Asymptotic properties of functionals of increments of a continuous semimartingale with stochastic sampling times

By

M. Levine, Purdue University
X. Wang, Purdue University
J. Zou, Worcester Polytechnic Institute

Technical Report #14-04

Department of Statistics
Purdue University

December 8, 2014

Asymptotic properties of functionals of increments of a continuous semimartingale with stochastic sampling times

M. Levine, Purdue University *
X. Wang, Purdue University
J. Zou, Worcester Polytechnic Institute

December 7, 2014

Abstract

This paper is concerned with asymptotic behavior of the normalized sums of functionals of a variety of continuous semimartingales where observations are sampled at stochastic times. Several laws of large numbers and a major central limit theorem are proved after an appropriate normalization. These results are connected to the needs of financial econometrics in that they provide future foundation for redefinition of the realized kernel estimation of integrated volatility. Realized kernel method is currently one of the most popular methods for estimation of integrated volatility of a price process for high-frequency financial data. The classical definition is based on equispaced data; however, it has to be redefined in cases where the trading times are stochastic. The stochastic trading duration assumption is typically true whenever the tick-by-tick trading data are recorded. Our results provide the foundation of asymptotic theory of the redefined realized kernel estimator which is the subject of our second (forthcoming) article.

Keywords: realized kernel, continuous semimartingale, high-frequency financial data, law of large numbers, central limit theorem

1 Introduction

Over the past decade, the field of volatility modeling and analysis for high-frequency financial data has developed optimistically. The class of realized kernel estimators was first introduced by (Barndorff-Nielsen et al., 2008) to estimate the quadratic variation of a price process from high-frequency data. The idea of realized kernel estimator extends an older kernel estimator proposed by (Zhou, 1996). (Barndorff-Nielsen et al., 2009) conducted an extensive empirical study of realized kernel estimators using real data. The method has been shown to be successful in applications; moreover, it has improved significantly our understanding of time-varying volatility of stochastic processes as well as the ability to predict future volatility. It also became clear that the realized kernel approach is closely related to the Two Scales Realized Volatility (TSRV) idea of (Zhang et al., 2005) and its extension to multiple scales known as MSRV; for details about the latter, see (Zhang et al., 2006) and (Ait-Sahalia et al., 2011). A comprehensive review of this literature was given by (Wang and Zou, 2014).

In this setting, it is usually assumed that the ex post variation of log prices over arbitrary fixed time period is of interest. Such a process is commonly assumed to be a Brownian semimartingale with the spot volatility σ_t. For simplicity, a fixed interval $[0, T]$ for some $T > 0$ is considered. Let $k(x)$ be the non-stochastic weight function defined on $[0, 1]$ and $\delta > 0$ the time gap. The number of observations over the interval $[0, T]$ is then $n = \lfloor \frac{T}{\delta} \rfloor$. Define the integer bandwidth $H > 0$; then, for a continuous time log price process X_t and a time gap $\delta > 0$, the realized autocovariance of order h, $h = -H, -H+1, \ldots, H-1, H$ is

$$\gamma_h(X_\delta) = \sum_{j=1}^{n} (X_{\delta j} - X_{\delta(j-1)})(X_{\delta(j-h)} - X_{\delta(j-h-1)}).$$

Note that, in this setting, the difference $(X_{\delta j} - X_{\delta(j-1)})$ represents jth high frequency return. Based on the above, the realized kernel estimator in its classical form is defined as

$$K(X_\delta) = \gamma_0(X_\delta) + \sum_{h=1}^{H} k\left(\frac{h-1}{H}\right)\{\gamma_h(X_\delta) + \gamma_{-h}(X_\delta)\}. \tag{1.1}$$

In a sense, the realized kernel estimator performs smoothing of autocovariances of the process similar to how they are smoothed to obtain a consistent estimator of the spectral density in the discrete time series process. This setting has been later generalized in (Barndorff-Nielsen et al., 2011) to the situation where the price process is d-dimensional with $d > 1$. In so doing, (Barndorff-Nielsen et al., 2011) have had to overcome the synchronicity problem between various assets while still taking into account market microstructure effects that may not be independent of the price process.

The classical setting assumes that the interval between successive observations is deterministic. This may not be the most realistic assumption since returns are commonly measured in tick time. Already (Barndorff-Nielsen et al., 2008) posted a question on what may happen if the duration times are, in fact, stochastic. To illustrate the situation, (Barndorff-Nielsen et al., 2008) assumes that the log return process X_t is a Brownian semimartingale; moreover, the measurement times are taken to be $T_{\delta j}$, $j = 1, 2, \ldots, n$ with $T = \int_0^t \tau_u^2\, du$ where τ has strictly positive cádlág paths. Under these assumptions, one can construct a new process $Z_t = Y_{T_t}$ such that $Z_{\delta j} = X_{t_{\delta j}}$ and one can now work with the process Z_t observed at equally spaced times. The result, however, is an inconsistent estimator of the quadratic variation over $[0, T]$. More specifically, let $v_t = \sigma_{T_t}\tau_t$. Then, direct application of the realized kernel approach to the process Z_t produces an estimate of the quantity $\int_0^t v_u^2\, du$ rather than that of the original quadratic variation $\int_0^t \sigma^2(u)\, du$.

To the best of our knowledge, a possibility of the truly stochastic duration times between adjacent observations has not been considered before. In particular, we again consider a finite interval $[0, T]$ with n transactions observed within that interval. The observation times $t_1, \ldots, t_n$ are stochastic. The durations $\tau_i = t_i - t_{i-1}\,, i = 2, \ldots, n$

are assumed to be iid with a continuous cumulative distribution function and with both expectation and variance going to zero as $n \to \infty$. In order to obtain a sensible realized kernel type estimator, we redefine realized autocovariances of the log return process X_t. To do this, consider first the realized cross-covariance between two distinct processes Z_t and X_t with common measurement times $\{t_i\}_{1 \le i \le n}$. For simplicity, we use the notation $\Delta_n = \mathbb{E}\,\tau_i$ for any $1 \le i \le n$. We define the realized cross-covariance as $\gamma_h(Z,X)_t = \Delta_n \sum_{j=1}^n \frac{(Z_{t_j} - Z_{t_{j-1}})}{\sqrt{\tau_j}} \cdot \frac{(X_{t_{j-h}} - X_{t_{j-h-1}})}{\sqrt{\tau_{j-h}}}$. If $X_t = Z_t$, the above definition provides the new definition of the realized autocovariance:

$$\gamma_h(Z)_t = \Delta_n \sum_{j=1}^{n} \frac{(Z_{t_j} - Z_{t_{j-1}})}{\sqrt{\tau_j}} \cdot \frac{(Z_{t_{j-h}} - Z_{t_{j-h-1}})}{\sqrt{\tau_{j-h}}}. \tag{1.2}$$

Now, the realized kernel estimator can be redefined by substituting (1.2) instead of the classic autocovariance definition into (1.1).

Although this definition seems rather sensible in the case of stochastically spaced measurement times, we need to establish a number of new asymptotic results of probabilistic nature first if we ever hope to characterize the large sample behavior of our new realized kernel estimator. In particular, one has to establish the law of large numbers and a central limit theorem for several functionals of increments of continuous semimartingales with observations being sampled stochastically with duration times τ_i, $i = 2, \ldots, n$. The current manuscript is dedicated specifically to these results while the second manuscript in our series will concentrate on the properties of the new estimator. The manuscript is structured as follows. Section (2) is concerned with the detailed model set-up. Section (3) discusses relevant laws of large numbers while section (4) covers a very important central limit theorem.

2 Model Set-up

1. Price model:
Assume that we have a probability space $(\Omega, P, \mathcal{F})$ and an assigned filtration $\{\mathcal{F}_t\}_{t \ge 0}$ containing all the information about market prices S_t up to time t; also, let $\{W_t\}$ be a Brownian Motion defined on this space. Let $X_t = ln(S_t)$ be the log price process such that $dX_t = b_t dt + \sigma_t dW_t$ with a drift process b_t and the volatility process σ_t. We assume that the drift process b_t and the volatility process σ_t are adapted to $\mathcal{F}_t$. For brevity, we denote the integrated volatility $\text{IV} = \int_0^T \sigma_t^2\, dt$.

Throughout this manuscript, we will use several important assumptions on the nature of the process X_t. For convenience, we start with enumerating all of them in one location.

1. **Assumption A:**
   Given any finite $T > 0$, we assume that the spot volatility σ_t^2, $0 \le t \le T$ can be bounded with probability 1:

$$P\{\sigma_t^2 \le M_T,\ 0 \le t \le T\} = 1$$

where M_T is a random variable with finite fourth moment:

$$E(M_T^4) < \infty$$

2. **Assumption B:**

$$P\{|b_t| \leq A_T, 0 \leq t \leq T\} = 1$$

for any fixed $T > 0$ where A_T is a random variable with finite fourth moment:

$$E(A_T^4) < \infty$$

Assumption H:
Let X_t be a continuous Itô semimartingale with the representation

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s$$

where W_t is a standard Wiener process and b_t, σ_t are locally bounded. Moreover, the volatility process σ_t is also an Itô semimartingale of the form

$$\sigma_t = \sigma_0 + \int_0^t \tilde{b}_s ds + \int_0^t \tilde{\sigma} dW_s + \tilde{\kappa}(\tilde{\delta}) \star (\underline{\mu} - \underline{\nu})_t + \tilde{\kappa}'(\tilde{\delta}) \star \underline{\mu}_t$$

where $\underline{\mu}$ is a Poisson random measure on $(0, \infty) \times E$ with intensity measure $\underline{\nu}(dt, dx) = dt \otimes \lambda(dx)$, where λ is a σ-finite and infinite measure without atom on an auxiliary measurable set $(E, \mathcal{E})$. $\tilde{\kappa}$ is a truncation function and $\tilde{\kappa}'(x) = x - \tilde{\kappa}(x)$. $\tilde{\delta}(\omega, t, x)$ is a predictable function on $\Omega \times R_+ \times E$. Moreover, we assume that

(a) Let $\tilde{\gamma}$ be a (non-random) nonnegative function such that $\int_E (\tilde{\gamma}(x)^2 \wedge 1)\lambda(dx) < \infty$. Then, the processes $\tilde{b}_t(\omega)$ and $\sup_{x \in E} \frac{\|\tilde{\delta}(\omega,t,x)\|}{\tilde{\gamma}(x)}$ are locally bounded, and

(b) All paths $t \to b_t(\omega)$, $t \to \tilde{\sigma}_t(\omega)$, $t \to \tilde{\delta}(\omega, t, x)$ are right-continuous with left limits (càdlàg).

Remark 2.1. *Recall that being locally bounded in this context means that a stopped version of a process is bounded. In other words, there exists a sequence of stopping times $\{T_n\}$, with $T_n \to \infty$, such that stopped process $b_{t \wedge T_n}$ is bounded by a constant that may depend on n but not on (ω, t).*

In what follows we will also use another, much stronger definition of what it means to be locally bounded.
Assumption SH:
In addition to the assumption (H) we have, for some constant Λ and all (ω, t, x):

$$\|b_t(\omega)\| \leq \Lambda, \|\sigma_t(\omega)\| \leq \Lambda, \|X_t(\omega)\| \leq \Lambda$$
$$\|\tilde{b}_t(\omega)\| \leq \Lambda, \|\tilde{\sigma}_t(\omega)\| \leq \Lambda, \|\tilde{\delta}(\omega, t, x)\| \leq \Lambda(\tilde{\gamma}(x) \wedge 1).$$

3. **Trading time model: Assumption T**
For a finite time interval $[0, T]$, define $\Delta_n = \frac{T}{n}$. We assume that n transactions occurred until the time T and that the transaction times $t_1, \ldots, t_n$ are stochastic. More specifically, the durations $\tau_i = t_i - t_{i-1}\,, i = 2, \ldots, n$ are assumed to be i.i.d. with some continuous cumulative distribution function, the mean

$$E[\tau_i] = \Delta_n \tag{2.1}$$

and variance

$$Var(\tau_i) = \Delta_n^{2+\epsilon}, \text{for some } \epsilon > 0 \tag{2.2}$$

.

Remark 2.2. *The Assumption T implies the following useful representation. If we select a random sequence ξ_i^n such that $\mathbb{E}\xi_i^n = 0$ and $Var\, \xi_i^n = \Delta_n^\varepsilon$ for some small $\varepsilon > 0$, the duration time τ_i satisfying* (2.1)-(2.2) *can be represented as $\tau_i = \Delta_n(1 + \xi_i^n)$. By Chebyshev's inequality, we also have $\xi_i^n = o_p(1)$.*

Remark 2.3. *Note that this assumption excludes, for example, the exponential distribution that is commonly used to model duration times since in that case the variance is equal to the mean. Historically, the assumption of exponential distribution for duration times was quite popular. As an example, a well known model of (Cont et al., 2010) models the trading times as a simple Poisson process which means that the trading durations are i.i.d. exponentially distributed with some parameter λ. Our assumption is of purely technical nature; as $\varepsilon \to 0$, the exponential model can be thought of as a limiting case of our model. Other alternative models of trading times may assume that the trading durations are correlated over time as in, for example, the autoregressive conditional duration (ACD) model introduced by (Engle and Russell, 1998). Moreover, (Bouchaud et al., 2002) offer a comprehensive study on the empirical properties of the whole order book. Since our main interest lies in estimation of realized volatility of the data, we are going to start with a simple assumption of independent duration times first. We will consider possible generalization to the ACD model as a next step in our research.*

Finally, the last assumption concerns the relationship between transaction times t_i and the price process X_t.

