E}\left(\prod_{k=1}^{n} e^{iu_k(Z_{\tau_k}-Z_{\tau_{k-1}})}\,\middle|\,\tau_0,\ldots,\tau_n\right)=M_{\tau_0,\ldots,\tau_n}(u_1,\ldots,u_n).$$

One could similarly show that the distribution of $Z_{\tau_1}-Z_{\tau_0},\ldots,Z_{\tau_n}-Z_{\tau_{n-1}}$ given $\mathcal{F}^\tau:=\sigma(\tau(t):t\geq 0)$ is the same distribution as that of $Z_{s_1}-Z_{s_0},\ldots,Z_{s_n}-Z_{s_{n-1}}$ at $s_0=\tau_0,\ldots,s_n=\tau_n$.

(2) As a consequence of the previous result, if $g:\mathbb{R}^n\to\mathbb{R}_+$ is such that

$$\mathbb{E}\,g(Z_{\tau_1}-Z_{\tau_0},\ldots,Z_{\tau_n}-Z_{\tau_{n-1}})<\infty, \tag{C.1}$$

then

$$\mathbb{E}\,g(Z_{\tau_1}-Z_{\tau_0},\ldots,Z_{\tau_n}-Z_{\tau_{n-1}})=\mathbb{E}\,G(\tau_0,\ldots,\tau_n), \tag{C.2}$$

where $G(s_0,\ldots,s_n):=\mathbb{E}\,g(Z_{s_1}-Z_{s_0},\ldots,Z_{s_n}-Z_{s_{n-1}})$. Furthermore, if $g:\mathbb{R}^n\to\mathbb{R}_+$ is continuous and there exists an $M<\infty$ such that $G(s_0,\ldots,s_n)\leq M$ whenever $0\leq s_0\leq\cdots\leq s_n$, then (C.1) holds and hence, (C.2) holds too. Indeed, by Fatou's lemma,

$$\mathbb{E}\,g(Z_{\tau_1}-Z_{\tau_0},\ldots,Z_{\tau_n}-Z_{\tau_{n-1}})$$
$$\leq\liminf_{m\to\infty}\mathbb{E}\,g(Z_{\kappa_m(\tau_1)}-Z_{\kappa_m(\tau_0)},\ldots,Z_{\kappa_m(\tau_n)}-Z_{\kappa_m(\tau_{n-1})})$$
$$\leq\liminf_{m\to\infty}\mathbb{E}\,G(\kappa_m(\tau_0),\ldots,\kappa_m(\tau_n))\leq M.$$

The above reasoning was used for instance to show (2.11) since, under the assumption (2.9), $G(s_0,\ldots,s_n):=(1/s_n)\sum_{k=1}^{n}H_{|\varphi|}(s_k-s_{k-1})\leq M_\varphi$.

(3) Let us show the identity (5.5). Let $\varphi_k, \psi_k : \mathbb{R} \to \mathbb{R}_+$, $k = 1, \ldots, n$, be continuous bounded functions and let $0 = t_0 \leq t_1 < \cdots < t_n \leq t < u < \infty$. Again, we write $\tau_k := \tau(t_k)$. Then, conditioning on $\mathcal{F}^\tau$,

$$\mathbb{E}\left[\varphi(Z_{\tau(u)} - Z_{\tau(t)}) \prod_{k=1}^{n} \varphi_k(Z_{\tau_k})\psi_k(\tau_k)\right] = \mathbb{E}\left[H_\varphi(\tau(u) - \tau(t)) m(\tau_1, \ldots, \tau_n)\right], \tag{C.3}$$

where $H_\varphi(t) := \mathbb{E}\,\varphi(Z_t)$, and $m(s_1, \ldots, s_n) = \mathbb{E}\prod_{k=1}^{n} \varphi_k(Z_{s_k})\psi_k(s_k)$. Since

$$m(\tau_1, \ldots, \tau_n) = \mathbb{E}\left[\prod_{k=1}^{n} \varphi_k(Z_{\tau_k})\psi_k(\tau_k) \middle| \mathcal{F}_t^\tau\right],$$

the right-hand side in (C.3) can be written as follows:

$$\mathbb{E}\left[\mathbb{E}\left[H_\varphi(\tau(u) - \tau(t)) | \mathcal{F}_t^\tau\right] \prod_{k=1}^{n} \varphi_k(Z_{\tau_k})\psi_k(\tau_k)\right].$$

Since $\mathcal{F}_t^\tau \subset \mathcal{F}_t^X \vee \mathcal{F}_t^\tau$, we conclude that

$$\mathbb{E}\left[\varphi(X_u - X_t) | \mathcal{F}_t^X \vee \mathcal{F}_t^\tau\right] = \mathbb{E}\left[H_\varphi(\tau(u) - \tau(t)) | \mathcal{F}_t^\tau\right].$$

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