4. **Independence Assumption C:**
Let $\{\mathcal{N}_t\}_{t \geq 0}$ be the filtration generated by transaction times $0 \leq t_1, \ldots, t_n \leq t$ for some $0 \leq t \leq T$. We assume that $\mathcal{N}_t$ is independent of $\mathcal{F}_t$.

3 Laws of large numbers (LLNs) for increments of functions of semi-martingales

In this section, we consider two continuous semi-martingale data processes. The first is a very simple constant volatility process $\tilde{X}_t = \sigma W_t$. The second is more complicated semimartingale process $X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s$. We also assume that all of the durations $\{\tau_i\}_{i=2}^n$ satisfy Assumption T. The increments of the first process are denoted $\Delta_i^n \tilde{X} = \tilde{X}_{t_i} - \tilde{X}_{t_{i-1}}$ while those of the second are $\Delta_i^n X = X_{t_i} - X_{t_{i-1}}$. For an arbitrary function f, functions of the increments of the simplified process $\tilde{X}_t$ are $\tilde{V}(f, \Delta_n)_t = \Sigma_{i=1}^n f(\Delta_i^n \tilde{X})$. We also consider these same increments in the normalized form as $\tilde{V'}(f, \Delta_n)_t = \Sigma_{i=1}^n f(\Delta_i^n \tilde{X}/\sqrt{\tau_i})$. Analogously, for the more complicated process X_t we have $V(f, \Delta_n)_t = \Sigma_{i=1}^n f(\Delta_i^n X)$ and, in the normalized form, $V'(f, \Delta_n)_t = \Sigma_{i=1}^n f(\Delta_i^n X/\sqrt{\tau_i})$. Finally, we also define the so-called approximate variation of the pth order for both processes X_t and $\tilde{X}_t$ as $B(p, \Delta_n)_t = \Sigma_{i=1}^n |\Delta_i^n X|^p$ and $\tilde{B}(p, \Delta_n) = \Sigma_{i=1}^n |\Delta_i^n \tilde{X}|^p$ for some positive integer p. Of course, if $p = 2$, these become approximate quadratic variations. Our ultimate goal is to derive certain laws of large numbers (LLN) and central limit theorems (CLT) for the functions of increments of X_t. As an intermediate step, both will be proved first for the functions of increments of $\tilde{X}_t$.

We begin with a simple lemma concerning the asymptotic behavior of the size of the time grid.

Lemma 3.1. *Under assumption T,* $\max \tau_i \xrightarrow{p} 0$ *as* $n \to \infty$.

Proof. Denote $\tau_{(n)} = \max_{1 \leq i \leq n} \tau_i$. Since $\tau_{(n)} \geq 0$, we have for any $a > 0$, by Markov inequality

$$P(\tau_{(n)} \geq a) \leq \frac{E(\tau_{(n)})}{a}$$

Then, by Hartley-David Inequality (see (Hartley et al., 1954)), we have $E(\tau_{(n)}) \leq \Delta_n + \frac{\sqrt{\Delta_n^{2+\epsilon}(n-1)}}{\sqrt{2}} = O\left(n^{-\frac{1+\epsilon}{2}}\right)$ if $0 < \varepsilon < 1$ and $O(n^{-1})$ if $\varepsilon \geq 1$. In either case, clearly, $E(\tau_{(n)}) \to 0$. Thus we have as $n \to \infty$,

$$\lim_{n \to \infty} P(\tau_{(n)} \geq a) = 0$$

which means that $\tau_{(n)} \xrightarrow{p} 0$. $\square$

Next, we will need the fact of asymptotic convergence of $B(2, \Delta_n)_t$. In other words, we need to see whether

$$B(2, \Delta_n)_t \xrightarrow{p} \int_0^t \sigma_s^2 ds.$$

For the fixed time grid this has been done in the literature earlier; see, for example, Theorem 2.10 of (Kessler et al., 2012)) and references therein. For convenience, we cite this result in full.

Theorem 3.2. *Let the grid of transaction times be*

$$\mathcal{G} = \{t_0, t_1, \cdots, t_n = T\};$$

define the maximum size of the grid to be $\Delta(\mathcal{G}) = max\{t_i - t_{i-1}\}$ *and assume that* $\Delta(\mathcal{G}) \to 0$ *as* $n \to \infty$. *We also define*

$$[X, Y]_t^{\mathcal{G}} = \Sigma_{t_{i+1} < t}(X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i})$$

for any two processes X_t and Y_t. Then, for any two semi-martingales X_t and Y_t, there is a process $[X,Y]_t$ such that

$$[X,Y]_t^{\mathcal{G}} \xrightarrow{p} [X,Y]_t$$

for all $t \in [0,T]$ as $\Delta(\mathcal{G}) \to 0$. Moreover, for an Itô process,

$$[X,X]_t = \int_0^t \sigma_s^2 ds$$

The resulting limit is independent of the sequence of grids $\mathcal{G}$.

Note that the theorem is still be true even when t_i's are stochastic, as long as they are stopping times and

$$\Delta(\mathcal{G}) = max_{1 \le i \le n}(t_i - t_{i-1}) \to 0$$

is still satisfied. The details can be found in (Jacod and Shiryaev, 2003), Theorem 4.47, page 52.

3.1 The first simple law of large numbers

As a first step, we establish a simple law of large numbers for functions of increments of an Itô semimartingale. It will serve as a stepping stone for later, more complicated results.

Lemma 3.3. *Let f be a continuous function: $\mathbb{R}^k \to \mathbb{R}$ and let X_t be a continuous Itô process defined as above. Then*

a) If $f(x) = o(\|x\|^2)$ as $x \to 0$, then

$$V(f, \Delta_n)_t \xrightarrow{p} 0$$

b) If there exists a neighborhood of 0 such that the function $f(x) \equiv g(x) = \gamma x^2$ for some constant γ, then

$$V(f, \Delta_n)_t \xrightarrow{p} \gamma \int_0^t \sigma_s^2 ds$$

Proof. a) If $f(x) = o(\|x\|^2)$ as $x \to 0$, then for any $\eta > 0$, there is an $\epsilon > 0$ such that function f can be represented as

$$f(x) = f_\epsilon(x) + f'_\epsilon(x)$$

where the first term is a continuous function $f_\epsilon(x)$ such that $f_\epsilon(x) = 0$ when $\|x\| \le \epsilon$ while $\|f'_\epsilon(x)\| \le \eta\|x\|^2$ for all x. Since $\Delta(\mathcal{G}) \xrightarrow{p} 0$ as $n \to \infty$ and X_t is continuous, then for each ω, there exists an integer $N(\omega)$ such that for all

$n \geq N(\omega)$, $\max\|\Delta_i^n X(\omega)\| \leq \epsilon$. Thus for each ω we have $V(f_\epsilon, \Delta_n)_t \to 0$.
Moreover, we have

$$\|V(f'_\epsilon, \Delta_n)_t\| \leq \eta \Sigma_{i=1}^{t/\Delta_n} \|\Delta_i^n X\|^2 \xrightarrow{p} \eta \int_0^t \sigma_s^2 ds$$

by Theorem (3.2).
Thus

$$\|V(f, \Delta_n)_t\| \leq \|V(f_\epsilon, \Delta_n)_t\| + \|V(f'_\epsilon, \Delta_n)_t\| \xrightarrow{p} \eta \int_0^t \sigma_s^2 ds$$

Since η can be arbitrarily small, the first statement is true.

b) Let $f' = f - g$, which is $o(\|x\|^2)$ on a neighborhood of 0, then from the results of (a), we obtain that

$$V(f', \Delta_n)_t = V(f, \Delta_n)_t - V(g, \Delta_n)_t \xrightarrow{p} 0$$

And

$$V(g, \Delta_n)_t = \gamma \Sigma_{i=1}^{t/\Delta_n} (\Delta_i^n X)^2 \xrightarrow{p} \gamma \int_0^t \sigma_s^2 ds$$

Thus we obtain the result (b) by combining these two equations together.

□

The LLN we just obtained in Lemma (3.3) is slightly weaker than what is needed. Before formulating a more general LLN, we need to define the idea of uniform convergence in probability.

Definition 3.4. *A sequence of jointly measurable stochastic processes ξ_t^n is said to converge locally uniformly in probability to a process ξ_t if $\lim_{n\to\infty} P\left(\sup_{s\leq t} |\xi_s^n - \xi_s| > K\right) = 0$ for any $K > 0$ and any finite t. This convergence is commonly denoted $\xi_t^n \overset{u.c.p.}{\to} \xi_t$.*

Now, armed with the new ideas, we can obtain a much stronger uniform law of large numbers.

Theorem 3.5. *Assume (H) and (T). Let f be a continuous function on R^k for some $k \geq 1$, which satisfies*

$$|f(x_1, \cdots, x_k)| \leq K_0 \prod_{j=1}^{k} (1 + \|x_j\|^p)$$

for some $p > 0$ and K_0. Define

$$V'(f, k, \Delta_n)_t = \sum_{i=1}^{[t/\Delta_n]} f\left(\Delta_i^n X/\sqrt{\Delta_n}, \cdots, \Delta_{i+k-1}^n X/\sqrt{\Delta_n}\right)$$

Then we have

$$\Delta_n V'(f, k, \Delta_n)_t \xrightarrow{u.c.p.} \int_0^t \rho_{\sigma_u}^{\otimes k}(f) du.$$

In the above, $\rho_\sigma^{\otimes k}(f) = E[f(X)]$ where $X = (x_1, x_2, \cdots, x_k) \sim N(0, \sigma^2 I)$ and I is a $k \times k$ identity matrix.

Proof. To prove this theorem, we will use the so-called localization procedure described in detail in (Kessler et al., 2012). Essentially, it is a very useful approach for proving limit theorems for discretized processes over a finite time interval. Our main tool in this undertaking is the following Lemma from (Kessler et al., 2012) that we show here in full for ease of exposition.

Lemma 3.6. *If X satisfies assumption H we can find a sequence of stopping times R_p increasing to $+\infty$ and a sequence of processes $X(p)$ satisfying assumption SH and with volatility process $\sigma(p)$, such that*

$$t < R_p \rightarrow X(p)_t = X_t \, , \sigma(p)_t = \sigma_t$$

Clearly, the assumptions of (3.6) are true in our case for the process X_t. Suppose that Theorem (3.5) has been proved when (SH) is satisfied. Let X now satisfy (H) only, and $(X(p), R_p)$ be as defined in the Lemma (3.6). If the process used in $V^{'}(f,k,\Delta_n)_t$ is $X(p)_t$, we use the modified notation $V^{'}(X(p);f,k,\Delta_n)_t$. We then know that for all p, T and all appropriate functions f,

$$\sup_{t\leq T}\left|\Delta_n V^{'}(X(p);f,k,\Delta_n)_t - \int_0^t \rho^{\otimes k}_{\sigma(p)_u}(f)du\right| \xrightarrow{P} 0$$

On the set $\{R_p > T+1\}$ we have, for any Δ_n such that $k\Delta_n \leq 1$, that $V^{'}(X(p);f,k,\Delta_n)_t = V^{'}(X;f,k,\Delta_n)_t$ and $\sigma(p)_t = \sigma_t$ for all $t \leq T$ by Lemma (3.6). Since $P(R_p > T+1) \rightarrow 1$ as $p \rightarrow \infty$, it readily follows that $\Delta_n V^{'}(f,k,\Delta_n)_t \xrightarrow{u.c.p.} \int_0^t \rho^{\otimes k}_{\sigma_u}(f)du$. This proves Theorem (3.5) under (H).
Thus, the only task remaining is to prove that the statement of the Theorem (3.5) is true when X satisfies (SH). For convenience purposes, from now on we denote t_i^n the time of the ith transaction within the interval $[0,T]$; the superscript n refers to the total number of transactions in this interval. Under SH, σ_t is a piecewise constant function equal to $\sigma_{t^n_{i-1}}$ on each of the intervals $[t^n_{i+l-1}, t^n_{i+l}]$ for $1 \leq l \leq n-i$. Define $\Delta^n_{i+l}W = W_{t^n_{i+l}} - W_{t^n_{i+l-1}}$. Then, defining

$$\beta^n_{i,l} = \sigma_{t^n_{i-1}}\Delta^n_{i+l}W/\sqrt{\tau_{i+l}},$$

and

$$x^n_{i,l} = \frac{1}{\sqrt{\tau_{i+l}}}\int_{t^n_{i+l-1}}^{t^n_{i+l}}\Big(b_s ds + (\sigma_s - \sigma_{t^n_{i-1}})dW_s\Big)$$

we obtain

$$\Delta^n_{i+l}X = \sqrt{\tau_{i+l}}(x^n_{i,l} + \beta^n_{i,l}).$$

Assuming (SH), note that we have $\Delta_n = \frac{T}{n} = O(n^{-1})$ and based on Assumption T, we have $\lim\limits_{n\to\infty}\frac{\tau_i}{\Delta_n} = 1$ in probability. Then, it's easy to check that, for any $q > 0$, there exists a constant K_q such that

$$E^n_{i+l-1}(\|\beta^n_{i,l}\|^q) \leq K_q$$

where

$$E^n_{i+l-1}(\cdot) = E\left(\cdot|\mathcal{F}_{t^n_{i+l-1}} \bigvee \mathcal{N}_{t^n_{i+l-1}}\right)$$

Using Doob's and Burkholder-Davis-Gundy inequalities (see (Burkholder et al., 2011)) repeatedly we obtain a sequence of inequalities. All of them are true for every ω and so can be interpreted in the almost sure sense. First, the direct application of the above cited inequalities gives us

$$E\left(\|\sigma_{t+s} - \sigma_t\|^q|\mathcal{F}_t\right) \leq K_q s^{1\wedge(q/2)};$$

this, in turn, lets us claim that

$$E^n_{i+l-1}\left(|x^n_{i,l}|\right) \leq K_q E^n_{i+l-1}(\sqrt{\tau_{i+l}}) \to 0$$

On the other hand, we have for function f satisfying the assumptions in Theorem (3.5), and any $A > 0$,

$$G_A(\epsilon) = \sup_{\{x_j, y_j : \|x_j\| \leq A, \|y_j\| \leq \epsilon\}} \|f(x_1 + y_1, \cdots, x_k + y_k) - f(x_1, \cdots, x_k)\| \xrightarrow{\epsilon \to 0} 0$$

Then we have

$$\sup_{i\geq 0, \omega\in\Omega} E^n_{i-1}\left(\left|f\left(\frac{\Delta^n_i X}{\sqrt{\tau_i}}, \cdots, \frac{\Delta^n_{i+k-1}X}{\sqrt{\tau_{i+k-1}}}\right) - f\left(\beta^n_{i,0}, \cdots, \beta^n_{i,k-1}\right)\right|\right)$$

$$= \sup_{i\geq 0, \omega\in\Omega} E^n_{i-1}\left(\left|f(x^n_{i,0} + \beta^n_{i,0}, \cdots, x^n_{i,k-1} + \beta^n_{i,k-1}) - f(\beta^n_{i,0}, \cdots, \beta^n_{i,k-1})\right|\right) \to 0$$

Let's denote

$$V''(f, k, \Delta_n)_t = \Sigma^{[t/\Delta_n]}_{i=1} f(\beta^n_{i,0}, \cdots, \beta^n_{i,k-1})$$

Then we have

$$\Delta_n\left(V'(f, k, \Delta_n)_t - V''(f, k, \Delta_n)_t\right) \xrightarrow{u.c.p.} 0$$

by the result above. Therefore it is enough to show the convergence for $\Delta_n V''(f, k, \Delta_n)$. Denote $\eta^n_i = \Delta_n f(\beta^n_{i,0}, \cdots, \beta^n_{i,k-1})$. Then we have

$$E^n_{i-1}(\eta^n_i) = \Delta_n \rho^{\otimes k}_{\sigma_{t^n_{i-1}}}(f) = (\tau_i + \Delta_n - \tau_i)\, \rho^{\otimes k}_{\sigma_{t^n_{i-1}}}(f) = \tau_i \rho^{\otimes k}_{\sigma_{t^n_{i-1}}}(f) + (\Delta_n - \tau_i)\rho^{\otimes k}_{\sigma_{t^n_{i-1}}}(f)$$

First of all, we show that

$$\sum_{i=1}^{[t/\Delta_n]} (\Delta_n - \tau_i)\rho^{\otimes k}_{\sigma_{t^n_{i-1}}}(f) \xrightarrow{u.c.p.} 0$$

because the conditions of Lemma 3.4 in (Kessler et al., 2012) are satisfied:

$$\sum_{i=1}^{[t/\Delta_n]} E^n_{i-1}\left((\Delta_n - \tau_i)\rho^{\otimes k}_{\sigma_{t^n_{i-1}}}(f)\right) = 0$$

$$\sum_{i=1}^{[t/\Delta_n]} E_{i-1}^n \left(|(\Delta_n - \tau_i)\rho_{\sigma_{t_{i-1}^n}}^{\otimes k}(f)|^2 \right) = \sum_{i=1}^{[t/\Delta_n]} \left(\rho_{\sigma_{t_{i-1}^n}}^{\otimes k}(f) \right)^2 Var(\tau_i) \leq K \sum_{i=1}^{[t/\Delta_n]} Var(\tau_i) \xrightarrow{P} 0$$

Note that we also have $E_{i-1}^n(|\eta_i^n|^2) \leq K\Delta_n^2$, thus by Riemann integration, we have

$$\Sigma_{i=1}^{[t/\Delta_n]} E_{i-1}^n(\eta_i^n) = \sum_{i=1}^{[t/\Delta_n]} \Delta_n \rho_{\sigma_{t_{i-1}^n}}^{\otimes k}(f) \xrightarrow{u.c.p.} \sum_{i=1}^{[t/\Delta_n]} \tau_i \rho_{\sigma_{t_{i-1}^n}}^{\otimes k}(f) \xrightarrow{u.c.p.} \int_0^t \rho_{\sigma_v}^{\otimes k}(f_v)dv$$

which concludes our proof. □

4 Main central limit theorem

4.1 A simple CLT for increments of the simplified process

To show the "flavor" of results we need to obtain, we state, as a first step, a simple central limit theorem for the normalized increments of the simplified process $\tilde{X}_t$. This CLT will not be used in the future to prove other results - it is simply an illustration of what we would like to establish for the normalized increments of the process X_t. As a first step, we need the following definition. A continuous function f is said to exhibit polynomial growth (grow at a polynomial rate) if

$$|f(x)| \leq K(1 + |x|^p) \leq K_0|x|^p \tag{4.1}$$

with some constants K, K_0 and some $p \geq 0$.

Theorem 4.1. *Let $\rho_\sigma(f) = \frac{1}{\sqrt{2\pi}\sigma} \int f(x) \exp\left(-x^2/2\sigma^2\right) dx$ be an integral of the function $f(x)$ with respect to the Gaussian law $N(0, \sigma^2)$. If the function f grows at a polynomial rate, we have*

$$\frac{1}{\sqrt{\Delta_n}} \left(\Delta_n \tilde{V}'(f, \Delta_n)_t - t\rho_\sigma(f) \right) \xrightarrow{L} N(0, t[\rho_\sigma(f^2) - \rho_\sigma(f)^2])$$

Proof. Note that any n variables $(\Delta_i^n \tilde{X}/\sqrt{\tau_i} : \quad i \geq 1)$ are i.i.d with law $\mathcal{N}(0, \sigma^2)$. Then the variables $f(\Delta_i^n \tilde{X}/\sqrt{\tau_i})$ when i varies are i.i.d with finite moments of all orders. An application of the standard CLT gives us the statement of this result. □

4.2 Main CLT

Now, we have to obtain the CLT for the increments of X_t. A major problem in doing so is to be able to characterize the limit, and, more specifically, the quadratic variation of the limiting process. As usual, we start with the necessary notation. Consider a sequence $(U_i)_{i\geq 1}$ of independent $\mathcal{N}(0,1)$ variables. Recall that ρ_σ, defined before, is actually the distribution law of σU_1, and so $\rho_\sigma(g) = E(g(\sigma U_1))$. Also recall that a function of k-dimensional argument $f(x_1,\ldots,x_k):\mathbb{R}^k\to\mathbb{R}$ exhibits polynomial growth if

$$|f(x_1,\ldots,x_k)|\le K_0\prod_{j=1}^{k}(1+|x_j|)^p$$

for a positive constant K_0 and some positive p. For such a function f on $\mathbb{R}^k$ we set

$$R_\sigma(f,k)=\sum_{l=-k+1}^{k-1}E\left[f^2(\sigma U_k,\cdots,\sigma U_{2k-1})\right]-(2k-1)E^2\left[f(\sigma U_1,\cdots,\sigma U_k)\right]$$

Our main result is as follows.

Theorem 4.2. *Assume (H) and (T). Let f satisfy either one of the two assumptions stated below.*

- *(a) f is a polynomial function on $\mathbb{R}^k$ for some $k\ge 1$, which is globally even, that is*
$$f(-x_1,\cdots,-x_l,\cdots,-x_k)=f(x_1,\cdots,x_l,\cdots,x_k)$$
- *(b) f is a continuous and once differentiable function with all derivatives exhibiting polynomial growth on $\mathbb{R}^k$ for some $k\ge 1$, which is even in each argument, i.e.*
$$f(x_1,\cdots,-x_l,\cdots,x_k)=f(x_1,\cdots,x_l,\cdots,x_k),\quad\forall\ 1\le l\le k$$

If X is continuous, then the process

$$\frac{1}{\sqrt{\Delta_n}}\left(\Delta_n V'(f,k,\Delta_n)_t-\int_0^t\rho_{\sigma_u}^{\otimes k}(f)du\right)$$

converge stably in law to a continuous process $U'(f,k)$ defined on an extension $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P})$ of the space $(\Omega,\mathcal{F},P)$. Such a process $U'(f,k)$ is a centered Gaussian $\mathbb{R}^1$-valued process with independent increments that, conditionally on the σ-field $\mathcal{F}$, satisfies

$$\tilde{E}(U'(f,k)_tU'(f,k)_t)=\int_0^t R_{\sigma_u}(f,k)du$$

where $\tilde{\mathbb{E}}$ refers to the expectation defined on an extended probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P})$. If $S_\sigma(f,k)$ is the square root of $R_\sigma(f,k)$, then there exists a 1-dimensional Brownian motion B on an extension of the space $(\Omega,\mathcal{F},P)$, independent of $\mathcal{F}$, such that $U'(f,k)$ is given by

$$U'(f,k)_t=\int_0^t S_{\sigma_u}(f,k)dB_u$$

Proof:
Firs, we define the following convenient notation:

$$\zeta_i^n = f(\Delta_i^n X/\sqrt{\tau_i}, \cdots, \Delta_{i+k-1}^n X/\sqrt{\tau_{i+k-1}}),$$

$$\zeta_i'^n = f(\beta_{i,0}^n, \cdots, \beta_{i,k-1}^n),$$

$$\zeta_i''^n = \zeta_i^n - \zeta_i'^n$$

The basic idea of the proof is to replace each normalized increment $\Delta_{i+l}^n X/\sqrt{\tau_i}$ by $\beta_{i,l}^n$, and show that CLT is true for that simpler process, then justify this replacement by showing that the simpler process converges to the original process we are really interested in. Since the proof is rather long and technical, we separate it into a sequence of lemmas that are proved separately. Then, they are combined to produce a proof of the general result.

Lemma 4.3.

$$\sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} \left(\zeta_i''^n - E_{i-1}^n(\zeta_i''^n) \right) \xrightarrow{u.c.p} 0$$

Lemma 4.4.

$$\frac{1}{\sqrt{\Delta_n}} \left(\Delta_n \sum_{i=1}^{[t/\Delta_n]} \rho_{\sigma_{t_{i-1}^n}}^{\otimes k}(f)du - \int_0^t \rho_{\sigma_u}^{\otimes k}(f)du \right) \xrightarrow{u.c.p} 0$$

Lemma 4.5. *The processes*

$$\bar{U}_t^n = \sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} \left(\zeta_i'^n - \rho_{\sigma_{t_{i-1}^n}}^{\otimes k}(f) \right)$$

converge stably in law to the process $U'(f,k)$ as defined in the Theorem (4.2).

Lemma 4.6.

$$\sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} E_{i-1}^n(\zeta_i''^n) \xrightarrow{u.c.p} 0$$

Once we prove these four lemmas, then our Theorem (4.2) follows rather easily. Indeed,

$$\frac{1}{\sqrt{\Delta_n}} \left(\Delta_n V'(f,k,\Delta_n)_t - \int_0^t \rho_{\sigma_u}^{\otimes k}(f)du \right) = \sqrt{\Delta_n} V'(f,k,\Delta_n)_t - \frac{1}{\sqrt{\Delta_n}} \int_0^t \rho_{\sigma_u}^{\otimes k}(f)du$$

$$= \sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} \zeta_i^n - \frac{1}{\sqrt{\Delta_n}} \int_0^t \rho_{\sigma_u}^{\otimes k}(f) du$$

$$= \sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} \left(\zeta_i'^n + \zeta_i''^n\right) - \frac{1}{\sqrt{\Delta_n}} \int_0^t \rho_{\sigma_u}^{\otimes k}(f) du$$

$$= \sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} \left(\zeta_i'^n + \zeta_i''^n\right) - \sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} \rho_{\sigma_{t_{i-1}^n}}^{\otimes k}(f) du + \sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} \rho_{\sigma_{t_{i-1}^n}}^{\otimes k}(f) du - \frac{1}{\sqrt{\Delta_n}} \int_0^t \rho_{\sigma_u}^{\otimes k}(f) du$$

$$= \bar{U}_t^n + \sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} \left(\zeta_i''^n - E_{i-1}^n(\zeta_i''^n) + E_{i-1}^n(\zeta_i''^n)\right) + \frac{1}{\sqrt{\Delta_n}} \left(\Delta_n \sum_{i=1}^{[t/\Delta_n]} \rho_{\sigma_{t_{i-1}^n}}^{\otimes k}(f) du - \int_0^t \rho_{\sigma_u}^{\otimes k}(f) du\right)$$

$$= \bar{U}_t^n + M_t^n$$

where M_t^n represents all the terms in the above equation besides $\bar{U}_t^n$. Due to Lemmas (4.3), (4.4) and (4.6), M_t^n converge to 0 uniformly in probability.

Proof of Lemma (4.3) In order to prove (4.3), we need to prove the following proposition

Proposition 4.7. *Assume (SH). Let $k \geq 1$ and let $q > 0$. Let f be a continuous function on $\mathbb{R}^k$, satisfying the condition in* (3.5) *for some $p \geq 0$ and $K_0 \geq 0$. If we further assume that X is continuous, then as $n \to \infty$:*

$$\sup_{i \geq 0, \omega \in \Omega} E_{i-1}^n \left(\left| f\left(\frac{\Delta_i^n X}{\sqrt{\tau_i}}, \cdots, \frac{\Delta_{i+k-1}^n X}{\sqrt{\tau_{i+k-1}}}\right) - f(\beta_{i,0}^n, \cdots, \beta_{i,k-1}^n) \right|^q \right) \to 0$$

Proof. First, for any polynomial growth exhibiting function f, we have that

$$G_A(\epsilon) = \sup_{\{x_j, y_j : \|x_j\| \leq A, \|y_j\| \leq \epsilon\}} \|f(x_1 + y_1, \cdots, x_k + y_k) - f(x_1, \cdots, x_k)\| \xrightarrow{\epsilon \to 0} 0$$

for any $A > 0$.
Then for all $A > 0$, $s \geq 0$ and $\epsilon > 0$, we have the same inequality as given in the proof of Lemma (3.17) in (Kessler et al., 2012); for convenience,we write it here in full:

$$|f(x_1 + y_1, \cdots, x_k + y_k) - f(x_1, \cdots, x_k)|^q$$

$$\leq G_A(\epsilon)^q + K \sum_{m=1}^{k} \left(h_{\epsilon,s,A,n}(x_m, y_m) \prod_{j=1,\cdots,k, j \neq m} g(x_j, y_j) \right)$$

where

$$h_{\epsilon,s,A,n}(x,y) = \frac{\|x\|^{pq+1}}{A} + \|x\|^{pq}(\|y\| \wedge 1) + A^{pq} \frac{\|y\|^2 \wedge 1}{\epsilon^2} + \frac{\|y\|^{pq+s}}{A^s}$$

$$g(x,y) = 1 + \|x\|^{qp} + \|y\|^{qp}$$

and K is a constant depending on K_0, q, k.
As a next step, we apply this inequality with $x_j = \beta^n_{i,j-1}$ and $y_j = \chi^n_{i,j-1}$, where $\chi^n_{i,j-1}$ is exactly the $x^n_{i,j-1}$ defined in the proof of Theorem (3.5). But since here we use x as symbol representing the general input for function f, to avoid confusion, we use $\chi^n_{i,j-1}$ instead from now on. Since we assume (SH) and X is continuous, the inequalities obtained earlier in the proof of Theorem (3.5) are still valid. In particular,

$$E^n_{i+l-1}(\|\beta^n_{i,l}\|^q) \leq K_q$$

$$E^n_{i+l-1}(\|\chi^n_{i,l}\|) \leq K_q E^n_{i+l-1}(\sqrt{\tau_{i+l}}) \to 0$$

Thus:

$$E^n_{i+j-2}(g(\beta^n_{i,j-1}, \chi^n_{i,j-1})) \leq K$$

for some constant K depending on K_0, q, k. Next consider $\nu^n_{i,j,\epsilon,A} = E^n_{i+j-2}(h_{\epsilon,s,A,n}(\beta^n_{i,j-1}, \chi^n_{i,j-1}))$ for $s = 1$. Applying Cauchy-Schwarz inequality we can obtain

$$\nu^n_{i,j,\epsilon,A} \leq \phi_n(A, \epsilon) = K(1/A + E\sqrt{\tau_{i+j-1}} + \Delta_n A^{pq}/\epsilon^2)$$

And

$$\lim_{A\to\infty} \limsup_{n\to\infty} \phi_n(A, \epsilon) = 0$$

Taking successive downward conditional expectations, we finally get

$$\sup_{i\geq 0,\omega\in\Omega} E^n_{i-1}\left(|f\left(\frac{\Delta^n_i X}{\sqrt{\tau_i}}, \cdots, \frac{\Delta^n_{i+k-1}X}{\sqrt{\tau_{i+k-1}}}\right) - f(\beta^n_{i,0}, \cdots, \beta^n_{i,k-1})|^q\right) \leq G_A(\epsilon) + K\phi_n(A, \epsilon)$$

for all $A > 1$ and $\epsilon > 0$. Then let $\epsilon \to 0$, we obtain the result we want to prove. □

Now, the combination of this result and the Lemma (3.4) from (Kessler et al., 2012) brings the needed conclusion.

Proof of Lemma (4.4)
As before, it is possible to establish this result under the assumption (SH). Recall that under (SH) $||\sigma_t|| \leq \Lambda$ and denote by $\mathcal{M}'$ the interval $(0, \Lambda]$. Consider the function $g(\sigma) = \rho^{\otimes k}_\sigma(f)$, defined on the set $\mathcal{M}'$ and denote $c_i = \frac{\tau_i}{\Delta_n}$. Then

$$\frac{1}{\sqrt{\Delta_n}}\left(\Delta_n \sum_{i=1}^{[t/\Delta_n]} \rho^{\otimes k}_{\sigma_{t^n_{i-1}}}(f)du - \int_0^t \rho^{\otimes k}_{\sigma_u}(f)du\right)$$

$$= \frac{1}{\sqrt{\Delta_n}}\left(\Delta_n \sum_{i=1}^{[t/\Delta_n]} g(\sigma^n_{t_{i-1}}) - \int_0^t g(\sigma_u)du\right)$$

$$= \frac{1}{\sqrt{\Delta_n}}\left(\sum_{i=1}^{[t/\Delta_n]} \frac{1}{c_i}\tau_i g(\sigma^n_{t_{i-1}}) - \int_0^t g(\sigma_u)du\right)$$

$$= \frac{1}{\sqrt{\Delta_n}} \left(\sum_{i=1}^{[t/\Delta_n]} \frac{1}{c_i} \int_{t_{i-1}}^{t_i} g(\sigma_{t_{i-1}}^n) du - \sum_{i=1}^{[t/\Delta_n]} \int_{t_{i-1}}^{t_i} g(\sigma_u) du \right)$$

$$= -\frac{1}{\sqrt{\Delta_n}} \sum_{i=1}^{[t/\Delta_n]} \left(\int_{t_{i-1}}^{t_i} \left(g(\sigma_u) - \frac{1}{c_i} g(\sigma_{t_{i-1}}^n) \right) du \right)$$

$$= -\frac{1}{\sqrt{\Delta_n}} \sum_{i=1}^{[t/\Delta_n]} \left(\int_{t_{i-1}}^{t_i} \left(g(\sigma_u) - g(\sigma_{t_{i-1}}^n) + g(\sigma_{t_{i-1}}^n) - \frac{1}{c_i} g(\sigma_{t_{i-1}}^n) \right) du \right)$$

$$= -\frac{1}{\sqrt{\Delta_n}} \sum_{i=1}^{[t/\Delta_n]} \left(\int_{t_{i-1}}^{t_i} \left(g(\sigma_u) - g(\sigma_{t_{i-1}}^n) \right) du \right) - \frac{1}{\sqrt{\Delta_n}} \sum_{i=1}^{[t/\Delta_n]} \int_{t_{i-1}}^{t_i} g(\sigma_{t_{i-1}}^n) \left(1 - \frac{1}{c_i} \right) du$$

$$= - \sum_{i=1}^{[t/\Delta_n]} \eta_i^n - \sum_{i=1}^{[t/\Delta_n]} \epsilon_i^n$$

where

$$\eta_i^n = \frac{1}{\sqrt{\Delta_n}} \int_{t_{i-1}}^{t_i} \left(g(\sigma_u) - g(\sigma_{t_{i-1}}^n) \right) du$$

and

$$\epsilon_i^n = \frac{1}{\sqrt{\Delta_n}} \int_{t_{i-1}}^{t_i} g(\sigma_{t_{i-1}}^n) \left(1 - \frac{1}{c_i} \right) du$$

So we only need to show

$$\sum_{i=1}^{[t/\Delta_n]} \eta_i^n \xrightarrow{u.c.p.} 0, \qquad \sum_{i=1}^{[t/\Delta_n]} \epsilon_i^n \xrightarrow{u.c.p.} 0$$

The proof of $\sum_{i=1}^{[t/\Delta_n]} \eta_i^n \xrightarrow{u.c.p.} 0$ is very similar to the proof of (3.7.3) in (Kessler et al., 2012). Let $\eta_i^n = \eta_i^{'n} + \eta_i^{''n}$, where

$$\eta_i^{'n} = \frac{1}{\sqrt{\Delta_n}} g'(\sigma_{t_{i-1}}^n) \int_{t_{i-1}}^{t_i} (\sigma_u - \sigma_{t_{i-1}}^n) du$$

$$\eta_i^{''n} = \frac{1}{\sqrt{\Delta_n}} \int_{t_{i-1}}^{t_i} \left[g(\sigma_u) - g(\sigma_{t_{i-1}}^n) - g'(\sigma_{t_{i-1}}^n)(\sigma_u - \sigma_{t_{i-1}}^n) \right] du$$

And we can further decompose $\eta_i^{'n}$ as $\eta_i^{'n} = \mu_i^n + \mu_i^{'n}$, where

$$\mu_i^n = \frac{1}{\sqrt{\Delta_n}} g'(\sigma_{t_{i-1}}^n) \int_{t_{i-1}}^{t_i} du \int_{t_{i-1}}^{u} \tilde{b}_s ds,$$

$$\mu_i^{'n} = \frac{1}{\sqrt{\Delta_n}} g'(\sigma_{t_{i-1}}^n) \int_{t_{i-1}}^{t_i} du \left(\int_{t_{i-1}}^{u} \tilde{\sigma}_s dW_s + \int_{t_{i-1}}^{u} \int \tilde{\delta}(s,x)(\underline{\mu} - \underline{\nu})(ds, dx) \right)$$

On the one hand, we have $|\mu_i^n| \leq \Lambda \frac{\tau_i^2}{\sqrt{\Delta_n}}$ (recall that g is C_b^1 and $\tilde{b}$ is bounded), so

$$\sum_{i=1}^{[t/\Delta_n]} |\mu_i^n| \leq \Lambda \frac{\sum \tau_i^2}{\sqrt{\Delta_n}}$$

Note that based on the Assumption T, we have

$$Var(\xi_i^n) = \frac{Var(\tau_i)}{\Delta_n^2} \to 0$$

Thus, $E(\tau_i^2) = O(\Delta_n^2)$, then we have

$$\sum_{i=1}^{[t/\Delta_n]} E_{i-1}^n(\|\mu_i^n\|) \xrightarrow{P} 0$$

Then, by Lemma (3.4) in (Kessler et al., 2012), we have

$$\sum_{i=1}^{[t/\Delta_n]} \mu_i^n \xrightarrow{u.c.p.} 0$$

On the other hand, we have $E_{i-1}^n(\mu_i^{'n}) = 0$ and $E_{i-1}^n((\mu_i^{'n})^2) \leq \Lambda E_{i-1}^n(\tau_i^2)$ by a similar arguments and through the use of Doob and Cauchy-Schwarz inequalities. Therefore, by Lemma (3.4) in (Kessler et al., 2012) again, we have

$$\sum_{i=1}^{[t/\Delta_n]} \mu_i^{'n} \xrightarrow{u.c.p.} 0$$

So we have already shown that $\sum_{i=1}^{[t/\Delta_n]} \eta_i^{'n} \xrightarrow{u.c.p.} 0$.
As for $\eta_i^{''n}$, since X is continuous and f is assumed to have polynomial growth, we further know that g is C_b^2 on the compact set $\mathcal{M}$. Then by Taylor expansion, we have $|g(\sigma') - g(\sigma) - g'(\sigma)(\sigma' - \sigma)| \leq \Lambda \|\sigma' - \sigma\|^2$ for all $\sigma, \sigma' \in \mathcal{M}$. Therefore,

$$\eta_i^{''n} \leq \frac{K}{\sqrt{\Delta_n}} \int_{t_{i-1}}^{t_i} |\sigma_u - \sigma_{t_{i-1}}^n|^2 du$$

Due to the inequality (3.73) from (Kessler et al., 2012) we have

$$E(\|\sigma_{s+t} - \sigma_t\|^q | \mathcal{F}_t) \leq K_q s^{1 \wedge (q/2)}$$

for a constant K_q that may depend on q. Threfore, for some $K > 0$, we have

$$E_{i-1}^n(|\eta_i^{''n}|) \leq \frac{\Lambda \cdot E(\tau_i^2)}{\sqrt{\Delta_n}} \leq K\Delta_n^{3/2}$$

So $\sum_{i=1}^{[t/\Delta_n]} E_{i-1}^n(|\eta_i^{''n}|) \to 0$. Then we have showed that

$$\sum_{i=1}^{[t/\Delta_n]} \eta_i^n \xrightarrow{u.c.p.} 0$$

At this point, it only remains to prove that

$$\sum_{i=1}^{[t/\Delta_n]} \epsilon_i^n \xrightarrow{u.c.p.} 0$$

First of all, ϵ_i^n can be further simplified:

$$\begin{aligned}
\epsilon_i^n &= \frac{1}{\sqrt{\Delta_n}} \int_{t_{i-1}}^{t_i} g(\sigma_{t_{i-1}}^n)(1 - \frac{1}{c_i})du \\
&= \frac{1}{\sqrt{\Delta_n}} g(\sigma_{t_{i-1}}^n) \left(\tau_i - \frac{\tau_i}{c_i}\right) \\
&= \frac{1}{\sqrt{\Delta_n}} g(\sigma_{t_{i-1}}^n) \left(\tau_i - \Delta_n\right)
\end{aligned}$$

Thus, obviously we have

$$\sum_{i=1}^{[t/\Delta_n]} E_{i-1}^n(\epsilon_i^n) = 0$$

since $E(\tau_i) = \Delta_n$. And

$$E_{i-1}^n(\|\epsilon_i^n\|^2) \leq \Lambda \frac{Var(\tau_i)}{\Delta_n} = \Lambda \Delta_n^{1+\epsilon}$$

for some $\epsilon > 0$ by Assumption T. Then

$$\sum_{i=1}^{[t/\Delta_n]} E_{i-1}^n(\|\epsilon_i^n\|^2) \xrightarrow{P} 0, \quad \forall t > 0$$

Again, by Lemma (3.4) in (Kessler et al., 2012), we have

$$\sum_{i=1}^{[t/\Delta_n]} \epsilon_i^n \xrightarrow{u.c.p.} 0$$

which marks the end of the proof of Lemma (4.4).

Proof of Lemma (4.5)
To make this proof simpler, we only consider the case $k = 2$. There are no conceptually new ideas needed to prove the case $k \geq 3$ but the derivations are much more involved and tedious.

Let $g_t(x) = \int \rho_{\sigma_t}(dy) f(x,y)$, we have

$$\bar{U}_t^n = \sum_{i=2}^{[t/\Delta_n]+1} \eta_i^n + \gamma_1'^n - \gamma_{[t/\Delta_n]+1}'^n$$

where $\eta_i^n = \gamma_i^n + \gamma_i'^n$ and

$$\gamma_i^n = \sqrt{\Delta_n}\left(f(\beta_{i-1,0}^n, \beta_{i-1,1}^n) - \int \rho_{\sigma_{t_{i-2}^n}}(dx) f(\beta_{i-1,0}^n, x)\right)$$

$$\gamma_i'^n = \sqrt{\Delta_n}\left(\int \rho_{\sigma_{t_{i-1}^n}}(dx) f(\beta_{i,0}^n, x) - \rho_{\sigma_{t_{i-1}^n}}^{\otimes 2}(f)\right)$$

Recall that $\beta_{i,l}^n = \sigma_{t_{i-1}^n} \Delta_{i+l}^n W / \sqrt{\tau_{i+l}}$. We use the localization procedure again, and, therefore, work under the assumption (SH). Note that this implies that all of the following integrals are taken on a closed interval of finite length. Also, recall that, while proving Theorem (3.5), we concluded that $E_{i+l-1}^n(|\beta_{i,l}^n|^q) \leq K_q$ for some constant K_q that depends on q; this makes it easy to show that

$$E(|\gamma_i'^n|) \leq K\sqrt{\Delta_n}$$

for some large enough constant K. For brevity, define $\bar{U}_t'^n = \sum_{i=2}^{[t/\Delta_n]+1} \eta_i^n$; now, it is enough to show that $\bar{U}_t'^n$ converges stably in law to the process $U'(f,2)_t$.
Note that η_i^n is $\mathcal{F}_{t_i^n}$ measurable. Combining the conclusion of Theorem (3.5) and Lemma (4.4), we show that

$$E_{i-1}^n(\eta_i^n) = 0$$

And based on the assumption (SH) and the polynomial growth of function f, it is also easy to check that

$$E_{i-1}^n(|\eta_i^n|^4) \leq K\Delta_n^2$$

Before calculating $E_{i-1}^n((\eta_i^n)^2)$, we first list several simple facts that can be used later:

$$E_{i-1}^n(\beta_{i-1,0}^n) = \beta_{i-1,0}^n$$

$$\beta_{i-1,1}^n | \mathcal{F}_{t_{i-1}^n} \sim N(0, \sigma_{t_{i-2}^n}^2) = \rho_{\sigma_{t_{i-2}^n}}$$

$$\beta_{i,0}^n | \mathcal{F}_{t_{i-1}^n} \sim N(0, \sigma_{t_{i-1}^n}^2) = \rho_{\sigma_{t_{i-1}^n}}$$

We only need to calculate $\sum_{i=2}^{[t/\Delta_n]+1} E_{i-1}^n((\eta_i^n)^2)$ for the variance term in order to apply Lemma (3.7) from (Kessler et al., 2012).

We have

$$E_{i-1}^n\left((\eta_i^n)^2\right) = \Delta_n \phi_i^n$$

where $\phi_i^n = g(t_{i-2}^n, t_{i-1}^n, \beta_{i-1,0}^n)$, and

$$g(s,t,x) = \int \rho_{\sigma_s}(dy) f^2(x,y) - \left(\int \rho_{\sigma_s}(dy) f(x,y)\right)^2$$

$$+\int \rho_{\sigma_t}(dy)\left(\rho_{\sigma_t}(dz)f(y,z)\right)^2 - \left(\rho_{\sigma_t}^{\otimes 2}(f)\right)^2 - 2\rho_{\sigma_t}^{\otimes 2}(f)\int \rho_{\sigma_s}(dy)f(x,y)$$

$$+2\int \rho(dy)\rho(dz)f(x,\sigma_s y)f(\sigma_t y,\sigma_t z)$$

Then if we can show the two properties:

$$\sum_{i=2}^{[t/\Delta_n]+1} E_{i-1}^n(\Delta_i^n N\eta_i^n) \xrightarrow{P} 0 \quad (A)$$

for any N which is a component of W (in the 1-dimensional case the W itself) or is a bounded martingale orthogonal to W, and

$$\Delta_n \sum_{i=2}^{[t/\Delta_n]+1} \phi_i^n \xrightarrow{P} \int_0^t R_{\sigma_u}(f,2)du \quad (B)$$

then the Lemma (3.7) from (Kessler et al., 2012) will yield the stable convergence in law of $\bar{U}_t^{\prime n}$ to $U'(f,2)$.
Let's first prove property (A). Recall $\eta_i^n = \gamma_i^n + \gamma_i^{\prime n}$, and observe that

$$\gamma_i^n = \sqrt{\Delta_n}h(\sigma_{t_{i-2}^n}, \Delta_{i-1}^n W/\sqrt{\tau_{i-1}}, \Delta_i^n W/\sqrt{\tau_i})$$

$$\gamma_i^{\prime n} = \sqrt{\Delta_n}h'(\sigma_{t_{i-1}^n}, \Delta_i^n W/\sqrt{\tau_i})$$

where $h(\sigma,x,y)$ and $h'(\sigma,x)$ are continuous functions with polynomial growth in x and y, uniform in $\sigma \in \mathcal{M}'$. Then the property (A) will be a direct conclusion of the following Proposition. In the statement of this Proposition (and occasionally afterwards as well), we sometimes use the notation β_i^n to replace $\beta_{i,0}^n$ since the second index is always zero.

Proposition 4.8. *Under (SH), for any function $(\omega,x) \mapsto g(\omega,x)$ on $\Omega \times R$ which is $\left(\mathcal{F}_{t_{i-1}^n} \bigvee \mathcal{N}_{t_{i-1}^n}\right) \otimes \mathcal{R}$-measurable and even, and with polynomial growth in x, we have*

$$E_{i-1}^n(\Delta_i^n N g(.,\beta_i^n)) = 0$$

where N can be either the process W itself or any bounded martingale orthogonal to both W and $\{\tau_i\}_{i\geq 1}$.

Proof. Assume that N is bounded and orthogonal to W. We consider the martingale $M_t = E\left(g(.,\beta_i^n)|\mathcal{F}_t \bigvee \mathcal{N}_{t_{i-1}^n}\right)$, for $t \geq t_{i-1}^n$. Since W is an $(\mathcal{F}_t)$-Brownian motion, and since β_i^n is a function of $\sigma_{t_{i-1}^n}$ and of $\Delta_i^n W$, we see that $(M_t)_{t\geq t_{i-1}^n}$ is also, conditionally on $\mathcal{F}_{t_{i-1}^n} \bigvee \mathcal{N}_{t_{i-1}^n}$, a martingale w.r.t. the filtration which is generated by $W_t - W_{t_{i-1}^n}$. By the martingale representation theorem the process M is thus of the form $M_t =$

$M_{t^n_{i-1}} + \int_{t^n_{i-1}}^t \upsilon_s dW_s$ for an appropriate predictable process υ. It follows that M is orthogonal to the process $N'_t = N_t - N_{t^n_{i-1}}$ for $t \geq t^n_{i-1}$. Hence

$$E^n_{i-1}\left(\Delta^n_i N g(., \sqrt{\Delta_n}\sigma_{t^n_{i-1}}\Delta^n_i W)\right) = E^n_{i-1}(\Delta^n_i N' M_{t^n_i})$$

$$E^n_{i-1}\left[\Delta^n_i N'\left(M_{t^n_{i-1}} + \int_{t^n_{i-1}}^{t^n_i} \upsilon_s dW_s\right)\right] = 0$$

Next assume that N is W itself. Then we have $\Delta^n_i N g(\beta^n_i)(\omega) = h(\sigma_{t_{i-1^n}}, \Delta^n_i W)(\omega)$ for a function $h(\omega, x, y)$which is odd and with polynomial growth in y, so obviously we have

$$E^n_{i-1}(\Delta^n_i N g(., \beta^n_i)) = 0$$

in this case. □

To prove the property (A), we just need to show that

$$E^n_{i-1}(\Delta^n_i N \gamma^n_i) = 0$$

and

$$E^n_{i-1}(\Delta^n_i N \gamma'^n_i) = 0$$

The part involving γ'^n_i is a direct consequence of Proposition (4.8). Furthermore, while N is a martingale orthogonal to W, we can derive $E^n_{i-1}(\Delta^n_i N \gamma^n_i) = 0$ following similar arguments as in the proof of Proposition (4.8). So it only remains to prove that while N is W itself,

$$\sum_{i=2}^{[t/\Delta_n]+1} \xi^n_i \xrightarrow{P} 0, \text{where } \ \xi^n_i = E^n_{i-1}(\gamma^n_i \Delta^n_i N) = E^n_{i-1}(\gamma^n_i \Delta^n_i W)$$

Since f is globally even and ρ_s is a measure symmetric about the origin, it is not hard to see that $h(\sigma, x, y)$ is globally even in (x, y), and thus $\int \rho_\sigma(dy) h(\sigma, x, y) y$ is odd in x. Further note that $\sigma_{t^n_{i-2}} \in \mathcal{F}_{t^n_{i-1}}$ and $\Delta^n_{i-1} W \in \mathcal{F}_{t^n_{i-1}}$, then it is obvious that

$$\xi^n_i = E^n_{i-1}(\gamma^n_i \Delta^n_i W) = E^n_{i-1}(\sqrt{\Delta_n} h(\sigma_{t^n_{i-2}}, \Delta^n_{i-1} W/\sqrt{\tau_{i-1}}, \Delta^n_i W/\sqrt{\tau_i})\Delta^n_i W) = 0$$

Thus we finish the proof of property (A).

In order to finish the proof of Lemma (4.5) we only need to verify the property (B):

$$\Delta_n \sum_{i=2}^{[t/\Delta_n]+1} \phi^n_i \xrightarrow{P} \int_0^t R_{\sigma_u}(f, 2) du \quad (B)$$

Recall that $E^n_{i-1}\left((\eta^n_i)^2\right) = \Delta_n \phi^n_i$. We have

$$\phi^n_i = g(t^n_{i-2}, t^n_{i-1}, \beta^n_{i-1})$$

where function $g(s,t,x)$ is as defined before. Observe that ϕ_i^n is $\mathcal{F}_{t_{i-1}^n} \bigvee \mathcal{N}_{t_{i-1}^n}$-measurable and

$$E_{i-2,i-1}^n(\phi_i^n) = E\left(\phi_i^n | \mathcal{F}_{t_{i-2}^n} \bigvee \mathcal{N}_{t_{i-1}^n}\right) = h(t_{i-2}^n, t_{i-1}^n), \quad E_{i-2,i-1}^n(|\phi_i^n|^2) \le K$$

where $h(s,t) = \int \rho_{\sigma_s}(dx) g(s,t,x)$.
Then by Lemma (3.4) from (Kessler et al., 2012), the property (B) would follow if we can show that

$$\Delta_n \sum_{i=1}^{[t/\Delta_n]} h(t_{i-1}^n, t_i^n) \xrightarrow{P} \int_0^t R_{\sigma_u}(f,2) du.$$

Since, due to Lemma (3.4) from (Kessler et al., 2012),

$$\Delta_n \sum_{i=1}^{[t/\Delta_n]} h(t_{i-1}^n, t_i^n) \xrightarrow{P} \sum_{i=1}^{[t/\Delta_n]} \tau_i h(t_{i-1}^n, t_i^n)$$

we only need to verify that

$$\sum_{i=1}^{[t/\Delta_n]} \tau_i h(t_{i-1}^n, t_i^n) \xrightarrow{P} \int_0^t R_{\sigma_u}(f,2) du. \tag{4.2}$$

To verify (4.2), we only have to show that

$$\sum_{i=1}^{[t/\Delta_n]} \tau_i h(t_{i-1}^n, t_i^n) \xrightarrow{P} \int_0^t h(u,u) du \tag{4.3}$$

since

$$h(t,t) = \rho_{\sigma_t}^{\otimes 2}(f^2) - 3\left(\rho_{\sigma_t}^{\otimes 2}(f)\right)^2 + 2\int \rho_{\sigma_t}(dx)\rho_{\sigma_t}(dy)\rho_{\sigma_t}(dz) f(x,y) f(y,z)$$

which is exactly $R_{\sigma_t}(f,2)$.
To show that (4.3) is true, we do Taylor expansion of function $h(t_{i-1}^n, y)$ at the point $y = t_{i-1}^n$:

$$\tau_i h(t_{i-1}^n, t_i^n) = \tau_i h(t_{i-1}^n, t_{i-1}^n) + \tau_i^2 h_y(t_{i-1}^n, t_{i-1}^n) + O(\tau_i^3)$$

Thus

$$\sum_{i=1}^{[t/\Delta_n]} \tau_i h(t_{i-1}^n, t_i^n) = \sum_{i=1}^{[t/\Delta_n]} \left(\tau_i h(t_{i-1}^n, t_{i-1}^n) + \tau_i^2 h_y(t_{i-1}^n, t_{i-1}^n) + O(\tau_i^3)\right)$$

By Riemann sum approximation, we know that

$$\sum_{i=1}^{[t/\Delta_n]} \tau_i h(t_{i-1}^n, t_{i-1}^n) \xrightarrow{P} \int_0^t h(u,u) du$$

So we just need to show

$$\sum_{i=1}^{[t/\Delta_n]} \left(\tau_i^2 h_y(t_{i-1}^n, t_{i-1}^n) + O(\tau_i^3)\right) \xrightarrow{P} 0$$

Since it is easy to check that function h has bounded second and third derivatives, we only need to show

$$\sum_{i=1}^{[t/\Delta_n]} \tau_i^2 \xrightarrow{P} 0$$

which is obviously true - just apply the Lemma (3.4) from (Kessler et al., 2012) one more time:

$$\sum_{i=1}^{[t/\Delta_n]} E_{i-1}^n(|\tau_i^2|) = \sum_{i=1}^{[t/\Delta_n]} \Delta_n^{2+\epsilon} \xrightarrow{P} 0$$

This finishes the proof of Lemma (4.5).

Proof of Lemma (4.6)

Define for $l = 0, \cdots, k-1$ the following functions:

$$g_{i,l}^n(x) = \int f\left(\frac{\Delta_i^n X}{\sqrt{\tau_i}}, \cdots, \frac{\Delta_{i+l-1}^n X}{\sqrt{\tau_{i+l-1}}}, x, x_{l+1}, \cdots, x_{k-1}\right) \rho_{\sigma_{t_{i-1}^n}}^{\otimes(k-l-1)}(dx_{l+1}, \cdots, dx_{k-1})$$

As a function of ω this is $\mathcal{F}_{t_{i+l-1}^n} \bigvee \mathcal{H}_{t_{i+l-1}^n}$-measurable. As a function of x it is C^1. Based on the assumptions on price process X_t and the assumption (SH), we further have

$$|g_{i,l}^n(x)| + |\bigtriangledown g_{i,l}^n(x)| \leq K Z_{i,l}^n(1 + |x|^r)$$

where $r \geq 0$, $E_{i-1}^n(|Z_{i,l}^n|^p) \leq K_p \quad \forall p > 0$, for some random variable $Z_{i,l}$ that is $\mathcal{F}_{t_{i+l-2}^n} \bigvee \mathcal{H}_{t_{i+l-2}^n}$-measurable.
For all $A \geq 1$ there is also a positive function $G_A(\epsilon)$ converging to 0 as $\epsilon \to 0$, such that with $Z_{i,l}^n$ as above:

$$|x| \leq A, Z_{i,l}^n \leq A, |y| \leq \epsilon \Longrightarrow |\bigtriangledown g_{i,l}^n(x+y) - \bigtriangledown g_{i,l}^n(x)| \leq G_A(\epsilon)$$

Defining is the sum over l from 0 to $k-1$ of

$$\zeta_i''^n = \sum_{l=0}^{k-1} f\left(\frac{\Delta_i^n X}{\sqrt{\tau_i}}, \cdots, \frac{\Delta_{i+l}^n X}{\sqrt{\tau_{i+l}}}, \beta_{i,l+1}^n, \cdots, \beta_{i,k-1}^n\right) - f\left(\frac{\Delta_i^n X}{\sqrt{\tau_i}}, \cdots, \frac{\Delta_{i+l-1}^n X}{\sqrt{\tau_{i+l-1}}}, \beta_{i,l}^n, \cdots, \beta_{i,k-1}^n\right)$$

we have

$$E_{i-1}^n(\zeta_i''^n) = \sum_{l=0}^{k-1} E_{i-1}^n\left(g_{i,l}^n(\Delta_{i+l}^n X/\sqrt{\tau_{i+l}}) - g_{i,l}^n(\beta_{i,l}^n)\right)$$

Therefore it is enough to prove that for any $l \geq 0$ we have

$$\sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} E_{i-1}^n \left(g_{i,l}^n(\Delta_{i+l}^n X/\sqrt{\tau_{i+l}}) - g_{i,l}^n(\beta_{i,l}^n) \right) \xrightarrow{u.c.p.} 0$$

If we define $\xi_{i,l}^n = \Delta_{i+l}^n X/\sqrt{\tau_{i+l}} - \beta_{i,l}^n$, we only need to show that

$$\sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} E_{i-1}^n \left(g_{i,l}^n(\beta_{i,l}^n + \xi_{i,l}^n) - g_{i,l}^n(\beta_{i,l}^n) \right) \xrightarrow{u.c.p.} 0$$

By Taylor expansion, the left side above can be further written as

$$\sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} E_{i-1}^n \left(g_{i,l}^n(\beta_{i,l}^n + \xi_{i,l}^n) - g_{i,l}^n(\beta_{i,l}^n) \right)$$

$$= \sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} E_{i-1}^n \left[\nabla g_{i,l}^n(\beta_{i,l}^n)\xi_{i,l}^n + \left(\nabla g_{i,l}^n(\beta_{i,l}^{\prime n}) - \nabla g_{i,l}^n(\beta_{i,l}^n) \right) \xi_{i,l}^n \right]$$

$$= \sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} E_{i-1}^n \left(\nabla g_{i,l}^n(\beta_{i,l}^n)\xi_{i,l}^n \right) + \sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} E_{i-1}^n \left(\left(\nabla g_{i,l}^n(\beta_{i,l}^{\prime n}) - \nabla g_{i,l}^n(\beta_{i,l}^n) \right) \xi_{i,l}^n \right)$$

Thus we only needs to show

$$\sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} E_{i-1}^n \left(\nabla g_{i,l}^n(\beta_{i,l}^n)\xi_{i,l}^n \right) \xrightarrow{u.c.p.} 0 \qquad (AA)$$

and

$$\sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} E_{i-1}^n \left(\left(\nabla g_{i,l}^n(\beta_{i,l}^{\prime n}) - \nabla g_{i,l}^n(\beta_{i,l}^n) \right) \xi_{i,l}^n \right) \xrightarrow{u.c.p.} 0 \qquad (BB)$$

separately.

Proof of BB:

$$| \left(\nabla g_{i,l}^n(\beta_{i,l}^{\prime n}) - \nabla g_{i,l}^n(\beta_{i,l}^n) \right) \xi_{i,l}^n |$$

$$\leq G_A(\epsilon)|\xi_{i,l}^n| + KZ_{i,l}^n \left(1 + |\beta_{i,l}^n|^r + |\xi_{i,l}^n|^r\right) |\xi_{i,l}^n| \left(1_{\{Z_{i,l}^n > A\}} + 1_{\{|\beta_{i,l}^n| > A\}} + 1_{\{|\xi_{i,l}^n| > \epsilon\}} \right)$$

$$\leq G_A(\epsilon)|\xi_{i,l}^n| + KZ_{i,l}^n \left(1 + |\beta_{i,l}^n|^r + |\xi_{i,l}^n|^r\right) |\xi_{i,l}^n| \left(\frac{|Z_{i,l}^n|}{A} + \frac{|\beta_{i,l}^n|}{A} + \frac{|\xi_{i,l}^n|}{\epsilon} \right)$$

Note that

$$E(|\xi_{i,l}^n|) \leq KE(\sqrt{\tau_{i+l}}) \leq K\sqrt{E(\tau_{i+l})} = K\sqrt{\Delta_n}$$

then

$$E_{i-1}^{n}\left(\left(\bigtriangledown g_{i,l}^{n}(\beta_{i,l}^{\prime n})-\bigtriangledown g_{i,l}^{n}(\beta_{i,l}^{n})\right)\xi_{i,l}^{n}\right)\leq K\sqrt{\Delta_n}\left(G_A(\epsilon)+\frac{1}{A}+\frac{\sqrt{\Delta_n}}{\epsilon}\right)$$

Thus we have

$$\sqrt{\Delta_n}\sum_{i=1}^{[t/\Delta_n]}E_{i-1}^{n}\left(\left(\bigtriangledown g_{i,l}^{n}(\beta_{i,l}^{\prime n})-\bigtriangledown g_{i,l}^{n}(\beta_{i,l}^{n})\right)\xi_{i,l}^{n}\right)\leq Kt\left(G_A(\epsilon)+\frac{1}{A}+\frac{\sqrt{\Delta_n}}{\epsilon}\right)$$

which will go to zero as n goes to infinity (choose A big and then ϵ small). Then we complete the proof of BB.

Proof of AA:

To prove (AA), following the same scheme as in (Kessler et al., 2012), we first further decompose $\xi_{i,l}^n$ into two parts as below:

$$\xi_{i,l}^{n}=\left(\hat{\xi}_{i,l}^{n}+\tilde{\xi}_{i,l}^{n}\right)/\sqrt{\tau_{i+l}}$$

where

$$\hat{\xi}_{i,l}^{n}=\int_{t_{i+l-1}^{n}}^{t_{i+l}^{n}}(b_s-b_{t_{i+l-1}^{n}})ds$$

$$+\int_{t_{i+l-1}^{n}}^{t_{i+l}^{n}}\left[\int_{t_{i+l-1}^{n}}^{s}\left(\tilde{b}_u du+(\tilde{\sigma}_u-\tilde{\sigma}_{t_{i+l-1}^{n}})dW_u\right)+\int_{t_{i+l-1}}^{s}\int_E\left(\tilde{\delta}(u,x)-\tilde{\delta}(t_{i+l-1}^{n},x)\right)(\underline{\mu}-\underline{\nu})(du,dx)\right]dW_s$$

$$\tilde{\xi}_{i,l}^{n}=b_{t_{i+l-1}^{n}}\tau_{i+l}+\int_{t_{i+l-1}^{n}}^{t_{i+l}^{n}}\left[\tilde{\sigma}_{t_{i+l-1}^{n}}\int_{t_{i+l-1}^{n}}^{s}dW_u+\int_{t_{i+l-1}^{n}}^{s}\int\tilde{\delta}(t_{i+l-1},x)(\underline{\mu}-\underline{\nu})(du,dx)\right]dW_s$$

Then (AA) amounts to the following two claims:

$$\sqrt{\Delta_n}\sum_{i=1}^{[t/\Delta_n]}E_{i-1}^{n}\left[\frac{1}{\sqrt{\tau_{i+l}}}\bigtriangledown g_{i,l}^{n}(\beta_{i,l}^{n})\tilde{\xi}_{i,l}^{n}\right]\xrightarrow{u.c.p.}0\qquad(A1)$$

$$\sqrt{\Delta_n}\sum_{i=1}^{[t/\Delta_n]}E_{i-1}^{n}\left[\frac{1}{\sqrt{\tau_{i+l}}}\bigtriangledown g_{i,l}^{n}(\beta_{i,l}^{n})\hat{\xi}_{i,l}^{n}\right]\xrightarrow{u.c.p.}0\qquad(A2)$$

Proof of A1:

Note that the restriction of $\underline{\mu}$ to $(t_{i+l-1},\infty)\times E$ and the increments of W after time t_{i+l-1} are independent, then conditional on $\mathcal{M}_{t_{i+l-1}^n}=\mathcal{F}_{t_{i+l-1}^n}\bigvee\sigma(W_t:t\geq 0)\bigvee\sigma(\tau_i:i\geq 0)$, we get

$$E(\tilde{\xi}_{i,l}^{n}|\mathcal{M}_{t_{i+l-1}})=b_{t_{i+l-1}}\tau_{i+l}+\tilde{\sigma}_{t_{i+l-1}^{n}}\int_{t_{i+l-1}}^{t_{i+l}}\left(\int_{t_{i+l-1}^{n}}^{s}dW_u\right)dW_s$$

which is even in W.
Thus for a function h which is odd with polynomial growth, we deduce

$$E^n_{i+l-1}(\tilde{\xi}^n_{i,l} h(\beta^n_{i,l})) = 0$$

Thus, if

- (a) function f is even in each argument, then $g^n_{i,l}(\beta^n_{i,l})$ is even and $\bigtriangledown g^n_{i,l}(\beta^n_{i,l})$ is odd. Then it is obvious that (A1) is true.
- (b) function f is a globally even polynomial function, then we follow the proof of (A1) as shown right below.

Define

$$h\left(\Delta_{i,l}X, x\right) = g^n_{i,l}(x) = \int f\left(\frac{\Delta^n_i X}{\sqrt{\tau_i}}, \cdots, \frac{\Delta^n_{i+l-1}X}{\sqrt{\tau_{i+l-1}}}, x, x_{l+1}, \cdots, x_{k-1}\right) \rho^{\otimes(k-l-1)}_{\sigma_{t^n_{i-1}}}(dx_{l+1}, \cdots, dx_{k-1})$$

where $\Delta_{i,l}X = \left(\frac{\Delta^n_i X}{\sqrt{\tau_i}}, \cdots, \frac{\Delta^n_{i+l-1}X}{\sqrt{\tau_{i+l-1}}}\right)$
Obviously we have function h is globally even in $(\Delta_{i,l}X, x)$ since f is globally even and the Gaussian law is symmetric.
Since f is a continuous function with at most polynomial growth, we can decompose the function h as below:

$$h(\Delta_{i,l}X, x) = a(\Delta_{i,l}X) + b_{i,l}(x) + c(\Delta_{i,l}X, x)$$

where function a only contains constant and terms with no x involved, $b_{i,l}$ only contains terms with only x (no any part of $\Delta_{i,l}X$) involved, and function c contains the rest, i.e. those terms with both x and part of $\Delta_{i,l}X$ involved.
Denote $\bigtriangledown_x$ as partial differential w.r.t x, then obviously we have

$$\bigtriangledown_x a(\Delta_{i,l}X) = 0$$

Since h is globally even, for those parts only contain x, they must be even in x, i.e. $b_{i,l}(x)$ is even is x. Thus $\bigtriangledown_x b_{i,l}(x)$ is odd in x and we have

$$E^n_{i-1}\left(\frac{1}{\sqrt{\tau_{i+l}}} \bigtriangledown_x b_{i,l}(x)\tilde{\xi}^n_{i,l}\right) = 0 \quad , \quad \text{while} \;\; x = \beta^n_{i,l}$$

from the arguments above.
For function c, based on the fact that function f is a polynomial function, we are able to write function c as the format below:

$$c(\Delta_{i,l}X, x) = \sum_{j=0}^{l-1} \left(\frac{\Delta^n_{i+j}X}{\sqrt{\tau_{i+j}}}\right)^{p_j} x^{q_j}$$

Since function c should still be globally even in $(\Delta_{i,l}X, x)$ (because function h is globally even), we must have, for any j, $p_j + q_j$ to be an even number. Thus $\bigtriangledown_x c(\Delta_{i,l}X, x)$ is globally odd in $(\Delta_{i,l}X, x)$. Now we treat function $\bigtriangledown_x c(\Delta_{i,l}X, \beta^n_{i,l})$ as a function of dW and τ_{i+j} as below:

$$\bigtriangledown_x c(\Delta_{i,l}X, \beta^n_{i,l}) = c_1(\tau, dW) + c_2(dW)$$

where τ here represents the vector $(\tau_i, \cdots, \tau_{i+l-1})$ and dW represents any terms contain integral w.r.t the Brownian motion. Recall

$$E(\tilde{\xi}^n_{i,l}|\mathcal{M}_{t_{i+l-1}}) = b_{t_{i+l-1}}\tau_{i+l} + \tilde{\sigma}_{t^n_{i+l-1}} \int_{t_{i+l-1}}^{t_{i+l}} \left(\int_{t^n_{i+l-1}}^{s} dW_u \right) dW_s$$

then

$$E^n_{i-1}\left(\frac{1}{\sqrt{\tau_{i+l}}} \bigtriangledown_x c(\Delta_{i,l}X, \beta^n_{i,l})\tilde{\xi}^n_{i,l}\right) = E^n_{i-1}\left(\left(\frac{1}{\sqrt{\tau_{i+l}}} c_1(\tau, dW) + \frac{1}{\sqrt{\tau_{i+l}}} c_2(dW)\right)\tilde{\xi}^n_{i,l}\right)$$

Since function $\bigtriangledown_x c(\Delta_{i,l}X, \beta^n_{i,l})$ is globally odd, then it is easy to check that c_2 is of odd power of dW and

$$E^n_{i-1}\left(c_2(dW)\tilde{\xi}^n_{i,l}\right) = 0$$

As for the term $E^n_{i-1}\left(\frac{1}{\sqrt{\tau_{i+l}}} c_1(\tau, dW)\tilde{\xi}^n_{i,l}\right)$, after simple calculations it is easy to check that those terms are all, at least, of order $O(n^{-\frac{3}{2}})$. (since $E^n_{i-1}\left(\frac{1}{\sqrt{\tau_{i+l}}}\tau^2_{i+j}\right) = O(n^{-\frac{3}{2}})$). Thus we still have

$$\sqrt{\Delta_n}\sum_{i=1}^{[t/\Delta_n]} E^n_{i-1}\left(\frac{1}{\sqrt{\tau_{i+l}}} \bigtriangledown_x c(\Delta_{i,l}X, \beta^n_{i,l})\tilde{\xi}^n_{i,l}\right) \xrightarrow{u.c.p.} 0$$

Thus, finally recall that

$$g^n_{i,l}(x) = h(\Delta_{i,l}X, x) = a(\Delta_{i,l}X, x) + b_{i,l}(x) + c(\Delta_{i,l}X, x)$$

we have already proved $(A1)$ by showing it converging to zero for each of the functions above in the decomposition.

Proof of A2:

To make our notation more convenient, denote $E\left(\cdot|\mathcal{F}_{i+l-1} \vee \mathcal{H}_T\right) = E^*_{i+l-1}$. Then, in order to establish the result needed, the following two Lemmas has to be proved.
Lemma A2A:
Assuming (SH), we have

$$E^*_{i+l-1}\left(|\hat{\xi}^n_{i,l}|^2\right) \leq K\tau_{i+l}\left(\tau^2_{i+l} + \alpha^n_{i,l}\right)$$

where

$$\alpha^n_{i,l} = E^*_{i+l-1}\left(\int_{t^n_{i+l-1}}^{t^n_{i+l}}\left(|b_s - b_{t_{i+l-1}}|^2 + |\tilde{\sigma}_s - \tilde{\sigma}_{t^n_{i+l-1}}|^2 + \int|\tilde{\delta}(s,x) - \tilde{\delta}(t^n_{i+l-1}, x)|^2\lambda(dx)\right)ds\right)$$

Proof of Lemma A2A:
Recall

$$\hat{\xi}_{i,l}^n = \int_{t_{i+l-1}^n}^{t_{i+l}^n} (b_s - b_{t_{i+l-1}^n})ds$$

$$+ \int_{t_{i+l-1}^n}^{t_{i+l}^n} \left[\int_{t_{i+l-1}^n}^{s} \left(\tilde{b}_u du + (\tilde{\sigma}_u - \tilde{\sigma}_{t_{i+l-1}^n})dW_u\right) + \int_{t_{i+l-1}}^{s} \int_E \left(\tilde{\delta}(u,x) - \tilde{\delta}(t_{i+l-1}^n, x)\right)(\underline{\mu} - \underline{\nu})(du,dx)\right] dW_s$$

Thus we have

$$E_{i+l-1}^*\left(|\hat{\xi}_{i,l}^n|^2\right) = E_{i+l-1}^*\left(\int_{t_{i+l-1}^n}^{t_{i+l}^n} (b_s - b_{t_{i+l-1}^n})ds\right)^2$$

$$+E_{i+l-1}^*\left(\int_{t_{i+l-1}^n}^{t_{i+l}^n} \left[\int_{t_{i+l-1}^n}^{s} \left(\tilde{b}_u du + (\tilde{\sigma}_u - \tilde{\sigma}_{t_{i+l-1}^n})dW_u\right) + \int_{t_{i+l-1}}^{s} \int_E \left(\tilde{\delta}(u,x) - \tilde{\delta}(t_{i+l-1}^n, x)\right)(\underline{\mu} - \underline{\nu})(du,dx)\right] dW_s\right)^2$$

Firstly, we have

$$E_{i+l-1}^*\left(\int_{t_{i+l-1}^n}^{t_{i+l}^n} (b_s - b_{t_{i+l-1}^n})ds\right)^2 \le E_{i+l-1}^*\left(\sqrt{\int_{t_{i+l-1}^n}^{t_{i+l}^n} (b_s - b_{t_{i+l-1}^n})^2 ds}\right)^2 \cdot E_{i+l-1}^*\left(\sqrt{\int_{t_{i+l-1}^n}^{t_{i+l}^n} 1 ds}\right)^2$$

$$= E_{i+l-1}^*\left(\int_{t_{i+l-1}^n}^{t_{i+l}^n} (b_s - b_{t_{i+l-1}^n})^2 ds\right) \cdot \tau_{i+l}$$

Secondly, we have

$$E_{i+l-1}^*\left(\int_{t_{i+l-1}^n}^{t_{i+l}^n} \left[\int_{t_{i+l-1}^n}^{s} \left(\tilde{b}_u du + (\tilde{\sigma}_u - \tilde{\sigma}_{t_{i+l-1}^n})dW_u\right) + \int_{t_{i+l-1}}^{s} \int_E \left(\tilde{\delta}(u,x) - \tilde{\delta}(t_{i+l-1}^n, x)\right)(\underline{\mu} - \underline{\nu})(du,dx)\right] dW_s\right)^2$$

$$= E_{i+l-1}^*\left(\int_{t_{i+l-1}^n}^{t_{i+l}^n} \left[\int_{t_{i+l-1}^n}^{s} \left(\tilde{b}_u du + (\tilde{\sigma}_u - \tilde{\sigma}_{t_{i+l-1}^n})dW_u\right) + \int_{t_{i+l-1}}^{s} \int_E \left(\tilde{\delta}(u,x) - \tilde{\delta}(t_{i+l-1}^n, x)\right)(\underline{\mu} - \underline{\nu})(du,dx)\right]^2 ds\right)$$

$$= E_{i+l-1}^*\left(\int_{t_{i+l-1}^n}^{t_{i+l}^n} \left[\int_{t_{i+l-1}^n}^{s} \tilde{b}_u du\right]^2 ds\right) + E_{i+l-1}^*\left[\int_{t_{i+l-1}^n}^{t_{i+l}^n} \left(\int_{t_{i+l-1}^n}^{s} \left(\tilde{\sigma}_u - \tilde{\sigma}_{t_{i+l-1}^n}\right) dW_u\right)^2 ds\right]$$

$$+E_{i+l-1}^*\left[\int_{t_{i+l-1}^n}^{t_{i+l}^n} \left(\int_{t_{i+l-1}}^{s} \int_E \left(\tilde{\delta}(u,x) - \tilde{\delta}(t_{i+l-1}^n, x)\right)(\underline{\mu} - \underline{\nu})(du,dx)\right)^2 ds\right]$$

There are three terms in the equation above, for the first one, we have

$$E_{i+l-1}^*\left(\int_{t_{i+l-1}^n}^{t_{i+l}^n} \left[\int_{t_{i+l-1}^n}^{s} \tilde{b}_u du\right]^2 ds\right) \le K E_{i+l-1}^*\left(\int_{t_{i+l-1}^n}^{t_{i+l}^n} \left[\int_{t_{i+l-1}^n}^{s} du\right]^2 ds\right) \le K \cdot \tau_{i+l}^3$$

For the second one and third one, we apply the Burkholder-Davis-Gundy inequality. More specifically, define $M_s = \int_{t_{i+l-1}^n}^s \left(\tilde{\sigma}_u - \tilde{\sigma}_{t_{i+l-1}^n}\right) dW_u$, whose quadratic variation is

$$[M_s] = \int_{t_{i+l-1}^n}^s \left(\tilde{\sigma}_u - \tilde{\sigma}_{t_{i+l-1}^n}\right)^2 du$$

Let $M_T^* = \sup_{s\leq T} M_s$, then we have

$$E_{i+l-1}^* \left[\int_{t_{i+l-1}^n}^{t_{i+l}^n} \left(\int_{t_{i+l-1}^n}^s \left(\tilde{\sigma}_u - \tilde{\sigma}_{t_{i+l-1}^n}\right) dW_u\right)^2 ds\right] = E_{i+l-1}^* \left(\int_{t_{i+l-1}^n}^{t_{i+l}^n} M_s^2 ds\right)$$

$$\leq E_{i+l-1}^* \left(\int_{t_{i+l-1}^n}^{t_{i+l}^n} M_T^{*2} ds\right) = E_{i+l-1}^* \left(M_T^{*2}\right) \cdot \tau_{i+l} \leq K E_{i+l-1}^* \left([M_T]\right) \cdot \tau_{i+l}$$

$$= K\tau_{i+l} E_{i+l-1}^* \left(\int_{t_{i+l-1}^n}^{t_{i+l}^n} (\tilde{\sigma}_u - \tilde{\sigma}_{t_{i+l-1}^n})^2 du\right)$$

Similarly, define $N_s = \int_{t_{i+l-1}^n}^s \int_E \left(\tilde{\delta}(u,x) - \tilde{\delta}(t_{i+l-1}^n, x)\right) (\underline{\mu} - \underline{\nu})(du, dx)$, which is also a martingale. Then we can apply the Burkholder-Davis-Gundy again and obtain:

$$E_{i+l-1}^* \left[\int_{t_{i+l-1}^n}^{t_{i+l}^n} \left(\int_{t_{i+l-1}}^s \int_E \left(\tilde{\delta}(u,x) - \tilde{\delta}(t_{i+l-1}^n, x)\right) (\underline{\mu} - \underline{\nu})(du, dx)\right)^2 ds\right]$$

$$\leq K\tau_{i+l} E_{i+l-1}^* \left(\int_{t_{i+l-1}^n}^{t_{i+l}^n} \int |\tilde{\delta}(s,x) - \tilde{\delta}(t_{i+l-1}^n, x)|^2 \lambda(dx) ds\right)$$

Combining these inequalities together and add the up we can get the result of Lemma A2A.

Lemma A2B: Under the same assumptions as the previous Lemma,

$$\sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} \sqrt{E(\alpha_{i,l}^n)} \to 0$$

Proof of Lemma A2B:
Denote

$$\lfloor s \rfloor = \max\{t_i^n : t_i^n \leq s\}$$

By Cauchy-Schwarz inequality, the square of the left side of the sum is smaller than

$$t \sum_{i=1}^{[t/\Delta_n]} E(\alpha_{i,l}^n) = tE\left(\int_{t_l^n}^{t_n^n} \left(|b_s - b_{\lfloor s \rfloor}|^2 + |\tilde{\sigma}_s - \tilde{\sigma}_{\lfloor s \rfloor}|^2 + \int |\tilde{\delta}(s,x) - \tilde{\delta}(\lfloor s \rfloor, x)|^2 \lambda(dx)\right) ds\right)$$

which goes to zero as n goes to infinity. (This is because $\lfloor s \rfloor \to s$ as $n \to \infty$. Then by the bounds given in (SH), $\int (\tilde{\gamma}(x) \wedge 1)\lambda(dx) < \infty$ and the dominated convergence theorem, we can achieve the conclusion.)

Now we can finally prove the statement A2. Combining Lemma A2A and A2B, we can show that

$$\sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} E^n_{i-1} \left[\frac{1}{\sqrt{\tau_{i+l}}} \bigtriangledown g^n_{i,l}(\beta^n_{i,l})\hat{\xi}^n_{i,l} \right] \leq \sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} E^n_{i-1} \left(\frac{K}{\sqrt{\tau_{i+l}}} Z^n_{i,l}(1 + |\beta^n_{i,l}|^r)|\hat{\xi}^n_{i,l}| \right)$$

$$\leq \sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} E^n_{i-1} \left(\frac{KZ^n_{i,l}}{\sqrt{\tau_{i+l}}} \sqrt{E^n_{i+l-1}(1 + |\beta^n_{i,l}|^r)^2} \cdot \sqrt{E^*_{i+l-1}|\hat{\xi}^n_{i,l}|^2} \right)$$

$$\leq \sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} E^n_{i-1} \left(KZ^n_{i,l}(\tau_{i+l} + \sqrt{\alpha^n_{i,l}}) \right)$$

$$\leq \sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} K \left(\Delta_n + \sqrt{E^n_{i-1}(\alpha^n_{i,l})} \right) \xrightarrow{u.c.p.} 0$$

Thus we finished the proof of (A2). Combining this with the proof of (A1) and (BB), the result of Lemma (4.6) is immediately obtained.

Note that in Theorem (4.2) the function f is a 1-dimensional function on $\mathbb{R}^k$. However, it is easy to check that the CLT should still be true even when f is a q-dimensional function on $\mathbb{R}^k$ as long as every assumption in Theorem (4.2) still holds true. Such a version may be more useful in many applications since it offers us more flexibility when constructing function f. We will state such a q-dimensional version as a Corollary here. Since its proof is almost the same as that of (4.2) but with an added layer of technical complexity, it will be omitted here.

Corollary 4.9. *Assume (H) and (T). Let $f = (f_1, \cdots, f_q)$ be a q-dimensional function on $\mathbb{R}^k$ satisfying any one of the two cases below*

- *(a) a polynomial function which is globally even, that is*

$$f(-x_1, \cdots, -x_l, \cdots, -x_k) = f(x_1, \cdots, x_l, \cdots, x_k)$$

- *(b) a C^1 function with derivatives having polynomial growth on R^k, which is even in each argument, i.e.*

$$f(x_1, \cdots, -x_l, \cdots, x_k) = f(x_1, \cdots, x_l, \cdots, x_k), \quad \forall\ 1 \leq l \leq k$$

If X is continuous, then the process

$$\frac{1}{\sqrt{\Delta_n}} \left(\Delta_n V'(f, k, \Delta_n)_t - \int_0^t \rho^{\otimes k}_{\sigma_u}(f) du \right)$$

converge stably in law to a continuous process $U'(f,k)$ defined on an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ of the space $(\Omega, \mathcal{F}, P)$, which conditionally on the σ-field $\mathcal{F}$ is a centered Gaussian $\mathbb{R}^q$-valued process with independent increments, satisfying

$$\tilde{E}(U'(f_i,k)_t U'(f_j,k)_t) = \int_0^t R^{i,j}_{\sigma_u}(f,k)du$$

where for $i, j = 1, \cdots, q$ we set:

$$R^{i,j}_{\sigma}(f,k) = \sum_{l=-k+1}^{k-1} E\left(f_i(\sigma U_k, \cdots, \sigma U_{2k-1}) f_j(\sigma U_{l+k}, \cdots, \sigma U_{l+2k-1})\right)$$
$$-(2k-1)E\left(f_i(\sigma U_1, \cdots, \sigma U_k)\right) E\left(f_j(\sigma U_1, \cdots, \sigma U_k)\right)$$

The proof of this Corollary is essentially the same as the proof of the Theorem (4.2). *Thus we will not show it again.*

5 Conclusion

In this paper, we study the asymptotic behavior of the normalized sums of functionals of a variety of continuous semimartingales where observations are sampled at stochastic times for financial assets based on high-frequency financial data. Unlike the usual assumptions of regularity or deterministic irregularity for trading times in realized kernel estimators suggested in (Barndorff-Nielsen et al., 2008), we allow the asset return observations to be nonequally spaced in time with stochastic (random) duration times τ_i between the two successive trading times t_{i-1} and t_i. This has practical advantages in the case of tick-by-tick high-frequency financial data, since the direct application of the realized kernel method then produces a biased estimator of the true underlying quadratic variation. Through delicate treatment of the functionals of the increments of the stochastic process for asset returns and duration times, we proved some important asymptotic results for the new estimator including the law of large numbers and the central limit theorem. This work builds the theoretical foundations for our redefined realized kernel estimator, and the subsequent statistical inferences for nonequally-spaced high-frequency financial data with random duration time.

There is also a large open field of research problems remaining for future researchers. As mentioned earlier, our next immediate project is dedicated to large sample asymptotics of the redefined realized kernel volatility estimator. Also, in our current work, we assumed that the stochastic trading times t_i are independent of the log price process X_t. This is fairly restrictive from the application viewpoint; thus, another step ahead would be to obtain a similar law of large numbers and the central limit theorem under a reasonable dependence assumption between the two. This will have to be followed by the re-consideration of the large sample asymptotics of the realized kernel volatility estimator in the dependence case.

6 Acknowledgements

M. Levine's research has been partially supported by the NSF-DMS grant # DMS-1208994.

Bibliography

Ait-Sahalia, Y., Mykland, P. A., and Zhang, L. (2011). Ultra high frequency volatility estimation with dependent microstructure noise. *Journal of Econometrics*, 160(1):160–175.

Barndorff-Nielsen, O. E., Hansen, P. R., Lunde, A., and Shephard, N. (2008). Designing realized kernels to measure the ex post variation of equity prices in the presence of noise. *Econometrica*, 76(6):1481–1536.

Barndorff-Nielsen, O. E., Hansen, P. R., Lunde, A., and Shephard, N. (2009). Realized kernels in practice: Trades and quotes. *The Econometrics Journal*, 12(3):C1–C32.

Barndorff-Nielsen, O. E., Hansen, P. R., Lunde, A., and Shephard, N. (2011). Multivariate realised kernels: consistent positive semi-definite estimators of the covariation of equity prices with noise and non-synchronous trading. *Journal of Econometrics*, 162(2):149–169.

Bouchaud, J.-P., Mézard, M., Potters, M., et al. (2002). Statistical properties of stock order books: empirical results and models. *Quantitative Finance*, 2(4):251–256.

Burkholder, D., Davis, B., and Gundy, R. (2011). Integral inequalities for convex functions of operators on martingales. *Selected Works of Donald L. Burkholder*, page 181.

Cont, R., Stoikov, S., and Talreja, R. (2010). A stochastic model for order book dynamics. *Operations research*, 58(3):549–563.

Engle, R. F. and Russell, J. R. (1998). Autoregressive conditional duration: a new model for irregularly spaced transaction data. *Econometrica*, pages 1127–1162.

Hartley, H., David, H., et al. (1954). Universal bounds for mean range and extreme observation. *The Annals of Mathematical Statistics*, 25(1):85–99.

Jacod, J. and Shiryaev, A. N. (2003). Limit theorems for stochastic processes.

Kessler, M., Lindner, A., and Sorensen, M. (2012). *Statistical methods for stochastic differential equations*, volume 124. CRC Press.

Wang, Y. and Zou, J. (2014). Volatility analysis in high-frequency financial data. *Wiley Interdisciplinary Reviews: Computational Statistics*, 6(6):393–404.

Zhang, L. et al. (2006). Efficient estimation of stochastic volatility using noisy observations: A multi-scale approach. *Bernoulli*, 12(6):1019–1043.

Zhang, L., Mykland, P. A., and Aït-Sahalia, Y. (2005). A tale of two time scales. *Journal of the American Statistical Association*, 100(472).

Zhou, B. (1996). High-frequency data and volatility in foreign-exchange rates. *Journal of Business & Economic Statistics*, 14(1):45–52.